Matrix rings generated by a companion matrix

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Introduction

Kronecker: $h(x)$ irreducible polynomial over field F

- $$\frac{F[x]}{\langle h(x)\rangle}$$ field

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- $$F \subseteq \frac{F[x]}{\langle h(x)\rangle}$$

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- $\dfrac{F[x]}{\langle h(x) \rangle}$ field
- $F \subseteq \dfrac{F[x]}{\langle h(x) \rangle}$
- $h(x)$ has a zero in this field.

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$[a\ b\ c][d\ e\ f] = [u\ v\ w]$ where

$$(a + bt + ct^2)(d + et + ft^2) = u + vt + wt^2$$

$$t^3 = h_2t^2 + h_1t + h_0$$

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$$\begin{bmatrix} a & b & c \\ h_0 c & a + h_1 c & b + h_2 c \\ h_0(b + h_2 c) & h_0 c + h_1(b + h_2 c) & a + h_1 c + h_2(b + h_2 c) \end{bmatrix}$$

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$$\cong \{f(E) \mid f(x) \in A[x]\} \qquad E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ h_0 & h_1 & h_2 \end{bmatrix}$$

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  $\gamma(A[x]) = \mathbb{M}_3(A,h)$ subring of $\mathbb{M}_3(A)$ generated by E

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- *Barnett matrices.*

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- quadratic extension
$$\mathbb{Q}(\sqrt{d}) = \frac{\mathbb{Q}[x]}{\langle x^2+d\rangle} \cong \left\{ \begin{bmatrix} a & b \\ -db & a \end{bmatrix} \mid a, b \in \mathbb{Q} \right\}$$

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- $$\frac{\mathbb{R}[x]}{\langle x^3-1\rangle} \cong \left\{ \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$ ring of 3×3 circulant matrices

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- radicals of $\mathbb{M}_{1,k}(A, h) \cong \mathbb{M}_k(A, h)$

General Radical Theory

- Kurosh-Amitsur radical α

  semisimple class $\mathcal{S}\alpha$

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  $\alpha(\mathbb{M}_k(A)) = \mathbb{M}_k(\alpha(A))$

- more useful $\dfrac{A[x]}{\langle h(x) \rangle} \cong \mathbb{M}_{1,k}(A, h)$

Theorem

α hypernilpotent radical

$$h(x) = x^k - h_{k-1}x^{k-1} - \ldots - h_1 x - h_0 \in A[x]$$

$h_{k-1}, h_{k-2}, \ldots, h_1, h_0 \in \alpha(A)$ *and*

$$[\alpha(A)\ 0\ \ldots\ 0] \subseteq \alpha(\mathbb{M}_{1,k}(A, h)).$$

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$$[\alpha(A)\ 0\ \ldots\ 0] \subseteq \alpha(\mathbb{M}_{1,k}(A,h)).$$

Then $\alpha(\mathbb{M}_{1,k}(A,h)) = [\alpha(A)\ A\ \ldots\ A]$ *and*

$$\frac{\mathbb{M}_{1,k}(A,h)}{\alpha(\mathbb{M}_{1,k}(A,h))} \cong A/\alpha(A).$$

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- *A* commutative ring with identity

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- ideals of $\mathbb{M}_{1,2}(A, h)$
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  - $[I \;\; J] \lhd \mathbb{M}_{1,2}(A, h) \Leftrightarrow h_0 J \subseteq I \subseteq J$

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  - $[I \;\; J] \lhd \mathbb{M}_{1,2}(A, h) \Leftrightarrow h_0 J \subseteq I \subseteq J$
  - $[I \;\; A] \lhd \mathbb{M}_{1,2}(A, h) \Leftrightarrow h_0 \in I$ and $\mathbb{M}_{1,2}(A, h)/[I \;\; A] \cong A/I$

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  $\overline{K} := K \cap [A\ 0]$,
  $K_1 := \{a \in A \mid \exists\ b \in A \text{ with } [a\ \ b] \in K\}$ and
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- $[\overline{K}\ \ \overline{K}]$ and $[K_1\ \ K_2]$ ideals of $\mathbb{M}_{1,2}(A, h)$; $[\overline{K}\ \ \overline{K}] \subseteq [K_1\ \ K_2]$

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- *$I \lhd A$ h-prime ideal of A*

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- K prime ideal of $\mathbb{M}_{1,2}(A,h)$. Then K homogeneous $\Leftrightarrow$ $\overline{K}$ is h-prime ideal of A

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- K prime ideal of $\mathbb{M}_{1,2}(A,h)$. Then K homogeneous $\Leftrightarrow \overline{K}$ is h-prime ideal of A

- $I \lhd A$. Then $[I\ \ I]$ maximal ideal in $\mathbb{M}_{1,2}(A,h)$

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$$\overline{K} = K \cap [A\ 0]$$

Ideals

- $K \lhd \mathbb{M}_{1,2}(A, h)$ maximal

  $\Leftrightarrow \overline{K}$ maximal ideal A and one of the following three cases hold:

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(i) $h_0 \in \overline{K}$ in which case $K = [\overline{K}\ A]$ or

(ii) $h_0 \notin \overline{K}$, $\overline{K}$ is h-prime and $K = [\overline{K}\ \overline{K}]$ or

(iii) $h_0 \notin \overline{K}$, $\overline{K}$ is not h-prime and $[\overline{K}\ \overline{K}] \subsetneq K$ with $K_1 = K_2 = A$.

h-prime integers

- h-prime maximal ideals $p\mathbb{Z}$ of the ring $\mathbb{Z}$ for $h(x) = x^2 - h_1 x - h_0 \in \mathbb{Z}[x]$

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- given polynomial $h(x)$, prime number $p \in \mathbb{Z}$ is called *h-prime* if $p\mathbb{Z}$ is h-prime ideal

$$\text{i.e. for all } a, b \in \mathbb{Z},\ p \mid a^2 + h_1 ab - h_0 b^2 \Rightarrow p \mid a \text{ and } p \mid b$$

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  $h(x) = x^2 + 3$, the primes $p = 5, 11, 17$ and 23 are h-prime in $\mathbb{Z}$ but 3 and 13 not h-prime

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Lemma

α radical with $R = \alpha(\mathbb{M}_{1,2}(A, h))$. Then

$$R \subseteq \{[a\ b] \mid \det[a\ b] \in \overline{R} \text{ and } trace[a\ b] \in \overline{R}\}$$
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- relationship between $\alpha(A)$ and $\overline{R}$ where $R = \alpha(\mathbb{M}_{1,2}(A, h))$

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- nil radical and Jacobson radical have ETP

Radicals of 2 x 2 Barnett matrix rings

Theorem

α hypernilpotent radical with ETP. Then:

$$\alpha(\mathbb{M}_{1,2}(A,h)) = \{[a\ b] \mid \det[a\ b] \in \alpha(A) \text{ and trace}[a\ b] \in \alpha(A)\}$$

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If $\mathbb{M}_{1,2}(\alpha(A),h)$ semiprime ideal of $\mathbb{M}_{1,2}(A,h)$, then $\alpha(\mathbb{M}_{1,2}(A,h)) = \mathbb{M}_{1,2}(\alpha(A),h)$.

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  - h can have at most two roots in A

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A integral domain, $\mathcal{N}$ nil radical. Then:

$$\mathcal{N}(\mathbb{M}_{1,2}(A,h)) = \begin{cases} \{0\} \text{ if } \Delta \neq 0 \\ \{[a \ b] \in \mathbb{M}_{1,2}(A,h) \mid 2a + h_1 b = 0\} \text{ if } \Delta = 0 \text{ and } char A \neq 2 \\ \{[a \ b] \in \mathbb{M}_{1,2}(A,h) \mid a^2 + h_0 b^2 = 0\} \text{ if } \Delta = 0 \text{ and } char A = 2 \end{cases}$$

Nil radical

Theorem

A integral domain, $h(x) = x^2 - h_1 x - h_0$ *with discriminant* $\Delta = h_1^2 + 4h_0$.
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If $\mathrm{char} A \neq 2$, *all four statements are equivalent.*

Nil radical

Theorem

Suppose $h(x) = x^2 - h_1 x - h_0 \in A[x]$ *has* $\Delta = 0$; A *integral domain.*
Then:
(1) $h(x) = (x - s)^2$ *for some* $s \in A \Leftrightarrow -h_0$ *is a square in* A.

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Suppose $h(x) = x^2 - h_1 x - h_0 \in A[x]$ *has* $\Delta = 0$; *A integral domain.*
Then:
(1) $h(x) = (x - s)^2$ *for some* $s \in A \Leftrightarrow -h_0$ *is a square in* A.
(2) If $\mathrm{char} A \neq 2$, *then* $h(x) = (x - s)^2$ *for some* $s \in A$
$\Leftrightarrow -h_0$ *is a square in* A
$\Leftrightarrow 2 \mid h_1$

Nil radical

- non-unit $p \in A$

  prime if for all $a, b \in A$, $p \mid ab \Rightarrow p \mid a$ or $p \mid b$

  semiprime if for all $a \in A$, $p \mid a^2 \Rightarrow p \mid a$

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Definition

non-unit $p \in A$ *weakly semiprime* if for all $a \in A$,

$$p^2 \mid a^2 \Rightarrow p \mid a$$

Nil radical

Theorem

A integral domain. Consider two conditions:

(1) Whenever $h(x) = x^2 - h_1 x - h_0 \in A[x]$ *with* $\Delta = 0$,
then $h(x) = (x - s)^2$ *for some* $s \in A$.

(2) 2 *is weakly semiprime in* A.

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Then $(1) \Rightarrow (2)$ *and when* $\mathrm{char} A \neq 2, (1) \Leftrightarrow (2)$.

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- $2 \in A$ prime or semiprime $\Rightarrow$ 2 is weakly semiprime

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Nil radical

- $2 \in A$ prime or semiprime $\Rightarrow$ 2 is weakly semiprime
- $\mathbb{Z}$ integers
  - prime has usual meaning
  - n $(\neq \pm 1)$ is semiprime if and only if n is square free
  - all $n \neq \pm 1$ are weakly semiprime

Nil radical

Theorem

$u \in \mathbb{Z}$, u not a square. Then 2 weakly semiprime in $\mathbb{Z}[\sqrt{u}]$
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- $\mathbb{Z}[\sqrt{-5}]$ integral domain

  2 not semiprime (eg. $2 \mid (1+\sqrt{-5})^2$ but $2 \nmid 1+\sqrt{-5}$)

  2 is weakly semiprime in $\mathbb{Z}[\sqrt{-5}]$.

von Neumann regular radical

- hypoidempotent radicals : nilpotent rings are semisimple

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- $\nu(\mathbb{M}_{1,2}(A,h)) = \{[a\ b] \mid [a\ 0] \text{ and } [b\ 0] \text{ are in } \nu(\mathbb{M}_{1,2}(A,h))\}$

  $\subseteq [\nu(A)\ \nu(A)]$

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  $\subseteq [\upsilon(A)\ \upsilon(A)]$

  $A \in \mathcal{S}\upsilon \Rightarrow \mathbb{M}_{1,2}(A,h) \in \mathcal{S}\upsilon$ and
  $\mathbb{M}_{1,2}(A,h) \in \upsilon \Rightarrow A \in \upsilon$