Integer-valued Polynomials on Algebras: New Results and New Questions

Nicholas J. Werner

SUNY College at Old Westbury

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Outline

Introduction

Background

When D is an integral domain with field of fractions K, the ring of integer-valued polynomials on D is defined to be

$$\text{Int}(D) = \{f(x) \in K[x] \mid f(d) \in D \text{ for all } d \in D\}$$

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Example

Let $n > 1$ and consider polynomials in $K[x]$ that map matrices in $M_n(D)$ back to $M_n(D)$ under evaluation:

$$\{f(x) \in K[x] \mid f(a) \in M_n(D) \text{ for all } a \in M_n(D)\}$$

This is the *ring of integer-valued polynomials on* $M_n(D)$ *with coefficients in* K.

Standard Assumptions

Throughout:

- D is an integral domain
- K is the field of fractions of D
- A denotes a torsion-free D-algebra such that $A \cap K = D$
- We say A is of *finite type* if A is finitely generated as a D-module

Two Ways to Generalize

One way to generalize $\text{Int}(D)$ is to replace D with A in the definition.

Definition

We define $\text{Int}_K(A) = \{f(x) \in K[x] \mid f(a) \in A \text{ for all } a \in A\}$

Since we are assuming that $A \cap K = D$, we always have

$$D[x] \subseteq \text{Int}_K(A) \subseteq \text{Int}(D)$$

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Another way to generalize $\text{Int}(D)$ is to replace D with A and replace K with a larger ring that contains A.

Definition

Let $B = K \otimes_D A$ be the extension of A to a K-algebra. We define

$$\text{Int}(A) = \{f(x) \in B[x] \mid f(a) \in A \text{ for all } a \in A\}$$

With this notation, $\text{Int}_K(A) = \text{Int}(A) \cap K[x]$

Is Int(A) a ring?

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$\text{Int}_K(A)$ is always a commutative ring.

If A is commutative, then $\text{Int}(A)$ is also a commutative ring.

Is $\text{Int}(A)$ a ring?

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If A is commutative, then $\text{Int}(A)$ is also a commutative ring.

But, what if A is noncommutative?

For instance, what happens when $A = M_n(D)$, so that $B = M_n(K)$?

In cases like these, $B[x]$ (and hence $\text{Int}(A)$) contains polynomials with coefficients from a noncommutative ring.

Polynomials over Noncommutative Rings

If B is a noncommutative ring, then we will add and multiply polynomials as we normally would: for all $a, b \in B$,

$$ax^n + bx^n = (a+b)x^n \qquad \text{and} \qquad (ax^n)(bx^m) = (ab)x^{n+m}$$

General conventions:

1. the indeterminate x commutes with everything
2. polynomials are evaluated with the indeterminate on the right

Evaluation can behave in unexpected ways. For example:

Let $a, b \in B$ be such that $ab \neq ba$.
Let $f(x) = x - a$ and $g(x) = x - b$ be elements of $B[x]$.
Let $h(x) = f(x)g(x) = x^2 - (a+b)x + ab$.
Then, $f(a)g(a) = 0$, but $h(a) = ab - ba \neq 0$.

Since $\text{Int}(A)$ is defined entirely in terms of evaluation, it is nontrivial to prove that it is closed under multiplication.

A Sufficient Condition for Int(A) to be a Ring

$$\text{Int}(A) = \{f(x) \in B[x] \mid f(a) \in A \text{ for all } a \in A\}$$

The set Int(A) is always closed under addition (and in fact has a left $\text{Int}_K(A)$-module structure).

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Theorem

Assume each $a \in A$ can be written as a finite sum $a = \sum_i c_i u_i$, where $c_i, u_i \in A$, each u_i is a unit in A, and each c_i is central in B. Then, Int(A) is a ring.

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In particular, Int(A) is a ring in the following cases:

- Matrix rings: $A = M_n(D)$
- Group rings: $A = DG$
- Lipschitz quaternions: $A = \mathbb{Z} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$, where $\mathbf{i}^2 = \mathbf{j}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$
- Hurwitz quaternions: $A = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2}\}$

Necessary and Sufficient Conditions?

The theorem stated on the previous slide is sufficient, but is not necessary.

Theorem (S. Frisch)

Let $n > 1$. Let A be the set of upper triangular matrices in $M_n(D)$. Then, $\text{Int}(A)$ is a ring.

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Open Problems

- If possible, find an example of a D-algebra A such that $\mathrm{Int}(A)$ is **not** a ring.
- Give necessary and sufficient conditions on A so that $\mathrm{Int}(A)$ is a ring.

Conjecture

Assume that A is of finite type. Then, $\mathrm{Int}(A)$ is a ring.

Non-triviality of $\text{Int}_K(A)$

Back to $\text{Int}_K(A)$

Recall that

$$\begin{aligned}\text{Int}_K(A) &= \{f(x) \in K[x] \mid f(a) \in A \text{ for all } a \in A\} \\ &= \text{Int}(A) \cap K[x]\end{aligned}$$

Also recall that we are assuming $A \cap K = D$. This condition is equivalent to having

$$D[x] \subseteq \text{Int}_K(A) \subseteq \text{Int}(D)$$

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Also recall that we are assuming $A \cap K = D$. This condition is equivalent to having

$$D[x] \subseteq \text{Int}_K(A) \subseteq \text{Int}(D)$$

It is natural to consider when these containments are proper. That is:

- When is $D[x] \subsetneqq \text{Int}_K(A)$?
- When is $\text{Int}_K(A) \subsetneqq \text{Int}(D)$?

We will investigate the first question now, and come back to the second question later.

Non-triviality for Int(D)

In the traditional setting, we say that Int(D) is *nontrivial* if $D[x] \subsetneqq \text{Int}(D)$. We adopt the same terminology for $\text{Int}_K(A)$.

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Definition

$\text{Int}_K(A)$ is *nontrivial* if $D[x] \subsetneq \text{Int}_K(A)$.

There are known characterizations of when Int(D) is nontrivial.

- For a Noetherian domain D, Int(D) is nontrivial if and only if there is a prime conductor ideal of D with finite residue field.
- D. Rush gave a double-boundedness condition on D that is necessary and sufficient for Int(D) to be nontrivial, and which holds for any domain D.

Non-triviality for $\text{Int}_K(A)$

The first of these non-triviality conditions carries over directly to $\text{Int}_K(A)$.

Theorem (S. Frisch)

Let D be Noetherian and let A be of finite type. Then, $\text{Int}_K(A)$ is nontrivial if and only if there is a prime conductor ideal of D with finite residue field.

By using Rush's criterion, we can drop the Noetherian condition on D and weaken the assumption that A is finitely generated.

Algebraic and Integral Algebras

Definition

Let R be a commutative and A an R-algebra.

We say that A is an *algebraic algebra* over R if every element of A satisfies a polynomial with coefficients in R.

We say that A is an *integral algebra* over R if every element of A satisfies a **monic** polynomial with coefficients in R.

We say that A is of *bounded degree* if there is a uniform bound on the degrees of the minimal polynomials of elements of A.

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Theorem

Let A be an integral D-algebra of bounded degree. Then, $\text{Int}_K(A)$ is nontrivial if and only if $\text{Int}(D)$ is nontrivial.

In particular, this theorem applies when A is finitely generated as a D-module.

Sketch of the proof

Theorem

Let A be an integral D-algebra of bounded degree. Then, $\text{Int}_K(A)$ is nontrivial if and only if $\text{Int}(D)$ is nontrivial.

Here is the idea of the proof:

- Assuming A is an integral D-algebra of bounded degree n, show that $\text{Int}_K(M_n(D)) \subseteq \text{Int}_K(A)$. Thus, we have the following containments:

$$D[x] \subseteq \text{Int}_K(M_n(D)) \subseteq \text{Int}_K(A) \subseteq \text{Int}(D)$$

- Use Rush's double-boundedness criteria to prove that if $\text{Int}(D)$ is nontrivial, then $\text{Int}_K(M_n(D))$ is nontrivial.

General Case

What if A is not finitely generated? It turns out that $\text{Int}_K(A)$ can still be nontrivial.

Example

Let $A = \prod_{i \in \mathbb{N}} \mathbb{Z}$. Then, $\text{Int}_{\mathbb{Q}}(A) = \text{Int}(Z)$, so $\text{Int}_{\mathbb{Q}}(A)$ is nontrivial.

In this example, A is not an algebraic $\mathbb{Z}$-algebra (let alone integral or of bounded degree).

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However, for each prime p, every element of A/pA is killed by $x^p - x$. So, A/pA is an algebraic algebra of bounded degree over $\mathbb{Z}/p\mathbb{Z}$.

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Theorem

Let D be a Dedekind domain. Then, $\text{Int}_K(A)$ is nontrivial if and only if there exists a prime ideal P of D of finite index such that A/PA is a D/P-algebraic algebra of bounded degree.

Examples

1. Let $D = \mathbb{Z}$ and $A = \overline{\mathbb{Z}}$, the absolute integral closure of $\mathbb{Z}$.
   Then, for each prime p, A/pA is an algebraic $\mathbb{Z}/p\mathbb{Z}$-algebra of unbounded degree.
   Thus, $\text{Int}_{\mathbb{Q}}(\overline{\mathbb{Z}}) = \mathbb{Z}[x]$.

2. Let $D = \mathbb{Z}_{(p)}$ and $A = \mathbb{Z}_p$, the p-adic integers.
   Then, $A/pA \cong D/pD \cong \mathbb{Z}/p\mathbb{Z}$, so $\mathbb{Z}_{(p)}[x] \subsetneqq \text{Int}_{\mathbb{Q}}(A)$.
   In fact, $\text{Int}_{\mathbb{Q}}(A) = \text{Int}(D)$ in this case.

3. Let D be a DVR with maximal ideal P and finite residue field.
   Let A be a D-algebra such that $\text{Int}_K(A) \subsetneqq \text{Int}(D)$, and let $\widehat{A}$ be the P-adic completion of A.
   Then, we have
   $$D[x] \subsetneqq \text{Int}_K(\widehat{A}) = \text{Int}_K(A) \subsetneqq \text{Int}(D)$$

Decomposition of $\text{Int}(A)$

Motivation: Matrix Rings

$\text{Int}_K(A)$ is commutative, so it should be easier to work with than $\text{Int}(A)$.
Question: What can $\text{Int}_K(A)$ tell us about $\text{Int}(A)$?
Answer: In some cases, quite a bit!

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Theorem (S. Frisch)

When $A = M_n(D)$, $\mathrm{Int}(A)$ is itself a matrix ring. Explicitly,

$$\mathrm{Int}(M_n(D)) \cong M_n(\mathrm{Int}_K(M_n(D)))$$

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The isomorphism in the theorem is achieved by associating a polynomial with matrix coefficients to a matrix with polynomial entries.
For example, with $M_2(\mathbb{Z})$,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{x^2(x-1)^2(x^2+x+1)}{2} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x^2 + 3x$$

corresponds to

$$\begin{pmatrix} \frac{x^2(x-1)^2(x^2+x+1)}{2} + 3x & x^2 \\ -x^2 & 3x \end{pmatrix}$$

Matrix Rings Redux

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Question: Are there other algebras that exhibit this behavior?

To find out, we will rephrase the theorem.

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For $1 \leq i, j \leq n$, let E_{ij} be the matrix with 1 in the (i, j)-entry and 0 elsewhere.

Then, $M_n(D) = \bigoplus_{i,j} DE_{ij}$ (direct sum as a D-module).

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Theorem (S. Frisch)

$$\text{Int}(M_n(D)) = \bigoplus_{i,j} \text{Int}_K(M_n(D))E_{ij} \text{ (direct sum as an } \text{Int}_K(M_n(D))\text{-module)}$$

Int-decomposable Algebras

Definition

Let $A = \bigoplus_{i=1}^{t} D\alpha_i$ be a free D-algebra. We say that A is *Int-decomposable* (with respect to $\{\alpha_i\}_{i=1}^{t}$) if $\text{Int}(A) = \bigoplus_{i=1}^{t} \text{Int}_K(A)\alpha_i$.

In other words, a (free) Int-decomposable algebra A is one with the following property:

Let $f \in \text{Int}(A)$

Write $f = \sum_i f_i\alpha_i$, where $f_i \in K[x]$

Then, each $f_i \in \text{Int}_K(A)$

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Lemma

Being Int-decomposable is independent of the D-basis we choose for A.

Examples and Non-examples

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Non-example: Gaussian Integers

Let $D = \mathbb{Z}$ and $A = \mathbb{Z}[\mathbf{i}]$.

Then, $\dfrac{(1+\mathbf{i})(x^2 - x)}{2} \in \text{Int}(\mathbb{Z}[\mathbf{i}])$, but $\dfrac{x^2 - x}{2} \notin \text{Int}_{\mathbb{Q}}(\mathbb{Z}[\mathbf{i}])$

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Non-example: Lipschitz Quaternions

Let $A = \mathbb{Z} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbf{k}$ with $\mathbf{i}^2 = \mathbf{j}^2 = -1$.

Then, $\dfrac{(1+\mathbf{i}+\mathbf{j}+\mathbf{k})(x^2-x)}{2} \in \text{Int}(A)$, but $\dfrac{x^2-x}{2} \notin \text{Int}_{\mathbb{Q}}(A)$

A Characterization Theorem

Example

Let p be an odd prime and $D = \mathbb{Z}_{(p)}$. Let A be the quaternion algebra $A = D \oplus D\mathbf{i} \oplus D\mathbf{j} \oplus D\mathbf{k}$ where $\mathbf{i}^2 = \mathbf{j}^2 = -1$.

Then, A is Int-decomposable.

The algebra in this example is not a matrix ring, but there is a connection to 2×2 matrices: $A/pA \cong M_2(\mathbb{F}_p)$.

This turns out to be what we need to classify Int-decomposable algebras.

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Theorem

Let D be a Dedekind domain with finite residue rings. Let A be a free D-algebra. Then, A is Int-decomposable if and only if for each nonzero prime P of D, there exist $n, t > 0$ and a finite field $\mathbb{F}_q$ such that $A/PA \cong \bigoplus_{i=1}^{t} M_n(\mathbb{F}_q)$.

Sketch of Proof

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The proof is involved! But here are the major steps.

- Localize at P. Thus, WLOG we can assume that D is a DVR with maximal ideal πD.
- There is a correspondence between polynomials in $\text{Int}(A)$ and polynomials in the *null ideals*

  $$N(A/\pi^k A) = \{g(x) \in (A/\pi^k A)[x] \mid g(a) = 0 \text{ for all } a \in A/\pi^k A\}$$

  Explicitly, $g(x)/\pi^k \in \text{Int}(A)$ if and only if $g(x) \in N(A/\pi^k A)$.

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- Prove that $N(A/\pi^k A)$ is "decomposable" if and only if $A/\pi^k A \cong \bigoplus_{i=1}^{t} M_n(T)$, where T is a commutative local rings of a certain form.

Sketch of Proof

Theorem

Let D be a Dedekind domain with finite residue rings. Let A be a free D-algebra. Then, A is Int-decomposable if and only if for each nonzero prime P of D, there exist $n, t > 0$ and a finite field $\mathbb{F}_q$ such that $A/PA \cong \bigoplus_{i=1}^{t} M_n(\mathbb{F}_q)$.

The proof is involved! But here are the major steps.

- Localize at P. Thus, WLOG we can assume that D is a DVR with maximal ideal πD.
- There is a correspondence between polynomials in $\mathrm{Int}(A)$ and polynomials in the *null ideals*

  $$N(A/\pi^k A) = \{g(x) \in (A/\pi^k A)[x] \mid g(a) = 0 \text{ for all } a \in A/\pi^k A\}$$

  Explicitly, $g(x)/\pi^k \in \mathrm{Int}(A)$ if and only if $g(x) \in N(A/\pi^k A)$.
- Develop an analogous notion of "decomposability" for the null ideals $N(A/\pi^k A)$.
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- Show that it is enough to check just $A/\pi A$.

Extending the Definition

The definition of Int-decomposable relies on the presence of a D-basis for A.

Can we make this notion work when A is not free?

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Key observation: Recall that $B = K \otimes_D A$.

Then, A, $\text{Int}(A)$, and $\text{Int}_K(A)$ are all contained in $B[x]$.

When $\text{Int}(A)$ is Int-decomposable, $\text{Int}(A)$ is equal to the subring of $B[x]$ generated by $\text{Int}_K(A)$ and A.

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Definition

We say that A (not necessarily free!) is *Int-decomposable* if

$$\text{Int}(A) \cong \text{Int}_K(A) \otimes_D A$$

Informally, $\text{Int}(A)$ is Int-decomposable if $\text{Int}(A)$ is equal to the subring of $B[x]$ generated by $\text{Int}_K(A)$ and A.

The Same Classification Theorem

Theorem

Let D be a Dedekind domain with finite residue rings. Assume that A is finitely generated as a D-module. Then, the following are equivalent.

1. A is Int-decomposable
2. For each nonzero prime P of D, there exist $n, t > 0$ and a finite field $\mathbb{F}_q$ such that $A/PA \cong \bigoplus_{i=1}^{t} M_n(\mathbb{F}_q)$

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3. For each nonzero prime P of D, there exist $n, t > 0$ such that the completion $\widehat{A}_P$ satisfies $\widehat{A}_P \cong \bigoplus_{i=1}^{t} M_n(\widehat{T}_P)$, where $\widehat{T}_P$ is a complete DVR with finite residue field and fraction field that is a finite unramified extension of $\widehat{K}_P$.

When $\mathrm{Int}_K(A) = \mathrm{Int}(D)$

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Theorem

Let D be a Dedekind domain with finite residue rings. Assume that A is of finite type. Then, the following are equivalent.

1. $\text{Int}_K(A) = \text{Int}(D)$
2. For each nonzero prime P of D, $A/PA \cong \bigoplus_{i=1}^{t} D/P$, for some $t > 0$.
3. For each nonzero prime P of D, $\widehat{A}_P \cong \bigoplus_{i=1}^{t} \widehat{D}_P$, for some $t > 0$.

A Crazy Theorem...

In the case where D is the ring of integers of a number field, we can also give a global characterization of Int-decomposability.

Theorem

Let K be a number field with ring of integers D. As usual, let A be a D-algebra of finite type and let $B = K \otimes_D A$.

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   (ii) F is a finite unramified Galois field extension of K
   (iii) each B_i is unramified at every finite place of F
   (iv) the degree of B_i as an F-central simple algebra is the same for each i

With Some Very Nice Corollaries

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Corollary

Let $K \subseteq L$ be number fields with rings of integers O_K and O_L. Consider O_L as an O_K-algebra. Then,

1. O_L is Int-decomposable if and only of L/K is an unramified Galois extension
2. $\text{Int}_K(O_L) = \text{Int}(O_K)$ if and only if $L = K$

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Corollary

Let A be a $\mathbb{Z}$-algebra that is finitely generated as a $\mathbb{Z}$-module.

1. A is Int-decomposable if and only if $A \cong \bigoplus_{i=1}^{t} M_n(\mathbb{Z})$ for some n and t
2. $\text{Int}_{\mathbb{Q}}(A) = \text{Int}(\mathbb{Z})$ if and only if $A \cong \bigoplus_{i=1}^{t} \mathbb{Z}$ for some t

Other Decompositions

We have classified the algebras such that $\text{Int}(A) \cong \text{Int}_K(A) \otimes_D A$

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Theorem (S. Frisch)

Let D be a domain. Let $T_n(D)$ be the ring of upper triangular matrices with entries in D. Then,

$$\text{Int}(T_n(D)) \cong \begin{pmatrix} \text{Int}_K(T_n(D)) & \text{Int}_K(T_{n-1}(D)) & \cdots & \text{Int}_K(T_2(D)) & \text{Int}_K(T_1(D)) \\ 0 & \text{Int}_K(T_{n-1}(D)) & \cdots & \text{Int}_K(T_2(D)) & \text{Int}_K(T_1(D)) \\ & & \ddots & & \\ 0 & 0 & \cdots & \text{Int}_K(T_2(D)) & \text{Int}_K(T_1(D)) \\ 0 & 0 & \cdots & 0 & \text{Int}_K(T_1(D)) \end{pmatrix}$$

Further Questions

What about Nonassociative Algebras?

Throughout, A has always been an associative algebra.

But, to define $\text{Int}_K(A)$, all we need is for A to be *power associative*, meaning that $a^n a^m = a^{n+m}$ for all $a \in A$.

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Question: Does anything interesting happen with $\text{Int}_K(A)$ if we allow A to be nonassociative?

Question: Let $\mathbb{O}_\mathbb{Q}$ be the rational octonions, and define

$$\text{Int}(\mathbb{O}_\mathbb{Z}) = \{f(x) \in \mathbb{O}_\mathbb{Q}[x] \mid f(a) \in \mathbb{O}_\mathbb{Z} \text{ for all } a \in \mathbb{O}_\mathbb{Z}\}$$

Does $\text{Int}(\mathbb{O}_\mathbb{Z})$ have a (nonassociative) ring structure? In other words, is $\text{Int}(\mathbb{O}_\mathbb{Z})$ closed under multiplication?

Integer-valued Rational Functions

Another variation on $\mathrm{Int}(D)$ is to study integer-valued rational functions. We define $\mathrm{Int}^{\mathrm{R}}(D) = \{\phi(x) \in K(x) \mid \phi(d) \in D \text{ for all } d \in D\}$.

There is nothing stopping us from doing the same thing with algebras:

$$\mathrm{Int}_K^{\mathrm{R}}(A) = \{\phi(x) \in K(x) \mid \phi(a) \in A \text{ for all } a \in A\}$$

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To get rational functions in $\text{Int}_K^{\text{R}}(A)$, we need to find polynomials that are unit-valued on A.

If $u(x) \in D[x]$ is unit-valued on A and $f(x) \in D[x]$, then $\phi(x) = \frac{f(x)}{u(x)} \in \text{Int}_K^{\text{R}}(A)$.

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Let $U = \{\text{unit-valued polynomials in } D[x]\}$. Then $\text{Int}_K^{\text{R}}(A)$ contains $U^{-1}D[x]$.

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Let $U = \{\text{unit-valued polynomials in } D[x]\}$. Then $\text{Int}_K^{\text{R}}(A)$ contains $U^{-1}D[x]$.

However, there are examples where $\text{Int}_K^{\text{R}}(A)$ strictly contains $U^{-1}D[x]$.

Proposition

Let $D = \mathbb{Q}[t]_{(t)}$ and $A = M_2(D)$. Then, the polynomial $x^4 + t$ is not unit-valued on A, but $t/(x^4 + t) \in \text{Int}_K^{\text{R}}(A)$.

Integer-valued Polynomials on Subsets

The traditional construction of $\mathrm{Int}(D)$,

$$\mathrm{Int}(D) = \{f \in k[x] \mid f(d) \in D \text{ for all } d \in D\}$$

can be extended to polynomials evaluated only on subsets of D.
For a subset $S \subseteq D$, we define

$$\mathrm{Int}(S, D) = \{f \in k[x] \mid f(s) \in D \text{ for all } s \in S\}$$

The rings $\mathrm{Int}(S, D)$ are well-studied, although in general they are harder to work with than $\mathrm{Int}(D)$.

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The rings Int(S, D) are well-studied, although in general they are harder to work with than Int(D).

Question: What happens if we attempt this with noncommutative rings?
For a subset $S \subseteq A$, we can define

$$\text{Int}(S, A) = \{f(x) \in B[x] \mid f(s) \in A \text{ for all } s \in S\}$$

Question: What can we prove about Int(S, A)? In particular, when is it a ring?

Example

Let A be a noncommutative $\mathbb{Z}$-algebra. Let $a, b \in A$ such that $ab \neq ba$.

Take $S = \{a\}$.

Then, $x - b \in \text{Int}(S, A)$ and $\frac{x-a}{n} \in \text{Int}(S, A)$ for all $n > 0$.

Since $ab - ba \neq 0$, there exists $m \in \mathbb{Z}$ such that $ab - ba \notin mA$.

Let $f(x) = \frac{x-a}{m}(x - b) = \frac{x^2-(a+b)x+ab}{m}$.

Then, $f(a) = \frac{ab-ba}{m} \notin A$.

Thus, $\text{Int}(S, A)$ is not closed under multiplication, and hence is not a ring.

A Sufficient Condition

Proposition

Assume that A is generated by a set of units U. If $uSu^{-1} \subseteq S$ for all $u \in U$, then $\text{Int}(S, A)$ is a ring.

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Let A be the Lipschitz quaternions: $A = \mathbb{Z} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$.

Let $S = \{\mathbf{i}, -\mathbf{i}\}$.

Then, $uSu^{-1} \subseteq S$ for all $u \in \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, so $\text{Int}(S, A)$ is a ring.

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Example

Let A be the Lipschitz quaternions and let $S = \{\mathbf{i}, \mathbf{j}\}$.

Then, one can prove that $\text{Int}(S, A)$ is a ring.

Union and Intersections

Lemma

Let A be a D-algebra. Let $S, T \subseteq A$ be such that both $\text{Int}(S, A)$ and $\text{Int}(T, A)$ are rings. Then, $\text{Int}(S \cup T, A)$ is a ring.

So, the collection of subsets S of A such that $\text{Int}(S, A)$ is a ring is closed under unions.

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Let A be the Lipschitz quaternions, $S = \{\mathbf{i}, -\mathbf{i}\}$, and $T = \{\mathbf{i}, \mathbf{j}\}$. Then, $S \cap T = \{\mathbf{i}\}$, and $\text{Int}(S \cap T, A)$ is not a ring.

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Question: What is going on?

Problem to work on: Which finite subsets S of $M_2(\mathbb{Z})$ are such that $\text{Int}(S, M_2(\mathbb{Z}))$ is a ring?

Thank you!