# Krull dimension and unique factorization in Hurwitz polynomial rings

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# The polynomial ring $R[x]$

## In this talk

In this talk all rings are commutative rings with identity.

## The polynomial ring $R[x]$

- Let $R$ be a ring and let
$$R[x] = \{\sum_{i=0}^{n} a_i x^i \mid n \geq 0, a_i \in R\}$$
be the set of polynomials with coefficients in $R$.
- With the usual addition $+$ and multiplication $\cdot$, $R[x]$ becomes a ring that contains $R$ as a subring.
- While the usual multiplication in $R[x]$ is usually considered, in general there do exist many other multiplications in $R[x]$ such that together with the usual addition, $R[x]$ is also a ring that contains $R$ as a subring.

# A generalization of the polynomial ring $R[x]$

## A generalization of the polynomial ring

- Let $\mathbb{N}_0$ (respectively $\mathbb{N}$) be the set of nonnegative (respectively positive) integers.
- Let $\lambda : \mathbb{N}_0 \to \mathbb{N}$ be any function such that

$$\lambda(i)\lambda(j) \text{ divides } \lambda(i+j) \text{ in } \mathbb{N} \text{ for each } i \text{ and } j \text{ in } \mathbb{N}_0.$$

- For each $i$ and $j$ in $\mathbb{N}_0$, let

$$\alpha_{i,j} = \frac{\lambda(i+j)}{\lambda(i)\lambda(j)} \in \mathbb{N}.$$

- We define a multiplication $*$ in $R[x]$ by

$$\left(\sum_{i=0}^{n} a_i x^i\right) * \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} \alpha_{i,j} a_i b_j\right) x^k.$$

- With this multiplication $*$ and the usual addition $+$, the set $R[x]$ becomes a ring that contains $R$ as a subring.
- We denote this ring by $(R[x], \lambda)$.

# A generalization of the polynomial ring $R[x]$

## The polynomial ring $R[x]$

- If $\lambda(i) = 1$ for all $i \in \mathbb{N}_0$, then $\alpha_{i,j} = \frac{\lambda(i+j)}{\lambda(i)\lambda(j)} = 1$ for each $i$ and $j$.

$$\left(\sum_{i=0}^{n} a_i x^i\right) * \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} 1 \cdot a_i b_j\right) x^k.$$

- In this case, the multiplication obtained from $\lambda$ is the usual multiplication in $R[x]$ and we get the usual polynomial ring $R[x]$.

## The Hurwitz polynomial ring $R_H[x]$

- Let $\lambda(i) = i!$ for all $i \in \mathbb{N}_0$. Then $\alpha_{i,j} = \frac{\lambda(i+j)}{\lambda(i)\lambda(j)} = \frac{(i+j)!}{i!j!} = \binom{i+j}{i}$ for each $i$ and $j$ in $\mathbb{N}$.

$$\left(\sum_{i=0}^{n} a_i x^i\right) * \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} \binom{i+j}{i} a_i b_j\right) x^k.$$

- In this case, the ring $(R[x], \lambda)$ is the well-known Hurwitz polynomial ring, which is denoted by $R_H[x]$ (some people use the notation $h(R)$).

# A generalization of the polynomial ring $R[x]$

**Theorem**

Let $R$ be an integral domain with quotient field $K$. If $\operatorname{char} R = 0$, then $(R[x], \lambda)$ is (isomorphic to) an intermediate ring between the usual polynomial rings $R[x]$ and $K[x]$.

**Proof**

- Define a map $\varphi : K[x] \to (K[x], \lambda)$ by
$$\varphi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \lambda(i) a_i x^i.$$
- Then $\varphi$ is a ring homomorphism.
- $\varphi$ is an isomorphism follows from the assumption that $R$ is an integral domain with $\operatorname{char} R = 0$.
- Since $\varphi(R[x]) \subseteq (R[x], \lambda) \subseteq (K[x], \lambda)$, we have $R[x] \subseteq \varphi^{-1}((R[x], \lambda)) \subseteq K[x]$.

**In this talk**

- Recall that if $\lambda(i) = 1$ for all $i \in \mathbb{N}_0$, then $(R[x], \lambda)$ is the usual polynomial ring $R[x]$.
- In the rest of this talk, we only focus on the case $\lambda(i) = i!$ for all $i \in \mathbb{N}_0$, i.e., we only consider the Hurwitz polynomial ring $R_H[x] := (R[x], \lambda)$.

# Some history

### A generalization of the power series ring $R[[x]]$

- Let $\lambda(i) = i!$ for all $i \in \mathbb{N}_0$. Then $\alpha_{i,j} = \frac{\lambda(i+j)}{\lambda(i)\lambda(j)} = \frac{(i+j)!}{i!j!} = \binom{i+j}{i}$ for each $i$ and $j$ in $\mathbb{N}$.
- Similarly, we define a multiplication $*$ in $R[[x]]$ by

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) * \left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} \binom{i+j}{i} a_i b_j\right) x^k.$$

- With this multiplication $*$ and the usual addition $+$, the set $R[[x]]$ becomes a ring that contains $R$ as a subring.
- The case when $\lambda(i) = i!$ for all $i \in \mathbb{N}_0$ gives the well-known Hurwitz power series ring, denoted by $R_H[[x]]$. This kind of multiplication was first considered by Hurwitz and was further studied by Bochner, Martin, Fliess, Taft, Benhissi, Koja, Ghanem, and Liu.
- Closely related to the power series ring, the Hurwitz power series ring has been shown to have many interesting properties, including applications in differential algebra.

# Krull dimension of the Hurwitz polynomial ring $R[x]$

**Proposition**

$R_H[x]$ is an integral domain if and only if $R$ is an integral domain with $\operatorname{char} R = 0$.

**Remark on** $\dim R_H[x]$

- Benhissi and Koja noted that $\operatorname{char} R \neq 0$, then $R_H[x]$ is integral over $R$ and hence $\dim R_H[x] = \dim R$.
- If $R$ is a ring such that $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$ and hence $\dim R_H[x] = \dim R[x]$.
- Hence, when studying the Krull dimension of $R_H[x]$ we can always assume that $\operatorname{char} R = 0$ (so that $\mathbb{Z} \subseteq R$) and that $\mathbb{Q} \not\subseteq R$.

**Well-known result on** $\dim R[x]$

It is well-known that if $R$ is a finite-dimensional ring, then

$$\dim R + 1 \leq \dim R[x] \leq 2 \dim R + 1.$$

# Krull dimension of the Hurwitz polynomial ring $R[x]$

**Lemma**

If $R$ is a ring, then any three different prime ideals $Q_1 \subset Q_2 \subset Q_3$ in $R_H[x]$ cannot contract to the same prime ideal in $R$.

**Theorem**

If $R$ is a finite-dimensional ring, then

$$\dim R \leq \dim R_H[x] \leq 2 \dim R + 1.$$

Furthermore, if $\mathbb{Q} \subseteq R$ or $R$ is an integral domain with $\operatorname{char} R = 0$, then $\dim R + 1 \leq \dim R_H[x]$.

**Proof**

- The above lemma shows that $\dim R_H[x] \leq 2 \dim R + 1$.
- Let $\phi : R_H[x] \to R$ be the natural ring homomorphism mapping each polynomial in $R_H[x]$ to its constant term. Hence, if $P$ is a prime ideal in $R$, then $\phi^{-1}(P)$ is a prime ideal in $R_H[x]$. This shows $\dim R_H[x] \geq \dim R$.
- If $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$ and hence $\dim R_H[x] = \dim R[x] \geq \dim R + 1$.
- If $R$ is an integral domain with $\operatorname{char} R = 0$, then $R_H[x]$ is also an integral domain, which means $(0)$ is a prime ideal in $R_H[x]$. It follows that $\dim R_H[x] \geq n + 1$.

# Krull dimension of the Hurwitz polynomial ring $R[x]$

**Well-known result on** $\dim R[x]$

It is well-known that if $R$ is a finite-dimensional Noetherian ring, then $\dim R[x] = \dim R + 1$, which is a nice application of Krull's Principal Ideal Theorem.

**Remark**

- The Hurwitz polynomial ring $R_H[x]$ is a Noetherian ring if and only if $R$ is a Noetherian ring containing $\mathbb{Q}$.
- Hence, Krull's Principal Ideal Theorem cannot be applied to $R_H[x]$ to show that $\dim R_H[x] \leq \dim R + 1$ when $R$ does not contain $\mathbb{Q}$.
- However, we can still show that $\dim R_H[x] \leq \dim R + 1$ if $R$ is a Noetherian ring.

**Theorem**

If $R$ is a finite-dimensional Noetherian ring, then

$$\dim R \leq \dim R_H[x] \leq \dim R + 1.$$

Furthermore, $\dim R_H[x] = \dim R + 1$ if one of the following holds.

**(1)** $\mathbb{Q} \subseteq R$.

**(2)** $R$ is an integral domain with $\operatorname{char} R = 0$.

**(3)** $\dim R = 0$ (i.e., $R$ is an Artinian ring) and $\operatorname{char} R = 0$.

# Krull dimension of the Hurwitz polynomial ring $R[x]$

**Proof**

The result is proved by using induction on $\dim R$ and the fact $P$ is a prime ideal of $R$ such that $\operatorname{ht} P = 1$ and $\operatorname{char} R/P = 0$, then $\operatorname{ht} P_H[x] = 1$.

**Theorem**

Let $R$ be a Noetherian ring with $\dim R = n \geq 1$. Then the following are equivalent.

**(1)** $\dim R_H[x] = \dim R = n$.

**(2)** For a minimal prime ideal $P$ of $R$, $\operatorname{char} R/P = 0$ implies $\dim R/P \leq n - 1$.

# Unique factorization in $R_H[x]$

**Lemma**

If $R$ is a domain with $\operatorname{char} R = 0$, then $x$ is an irreducible element in $R_H[x]$.

**Proof**

- Suppose that there exist $f = \sum_{i=0}^{r} b_i x^i$ and $g = \sum_{j=0}^{s} c_j x^j$ in $R_H[x]$ such that $x = f * g$.
- We may assume that $r \leq s$.
- Since $R_H[x]$ is a domain, by comparing the degree on both sides of $x = f * g$, we get $r = 0$ and $s = 1$.
- It follows that $1 = b_0 c_1$ and hence $f = b_0$ is a unit.

**Theorem**

The following are equivalent for a ring $R$.

**(1)** $R_H[x]$ is a UFD.

**(2)** $R$ is a UFD and $\mathbb{Q} \subseteq R$.

**(3)** $R$ is a UFD and $R_H[x] \cong R[x]$.

# Unique factorization in $R_H[x]$

**Proof**

(1) $\Rightarrow$ (2) : Suppose that $R_H[x]$ is a UFD. In particular, $R_H[x]$ is a domain. Thus, $R$ is a domain with char $R = 0$.

- If $\mathbb{Q} \subseteq R$, then $R[x] \cong R_H[x]$ is a UFD and hence $R$ is a UFD.
- We now show that $\mathbb{Q} \subseteq R$. Suppose on the contrary that $\mathbb{Q} \not\subseteq R$. Then there exists a prime number $p$ that is not a unit in $R$. We have
$$\underbrace{x * x * \cdots * x}_{p \text{ times}} = p!x^p = (p!) * x^p.$$
- By the above lemma, $x$ is a prime element in $R_H[x]$ (since $R_H[x]$ is a UFD). Thus, $x$ divides either $p!$ or $x^p$ in $R_H[x]$. It is easy to see that $x$ cannot divide $p!$. So $x$ divides $x^p$.
- Therefore, there exists an element $f$ in $R_H[x]$ such that $x * f = x^p$ and hence $f$ must have the form $f = bx^{p-1}$ for some $b \in R$. We have
$$pbx^p = x * (bx^{p-1}) = x * f = x^p.$$
This means $pb = 1$ and $p$ is a unit in $R$, a contradiction.

(2) $\Rightarrow$ (3) : If $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$.

(3) $\Rightarrow$ (1) : It follows from the well known result that if $R$ is a UFD, then so is $R[x]$.

# Unique factorization in $R_H[x]$

**Theorem**

The following are equivalent for a ring $R$.

**(1)** $R_H[x]$ is a Krull domain.

**(2)** $R$ is a Krull domain and $\mathbb{Q} \subseteq R$.

**(3)** $R$ is a Krull domain and $R_H[x] \cong R[x]$.

# Unique factorization in $R_H[x]$

**For more please see**

B. G. Kang and P. T. Toan, Krull dimension and unique factorization in Hurwitz polynomial rings, to appear in Rocky Mountain J. Math..

# Thank you!