Eggert's conjecture and a structure theorem

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An unsolved important question in the theory of finite rings is the stucture of the unit group of a finite commutative ring of characteristic a prime p, $p > 0$. In this talk we discuss some questions relating to a conjecture by N Eggert (Pacific Journal of Maths 1971) which attempted to provide an answer to this question.

Paul M Cohn 2004 Basic Algebra Groups Rings and Fields:

A ring with identity is completely primary if each of its elements is either invertible or nilpotent. It is is given as an exercise in this book to prove that every commutative Artinian ring with identity can be written as a finite direct product of (commutative) completely primary rings. In particular if R is a finite commutative ring with identity of prime characteristic $p, p > 0$ then

$$R = R_1 \times R_2 \times ....R_n$$

with each R_i cummutative and completely primary. This is proved by B MacDonald in Finite Rings with Identity

Also when R is finite and commutative with identity having characteristic p

$$U(R) = U(R_1) \times U(R_2) \times ..... \times U(R_n)$$

Furthermore the unit group of a finite commutative completely primary ring is simply a direct product of a p group its unique Sylow p group, with a cyclic group of order a power of the prime p -1. So

$$\mathrm{U}(\mathrm{R}_i) \cong Sylow_p U(R_i) \times C^{p^m - 1}$$

some $m \geq 1$

This raises the question what is the stucture of these Sylow p groups. Effectively to solve the unit group question in case the ring is finite with identity and commuative having characteristic a prime p, $p > 0$ we need to classifiy these abelian groups i.e find out what they are.

Which abelian p - groups can arise as the unique Sylow p subgroups of the unit groups of a completely primary ring.

Another characterisation of a finite completely primary ring is a ring with identity in which the zero divisors form an ideal that is the sum of two zero divisors forms another zero divisor. If M is the set of zero divisors then M is the unique maximal ideal of R and is therefore nilpotent because it being unique is the intersection of all the left maximal ideals and is therefore the Jacobson Radical and it is a well known result that the Jacobson Radical is nilpotent

Indeed

$$1 + M \cong Sylow_p(U(R))$$

if we consider the circle operation $\circ$, $x \circ y = x + y - xy$ then $(M, \circ)$ forms an abelian group and $(1 + M, \times) \cong (M, \circ)$.

This circle group is also known as the adjoint or quasi-regular group of a nilpotent ring.

So we could ask which finite abelian p groups arise as the adjoint or quasi- regular or circle groups of fintie commutative algebras (or rings with prime characteristic)

This was Eggert's question.

An algebra M is nilpotent if $M^n = 0$ some $n \geq 1$.

$M^j =<< x_1 x_2 .... x_j, ... x_j \in M, >>$ is the subalgebra generated by monomials of length j or greater in the elements of M.

If M is nilpotent then for some $n \geq 1$,

$x_1 x_2 .... x_n = 0$

$x^n = 0$.

if $M^{n-1} \neq 0$ then the index of nilpotency is defined as n. So the index of nilpotency if n is n is the first power for which

$$M^n = 0$$

In particular if M is finite dimensional then

$$M \supseteq M^2 \supseteq M^3 ..... \supseteq M^{n-1} \supseteq M^n = 0$$

and M^i/M^{n-1} is a non trivial vector space if $i + 1 \leq n$

$d_i = dim_F(M^i/M^{i+1})$ Consider $d_1, d_2.....d_{n-1}$ How are these related? What is the vector space structure of M^i/M^{i+1}?

Clearly $dimM = \sum_{i=1}^{n-1} d_i$.

Eggert showed in 1971 that is had the following result then we would be able to classify these adjoint groups but didn't spefify them exactly.

Eggert's conjecture.

If M is a finite dimensional commutative nilpotent algebra over a field of characteristic a prime p, $p > 0$ then if

$$x_1^p, x_2^p, x_3^p, ...x_n^p$$

are linearly indepdendent in M then

$$dimM \geq pn$$

.

so for example

if $x^p \neq 0$ then $dim M \geq p$

if $\{x^p, y^p\}$ is linearly independent then $dim M \geq 2p$ and

if $\{x^p, y^p, z^p\}$ is linearly independent then $dim M \geq 3p$ and so on.

This is known as Eggert's conjecture for finite dimensional commutative nilpotent algebras.

The Sylow p - groups arise as adjoint groups of cyclic nilpotent or power algebras and their direct products. These abelian p -g roups arise for instance as the circle groups of

$$<< x >>, x^2 = 0, dim = 1$$

$$<< x, x^2 >>, x^3 = 0, dim = 2$$

$$<< x, x^2 x^3 x^3 >>, x^4 = 0, dim = 3$$

$$<< x, x^2, x^3, x^4, >> x^4 = 0, dim = 4$$

....

What Eggert's conjecture is saying is that the only abelian p groups which can arise as quasi regular or adjoint groups of of any finite commutative algebra are the adjoint groups of power or cyclic algebras or their direct products. What are these groups.

Consider the case $p = 2$ and the one dimensional case $<< x >> x^2 = 0$

then $(M, \circ) \cong C_2$ since $x \circ x = x + x - x^2 = 2x = 0$

If we look at the two dimensional case $M =<< x, x^2 >>$ $, x^3 = 0$ then $x \circ x = x + x - x^2 = 2x - x^2 = x^2$

and $x \circ x \circ x = x^2 \circ x = x^2 + x - x^3$ and

$x \circ x \circ x \circ x = x + x^2 + x - x^2 - x^3 = 0$

there is no element of order $2^3 = 8$ and so $(M, \circ) \cong C_4 \times C_2$.

A similar exercise shows that in the four dimensional case $(M, \circ) \cong C_8 \times C_2$.

There are a very restrictive class of abelian p groups. Why?? Beccause of the commutativity together with the vector space structure?

Eggert 1971 establised the equivalence between a conjecture for the structure of the adjoint groups of finite commutative nilpotent algebras and the dimension of nilpotent algebras. This was a key result in this paper. Quasi Regular Groups of Finite Commutative nilpotent algebras. N H Eggert Pac J of Maths 1971

Eggert 1971.

If M is a finite commutative nilpotent algebra over a perfect field of characteristic p $p > 0$ then if $\{x^p, y^p\}$ is linearly independent then $dimM \geq 2p$.

Frobenius 1912 proved that

if $x^n \neq 0$ in any nilpotent algebra then $\{x, x^2, ...x^n\}$ is linearly independent.

The proof is elementary but important

Proof

Suppose $x^n \neq 0$ and $x^m = 0$ but $x^{m-1} \neq 0$ and suppose

$$\lambda_1 x + \lambda_2 x^2 + ...\lambda_n x^n = 0$$

Multiply by x^{m-2} we get

$$\lambda_1 x^{m-1} + \lambda_2 x^m + ...\lambda_n x^{n+m-2} = 0$$

thus

$$\lambda_1 x^{m-1} = 0$$

and thus $\lambda_1 = 0$.

$\lambda_2 x^2 + ... + \lambda_n x^n = 0$

continuing in this manner... multiplying by x^{m-3} we obtain

$\lambda_1 = \lambda_2 = ... = \lambda_n = 0$ so $\{x, x^2.....x^n\}$ are linearly independent.

In particular if $x^p \neq 0$ then $dimM \geq p$.

Anything Frobenius observed must be important!

We could say then that Eggert's conjecture extends Frobenius's 1912 result.

Frobenius also looked at power or cyclic algebra and proved that they are unique up to isomorphism and are precisely those given earlier.

Exercise work out the adjoint group of a five dimensional singly generated power algebra that is one in which $x^5 \neq 0$ but $x^6 = 0$.

Some notation

Let M be a finite dimensional nilpotent p algebra then

$$N^{(p)} =< x^p, x \in M >$$

is the subspace generated by $x^p, x \in M$ and

$M^p =<< x^p, x \in M >>$

is the subalgebra of M generated by $x^p, x \in M$.

Stack 1996 gave an algebraic proof of Eggert's 1971 theorem.

If M is a finite dimensonal nilpotent algebra over a field of characteristic p, $p > 0$ and if $dim M^{(p)} \leq 2$ then $dim M \leq 2p$

Proof.

The main part of the prooof is ths nice structure theorem.............

Theorem 1.

If M is a finite nilpotent algebra and over a field F of characteristic p, $p > 0$ and if $M^p \neq 0$ then if $d_i = dim M^i/M^{i+1}$ for some $1 \leq i \leq p-1$ then $M = Y + A$ where $Y = << y >>$, some $y \in M$ is singly generated of maximal lengh in M and $A = Annil_L(y^{p-1})$.

Proof.

Suppose $d_i = 1$ so $dim M^i/M^{i+1} = 1$ and $i + 1 \leq p$

so there exists a mononial $xx_2...x_i$ such that

$M^i = Kxx_2x_3....x_i + M^{i+1}$.

Choose j maximal such that

$M^i = Kx^j m_2...m_{i-j} + M^{i+1}$

we claim that $j = i$. (this is a key step here) so that

$M^i = Kx^i + M^{i+1}, x \in M$.

To see this observe that

$$M^{i+1} = M^i M = Kxx_2...x_i M + M^{i+2}$$

$$\subseteq xM^i + M^{i+2} \subseteq x^{j+1} M^{i-j-} + M^{i+2}$$

Thus

$$M^{i+1} \subseteq x^{j+1} M^{i-j} + M^{i+2}.$$

Since $M^{i+1} \neq 0$ there exists $x^{j+1} x_2...x_{i-j} \in M^{i+1} - M^{i+2}$ so

x$^{j+1} x_2...x_{i-j-1} \in M^i - M^{i+1}$ and so since $d_i = 1$

$$M^{i+1} = Kx^{j+1}x_2 ... x_{i-j-1} + M^{i+1}$$

This contracdicts the maximality of j and so there exists y such that M$^i = Ky^i + M^{i+1}$.

It is easy to show that this implies that

$$M^i = < y^i, y^2, ...n^{n-1} > \text{ if } M^n = 0 \text{ and } M^{n-1} \neq 0..$$

In particular if $d_i = 1$, we see that $d_j = 1$ and that $M^j/M^{j+1} = Ky^j + M^{j+1}$.

Let $Y =<< y >>$ and consider the map $f : M \to M^{p-1}$

$f : x \to x^{p-1}$

f is a morphism of algebras. $Ker f = A = Annil_L(y^{p-1})$

also $f(Y) = M^{p-1} =< y^{p-1}, y^p....y^{n-1} >$ so f maps Y onto M this gives $Y + A = M$.

Corollary

if $d_i = 1$ $1 \leq i \leq p-1$ and if $N^{(p)} =< x^p, x \in M >$ then $N^{(p)}) = < y^{pi}, i \geq 1 >$

Proof By previous theorem

$M = Y + A$ so $m = y + a$ and so

$m^p = y^p + \sum$ (monomials in the elements of Y and A of length p with at least one element from A.

Consider such a monomial eg $y_1 y_2 .... y_{i-1} a y_{i+1} \cdots y_p$

If $y_1 \in A$ then this monomial $\in AN^{p-1} = 0$. so assume $y_1\ inY$ so $y_1 \in << y >>$.

Now since $M^{p-1} =< y^{p-1}, y^{p-2} ... > Y$ centralises M^{p-1}

so $y_1 y_2 \ldots y_{i-1} a y_{i+1} \ldots y_p$

$= y_2 \ldots\ldots y_{i-1} \ldots y_p y_1 \cdot$

$= \ldots.$

$= a y_{i-1} \ldots y_{i-1} \in AM^{p-1} = 0$

sp $m^p = y^p, y \in Y$

thus $N^{(p)} \subset Y^{(p)} =< y^p, y^{2p}, y^{3p} \ldots\ldots\ldots\ldots\ldots >$ as claimed.

Corollary Eggert's theorem. (This version does not require the ring to be commutative or the field to be perfect).

Theorem 2.

If $M^{(p)} = 2$ then $dimM \geq pdimM^{(p)} = 2p$.

Proof.

Assume the theorem if false and let M be a counterexample of least dimension. Then $dimM \leq 2p$ and so $M^{2p} = 0$ so $m^p n^p = 0$ and so $M^p =< x^p, x \in M >= N^{(p)}$. By the previous corollary $N^{(p})=< y^{pi}, i \geq 1 >$ so since $dimM < 2p$ $M^{2p} = 0$ and so

$M^{(p)} = Ky^p$ and so $diimM^{(p)} = 1$ this is a contradiction which completes the proof.

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