Polynomial constants of cyclic factorizable derivations

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Notation

$k$ - a field of characteristic zero
$\mathbb{N}$ - the set of nonnegative integers
$\mathbb{N}_+$ - the set of positive integers
$\mathbb{Q}_+$ - the set of positive rationals
$n$ - an integer $\geq 3$

$k[X] = k[x_1, \ldots, x_n]$

$k(X) = k(x_1, \ldots, x_n)$
Motivation

- applications in population biology, laser physics, plasma physics, hydrodynamics, chemistry, biochemistry, neural networks, economics
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- Lagutinskii’s procedure of association of the factorizable derivation with any given derivation
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- applications in population biology, laser physics, plasma physics, hydrodynamics, chemistry, biochemistry, neural networks, economics
- Lagutinskii’s procedure of association of the factorizable derivation with any given derivation
- link to the invariant theory (for every connected algebraic group $G \subseteq \text{Gl}_n(k)$ there exists a derivation $d$ such that $k[x_1, \ldots, x_n]^G = k[x_1, \ldots, x_n]^d$)
If $R$ is a commutative $k$-algebra, then a $k$-linear map $d : R \rightarrow R$ is called a \textit{derivation} of $R$ if for all $a, b \in R$
\[
d(ab) = ad(b) + d(a)b.
\]

We call $R^d = \ker d$ the \textit{ring of constants} of the derivation $d$.

Then $k \subseteq R^d$ and a \textit{nontrivial} constant of $d$ is an element of the set $R^d \setminus k$.

If $f_1, \ldots, f_n \in k[X]$, then there exists exactly one derivation $d : k[X] \rightarrow k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$. 
Example 1. Hilbert’s fourteenth problem.

Let \( L \) be a field such that \( k \subseteq L \subseteq k(X) \).

Problem (1900, Hilbert): Is the ring \( L \cap k[X] \) finitely generated over \( k \)?

1958, Nagata: no.
1993, Derksen: that counterexample is of the form \( k[X]^d \) for \( n = 32 \).
1994, Roberts, Deveney, Finston: a counterexample for \( n = 7 \).
1998, Freudenburg: for \( n = 6 \).
1999, Daigle, Freudenburg: for \( n = 5 \).
2005, Kuroda: for \( n = 4 \).
1988, Nagata, Nowicki: \( k[X]^d \) is finitely generated over \( k \) for \( n \leq 3 \).
Example 2. Jouanolou derivations.

Let $s, n \in \mathbb{N}$, $s \geq 2$, $n \geq 3$

$$d : \mathbb{C}[x_1, \ldots, x_n] \rightarrow \mathbb{C}[x_1, \ldots, x_n]$$

$$d(x_i) = x_{i+1}^s$$

(in a cyclic sense, that is, $x_{n+1} = x_1$).

1979, Jouanolou: if $n = 3$, then $\mathbb{C}[x_1, \ldots, x_n]^d = \mathbb{C}$.
2000, Maciejewski, Moulin Ollagnier, Nowicki, Strelcyn: if $n$ is a prime number $\geq 3$, then $\mathbb{C}[x_1, \ldots, x_n]^d = \mathbb{C}$.
2003, Žoładek: if $n \geq 3$, then $\mathbb{C}[x_1, \ldots, x_n]^d = \mathbb{C}$. 
A derivation $d : k[X] \rightarrow k[X]$ is called \textit{factorizable} if $d(x_i) = x_if_i$, where the polynomials $f_i$ are of degree 1 for $i = 1, \ldots, n$.

A derivation $d : k[X] \rightarrow k[X]$ is said to be \textit{cyclic factorizable} if $d(x_i) = x_i(A_ix_{i-1} + B_ix_{i+1})$, where $A_i, B_i \in k$ for $i = 1, \ldots, n$ (we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$).

The task: determine $k[X]^d$ (and then the field of constants).
We call a derivation $d : k[X] \rightarrow k[X]$ Lotka-Volterra with parameters $C_1, \ldots, C_n \in k$ if

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$$

for $i = 1, \ldots, n$ (in a cyclic sense).
J. Moulin Ollagnier, A. Nowicki (1999): for $n = 3$ and arbitrary parameters $C_i$

P. Hegedűs (2012): for an arbitrary $n$ but all $C_i$ equal to 1

J. Z. (2013): for $n = 4$ and parameters $C_i$

P. Hegedűs, J. Z. (2016): for an arbitrary $n$ and arbitrary parameters
Main results

If $C_i = 1$ for all $i$, then we call $d$ a Volterra derivation.

**Theorem (Hegedűs)**

The ring of constants of Volterra derivation in $n$ variables is finitely generated over $k$ with $[(n + 2)/2]$ generators. In each case it is a polynomial ring.

**Theorem (Hegedűs and Z.)**

The ring of constants of Lotka-Volterra derivation in $n \geq 4$ variables is finitely generated over $k$ with at most three generators if there exists $i$ such that $C_i \neq 1$. In each case it is a polynomial ring.
Let \( f = \sum_{i=1}^{n}(\prod_{j=1}^{i-1} C_j)x_i = x_1 + C_1x_2 + C_1 C_2 x_3 + \ldots + C_1 C_2 \cdots C_{n-1} x_n. \)

Consider nonempty subsets \( A \subseteq \mathbb{Z}_n \) of integers mod \( n \) closed under \( i \mapsto i + 2 \). If \( n \) is odd, then \( A = \mathbb{Z}_n \). If \( n \) is even, then we also have subsets \( \mathcal{E} = \{2i; \ i \leq n/2\} \) and \( \mathcal{O} = \{2i - 1; \ i \leq n/2\} \). For a given \( A \) we define a polynomial \( g_A \) if \( C_i \in \mathbb{Q}_+ \) for all \( i \in A \) and \( \prod_{i \in A} C_i = 1 \). Then there exist unique coprime \( \theta_i \in \mathbb{N}_+ \) for \( i \in A \) such that \( \theta_{i+2} = C_i \theta_i \). Let then \( g_A = \prod_{i \in A} x_i^{\theta_i} \).
Theorem (Hegedűs and Z.)

Let $n > 4$ and there exists $i$ such that $C_i \neq 1$. Then the number of generators of the Lotka-Volterra derivation with respect to the parameters $C_1, \ldots, C_n$ is equal to:

- $0$, if $\prod C_i \neq 1$ and no $g_A$ is defined,
- $3$, if $n$ is even with both $g_E$ and $g_O$ are defined,
- $2$, if $n$ is odd and $g_{\mathbb{Z}_n}$ is defined, or $n$ is even and $\prod C_i = 1$ but only one of $g_E$ and $g_O$ is defined,
- $1$ in every other case.

In each case the generators are those polynomials $g_A$ that are defined together with $f$ if $\prod C_i = 1$. 
Let $n = 4$ and let $C_1 C_2 C_3 C_4 = -1$.

If there are two consecutive indices $i$ such that $C_i = 1$, then there is a constant $f_4$. We define it for $C_1 = C_2 = 1$ and $C_4 = -1/C_3$, for the other possibilities one has to rotate the indices appropriately.

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2 x_4^2 + 2x_1 x_2 - 2x_1 x_3 - 2C_3 x_1 x_4 + 2x_2 x_3 - 2C_3 x_2 x_4 + 2C_3 x_3 x_4.$$
Theorem (Hegedűs and Z.)

Let $n = 4$ and there exists $i$ such that $C_i \neq 1$. Then the number of generators of the Lotka-Volterra derivation with respect to the parameters $C_1, \ldots, C_n$ equals:

- 0, if $\prod C_i \neq 1$ and none of $g_O$, $g_E$, $f_4$ is defined,
- 3, if both $g_E$ and $g_O$ are defined,
- 2, if $\prod C_i = 1$ but only one of $g_E$ and $g_O$ is defined, or one of the parameters is $-1$ and the other three are $1$,
- 1 in every other case.

In each case the generators are those polynomials $g_A$ that are defined together with $f$ if $\prod C_i = 1$ or together with $f_4$ if $C_1 C_2 C_3 C_4 = -1$ and two consecutive parameters equals $1$. 
Further notation

Denote by $k[X]_{(m)}$ the homogeneous component of $k[X]$ of degree $m$.

Let $k[X]^d_{(m)} = k[X]_{(m)} \cap k[X]^d$.

Since $d$ is homogeneous, we have $k[X]^d = \bigoplus_{m=0}^{\infty} k[X]_{(m)}$. 
Exemplary lemmas

Let \( C_n \neq 1 \). Consider the standard lexicographic ordering on the monomials of \( k[X]_n \). Suppose that \( h \) has lexicographically the smallest leading monomial among all homogenous polynomials that are counterexamples to the theorem. Let \( m_1 = \prod_{i=1}^{n} x_i^{\alpha_i} \) be the leading monomial of \( h \). Let \( M(h) \) denote the set of monomials occurring in \( h \) with nonzero coefficient.

**Lemma (Hegedűs and Z.)**

Suppose \( m = \prod_{i=1}^{n} x_i^{\gamma_i} \) is a monomial and \( r \) is a positive integer with the following properties:

1. \( \gamma_n = \alpha_n \),
2. \( \gamma_{2i-1} = \alpha_{2i-1} \) for \( 1 \leq i \leq r \),
3. \( \gamma_{2i} = C_{2i-2} \gamma_{2i-2} \) for \( 1 \leq i \leq r - 1 \),
4. \( \gamma_{2r} \neq C_{2r-2} \gamma_{2r-2} \).

Then \( m \notin M(h) \).
Lemma (Hegedűs and Z.)

Suppose $m = \prod_{i=1}^{n} x_i^{\gamma_i} \in M(h)$ is a monomial and $r < n/2$ is a positive integer with the following properties:

1. $\gamma_n = \alpha_n$ (or $C_n = 0$),
2. $\gamma_{2i-1} = \alpha_{2i-1}$ for $1 \leq i \leq r$,
3. $\gamma_{2i} = \alpha_{2i}$ for $1 \leq i \leq r$.

Then there exists a nonnegative integer $\beta_{2r-1}$ such that

\[ C_{2r-1}(\gamma_{2r-1} - \beta_{2r-1}) = \gamma_{2r+1} \] and $m' = m(x_{2r}/x_{2r-1})^{\beta_{2r-1}} \in M(h)$. In particular, there exist nonnegative integers $\beta'_{2i-1}$ such that

\[ C_{2i-1}(\alpha_{2i-1} - \beta'_{2i-1}) = \alpha_{2i+1} \] for $1 \leq i < n/2$. 

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Lemma (Hegedűs and Z.)

Suppose $C_n \neq 0$ and $m = \prod x_i^{\gamma_i} \in M(h)$ is such that $\gamma_n = \alpha_n$, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2 = C_n \alpha_n$. Then $l = \gamma_1 - C_{n-1} \gamma_{n-1}$ is a nonnegative integer and $m' = m(x_n/x_1)^l \in M(h)$. In particular, $C_1 - C_{n-1} \alpha_{n-1}$ is a nonnegative integer.

Lemma (Hegedűs and Z.)

Let $h, g \in R^d$, where $g$ is a monomial. If the leading monomial of $h$ is $m_1 = x_1^k g$ with $k > 0$ then $C_1 C_2 \cdots C_n$ is a $k$-th root of unity, in particular all $C_i \neq 0$. If further $n > 4$ then $C_1 C_2 \cdots C_n = 1$. 
We call a polynomial $g \in k[X]$ strict if it is homogeneous and not divisible by the variables $x_1, \ldots, x_n$.

Every nonzero homogeneous polynomial $f \in k[X]$ has the unique representation $f = X^\alpha g$, where $X^\alpha$ is a monomial and $g$ is strict.
A nonzero polynomial $f$ is said to be a *Darboux polynomial* of a derivation $\delta : k[X] \to k[X]$ if $\delta(f) = \Lambda f$ for some $\Lambda \in k[X]$. We will call $\Lambda$ a *cofactor* of $f$.

**Lemma (Ossowski and Z.)**

Let $n = 4$. If $g \in k[X]$ is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in $\mathbb{N}$.

**Lemma (Ossowski and Z.)**

Let $n = 4$. If $d(f) = 0$ and $f = X^\alpha g$, where $g$ is strict, then $d(X^\alpha) = 0$ and $d(g) = 0$. 
Let $\varphi \in k[X]$. For every $A \subseteq \{1, \ldots, n\}$ we denote by $\varphi^A$ the sum of terms of $\varphi$ that depend on variables $x_i$ for $i \in A$, that is, $\varphi^A = \varphi \mid_{x_j=0}$ for $j \not\in A$.

**Lemma (Ossowski and Z.)**

If $A \subseteq \{1, \ldots, n\}$, then for every $\varphi \in k[X]_d^m$ we have $d(\varphi^A)^A = 0$.

**Lemma (Ossowski and Z.)**

Let $\varphi \in k[X]_m$ and $A = \{i, i+1\} \subseteq \{1, \ldots, n\}$. If $d(\varphi^A)^A = 0$, then $\varphi^A = a(x_i + C_ix_{i+1})^m$ for $a \in k$.

**Lemma (Ossowski and Z.)**

Let $n \geq 4$, $\varphi \in k[X]_m$ and $i \in \{1, \ldots, n\}$. Let $C_i \notin \mathbb{Q}_+$ and $A = \{i, i+1, i+2\}$. If $d(\varphi^A)^A = 0$, then $\varphi^A \in k[x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2}]$. 

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Field of rational constants

For any derivation \( \delta : k[X] \rightarrow k[X] \) there exists exactly one derivation \( \bar{\delta} : k(X) \rightarrow k(X) \) such that \( \bar{\delta}|_{k[X]} = \delta \).

By a rational constant of the derivation \( \delta : k[X] \rightarrow k[X] \) we mean the constant of its corresponding derivation \( \bar{\delta} : k(X) \rightarrow k(X) \).

The rational constants of \( \delta \) form a field.

**Theorem**

*If \( d \) is the four-variable Volterra derivation, then*

\[
k(X)^d = k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4).
\]

Denote by \( D_0 \) the field of fractions of an integral domain \( D \).

**Corollary**

*If \( d \) is the four-variable Volterra derivation, then* \( k(X)^d = (k[X]^d)_0 \).
Theorem

If $d$ is a four-variable Lotka-Volterra derivation, then $k(X)^d$ contains a nontrivial rational constant if and only if at least one of the following conditions is fulfilled:

1. $C_1 C_2 C_3 C_4 = 1$,
2. $C_1 C_2 C_3 C_4 = -1$ and $C_i = 1$ for two consecutive indices,
3. $C_1, C_3 \in \mathbb{Q}$ and $C_1 C_3 = 1$,
4. $C_2, C_4 \in \mathbb{Q}$ and $C_2 C_4 = 1$. 
Theorem

Let $s_1, \ldots, s_4 \in \mathbb{N}_+$, where $(s_1, s_3) \neq (1, 1)$ and $(s_2, s_4) \neq (1, 1)$. Let $D : k(X) \to k(X)$ be a derivation of the form

$$D(x_i) = x_i^{s_{i-1}+1} x_i^{s_i+1} x_i^{s_{i+2}}$$

for $i = 1, \ldots, 4$ (in a cyclic sense). Then $k(X)^D = k$. 

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Thank you very much :) 
Danke schön :)