Polynomials Inducing the Zero Function on Local Rings

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Definition

For a subset S and ideal J of R, we denote by $\mathcal{Z}(S, J)$ the ideal of polynomials in $R[x]$ which map S into J. When $J = 0$, we simply write $\mathcal{Z}(S)$. The focus of this talk is on the connection between $\mathcal{Z}(R)$, which we call the *zero-function ideal* of R, and $\mathcal{Z}(\mathfrak{m})$.

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$\mathcal{Z}(R)$ is the kernel of the map

$$R[x] \to \{\text{polynomial functions on } R\}$$

Some Previous Results

K Kempner, A., *Polynomials and their Residual Systems*, Trans. Amer. Math. Soc. **22** (1921), 240–288.

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$$\mathcal{Z}\left(\mathbb{Z}_9\right) = (x(x-1)(x-2)(x-3)(x-4)(x-5), 3x(x-1)(x-2))$$

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One can construct polynomials in $\mathcal{Z}(R)$ by composing a polynomial in $\mathcal{Z}(\mathfrak{m})$ with a polynomial in $\mathcal{Z}(R, \mathfrak{m})$.

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$\mathcal{Z}(\mathfrak{m})$ is often easier to work with than $\mathcal{Z}(R)$.

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- If R is an infinite domain, then $\mathcal{Z}(R) = 0$.

Example. If $R = \mathbb{F}_q[[S,T]]/(S^2, ST)$, then $\mathcal{Z}(\mathfrak{m}) = (sx)$, so $\mathcal{Z}(R) \supseteq (s(x^q - x))$.

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Theorem

Let $(R, \mathfrak{m})$ be a Noetherian local ring.

1. *$\mathcal{Z}(\mathfrak{m})$ contains nonzero polynomials if and only if $\operatorname{depth} R = 0$.*
2. *$\mathcal{Z}(R)$ contains nonzero polynomials if and only if $\operatorname{depth} R = 0$ and $\overline{R}$ is finite.*

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3. $\mathcal{Z}(\mathfrak{m})$ *contains regular polynomials if and only if* $\dim R = 0$.
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π-polynomials

Definition

Suppose the local ring $(R, \mathfrak{m})$ has a finite residue field. If $c_1, \ldots, c_q$ is any set of representatives of the residue classes of $\mathfrak{m}$, then we call the polynomial $\pi(x) = \prod_{i=1}^{q}(x - c_i)$ a π-*polynomial* for R.

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If R is a local ring with residue field $\overline{R}$ of cardinality q, then $\mathcal{Z}(R, \mathfrak{m}) = (\pi(x), \mathfrak{m})$ for any π-polynomial.

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$$\pi(R) = \pi(c + \mathfrak{m}) = \mathfrak{m}$$

for any $c \in R$. (Bandini (2002) showed $x^p - x$ maps $\mathbb{Z}_{p^n}$ onto $p\mathbb{Z}_{p^n}$).

Main Result

(Roughly, $\mathcal{Z}(\mathfrak{m}) \circ \mathcal{Z}(R, \mathfrak{m}) = \mathcal{Z}(R)$)

Theorem

Suppose $(R, \mathfrak{m})$ *is a Henselian local ring with finite residue field* $\overline{R}$ *of cardinality* q *and let* $\pi(x)$ *be an arbitrary* π*-polynomial. If* $\mathcal{Z}(\mathfrak{m}) = (F_1(x), \ldots, F_n(x))$ *then* $\mathcal{Z}(R) = (F_1(\pi(x)), \ldots, F_n(\pi(x)))$.

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- Let $R = \mathbb{Z}_{p^2}$, so $\mathfrak{m} = (p)$. Then $\mathcal{Z}(\mathfrak{m}) = (x^2, px)$, so

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The proof requires the following:

Proposition

Let $(R, \mathfrak{m})$ *be a Henselian local ring with finite residue field* $\overline{R}$ *and let* $\pi(x)$ *be an arbitrary* π*-polynomial. Any* $f(x) \in \mathcal{Z}(R)$ *may be written in the form*

$$f(x) = p_0(\pi(x)) + xp_1(\pi(x)) + x^2 p_2(\pi(x)) + \cdots + x^{q-1} p_{q-1}(\pi(x))$$

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Part of proof: Since $\pi(x)$ maps each coset of $\mathfrak{m}$ onto $\mathfrak{m}$, for each $m \in \mathfrak{m}$ there exist a complete set of representatives $c_1, \ldots, c_q \in R$ of the residue field with $\pi(c_i) = m$.

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whose coefficient matrix is a Vandermonde matrix with unit determinant. Hence $p_i(m) = 0$ for all i and $m \in \mathfrak{m}$.

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Theorem

Let $(R, \mathfrak{m})$ be a finite local ring with principal maximal ideal $\mathfrak{m} = (m)$; set $q = |R/\mathfrak{m}|$. Suppose e is the index of nilpotency of $\mathfrak{m}$. If $e \leqslant q$ then $\mathcal{Z}(\mathfrak{m}) = (x, m)^e$; if $e = q + 1$, then $\mathcal{Z}(\mathfrak{m}) = (x, m)^e + (x^q - m^{q-1}x)$.

This is a version of results of Dickson (1929), Lewis (1956), and Bandini (2002), adapted for $\mathcal{Z}(\mathfrak{m})$ rather than $\mathcal{Z}(R)$, and for finite local rings rather than specific rings.

Other Results

Gilmer (2000) showed that if $(R, \mathfrak{m})$ is a zero-dimensional local ring, then $\mathcal{Z}(R)$ is principal if and only if either $\overline{R}$ is infinite (when $\mathcal{Z}(R) = 0$) or R is a finite field (when $\mathcal{Z}(R)$ is generated by $x^q - x$.)

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Theorem

Let $(R, \mathfrak{m})$ be a finite local ring and let $\pi(x)$ be any π-polynomial for R. The following statements are equivalent:

1. *R is a field.*
2. *$\mathcal{Z}(R)$ is principal.*
3. *$\mathcal{Z}(R) = (\pi(x))$.*
4. *$\mathcal{Z}(\mathfrak{m})$ is principal.*
5. *$\mathcal{Z}(\mathfrak{m}) = (x)$.*

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5. $\mathcal{Z}(\mathfrak{m})$ *is generated by a regular polynomial if and only if* $\operatorname{edim} R = 0$*.*
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Note: For I and ideal of R and $c \in R$, it's clear that $f(x) \in \mathcal{Z}(I)$ if and only if $f(x-c) \in \mathcal{Z}(c+I)$.

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Proposition

Let $(R, \mathfrak{m})$ be a (finite) local ring with residue field $\overline{R}$ of cardinality q. Let $c_1, \ldots, c_q$ be a set of representatives of the residue classes of $\mathfrak{m}$. Then $\mathcal{Z}(R) = \bigcap_{i=1}^{q} \mathcal{Z}(c_i + \mathfrak{m})$ is a minimal primary decomposition of $\mathcal{Z}(R)$. For each i, the associated prime of $\mathcal{Z}(c_i + \mathfrak{m})$ is the maximal ideal $(x - c_i, \mathfrak{m})$.

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Peruginelli's paper (2014) gives this result for $R = \mathbb{Z}/p^n\mathbb{Z}$, p prime.

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2. The elements of $\mathcal{Z}(\mathfrak{m})$ reflect the same properties of R as do the elements of $\mathcal{Z}(R)$.
3. Constructing $\mathcal{Z}(\mathfrak{m})$ constructs a minimal primary decomposition of $\mathcal{Z}(R)$.

Bibliography

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