Polynomial constants of cyclic factorizable derivations

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Conference on Rings and Polynomials
July 3-8, 2016, Graz, Austria

Notation

k - a field of characteristic zero
$\mathbb{N}$ - the set of nonnegative integers
$\mathbb{N}_+$ - the set of positive integers
$\mathbb{Q}_+$ - the set of positive rationals
n - an integer $\geqslant 3$
$k[X] = k[x_1, \ldots, x_n]$
$k(X) = k(x_1, \ldots, x_n)$

Motivation

- applications in population biology, laser physics, plasma physics, hydrodynamics, chemistry, biochemistry, neural networks, economics

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- applications in population biology, laser physics, plasma physics, hydrodynamics, chemistry, biochemistry, neural networks, economics
- Lagutinskii's procedure of association of the factorizable derivation with any given derivation
- link to the invariant theory (for every connected algebraic group $G \subseteq \mathrm{Gl}_n(k)$ there exists a derivation d such that $k[x_1, \ldots, x_n]^G = k[x_1, \ldots, x_n]^d$)

Derivations

If R is a commutative k-algebra, then a k-linear map $d : R \to R$ is called a *derivation* of R if for all $a, b \in R$

$$d(ab) = ad(b) + d(a)b.$$

We call $R^d = \ker d$ the *ring of constants* of the derivation d.

Then $k \subseteq R^d$ and a *nontrivial* constant of d is an element of the set $R^d \setminus k$.

If $f_1, \ldots, f_n \in k[X]$, then there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$.

Examples

- Example 1. Hilbert's fourteenth problem.

Let L be a field such that $k \subseteq L \subseteq k(X)$.

Problem (1900, Hilbert): Is the ring $L \cap k[X]$ finitely generated over k?

1958, Nagata: no.
1993, Derksen: that counterexample is of the form $k[X]^d$ for $n = 32$.
1994, Roberts, Deveney, Finston: a counterexample for $n = 7$.
1998, Freudenburg: for $n = 6$.
1999, Daigle, Freudenburg: for $n = 5$.
2005, Kuroda: for $n = 4$.
1988, Nagata, Nowicki: $k[X]^d$ is finitely generated over k for $n \leqslant 3$.

Examples

- Example 2. Jouanolou derivations.

Let $s, n \in \mathbb{N}$, $s \geqslant 2$, $n \geqslant 3$

$$d : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$$

$$d(x_i) = x_{i+1}^s$$

(in a cyclic sense, that is, $x_{n+1} = x_1$).

1979, Jouanolou: if $n = 3$, then $\mathbb{C}[x_1, \dots, x_n]^d = \mathbb{C}$.
2000, Maciejewski, Moulin Ollagnier, Nowicki, Strelcyn: if n is a prime number $\geqslant 3$, then $\mathbb{C}[x_1, \dots, x_n]^d = \mathbb{C}$.
2003, Żołądek: if $n \geqslant 3$, then $\mathbb{C}[x_1, \dots, x_n]^d = \mathbb{C}$.

Formulation of the problem

A derivation $d : k[X] \to k[X]$ is called *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1, \dots, n$.

A derivation $d : k[X] \to k[X]$ is said to be *cyclic factorizable* if $d(x_i) = x_i(A_i x_{i-1} + B_i x_{i+1})$, where $A_i, B_i \in k$ for $i = 1, \dots, n$ (we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$).

The task: determine $k[X]^d$ (and then the field of constants).

Formulation of the problem

We call a derivation $d : k[X] \to k[X]$ *Lotka-Volterra* with parameters $C_1, \ldots, C_n \in k$ if

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$$

for $i = 1, \ldots, n$ (in a cyclic sense).

Solutions

- J. Moulin Ollagnier, A. Nowicki (1999): for $n = 3$ and arbitrary parameters C_i
- P. Hegedűs (2012): for an arbitrary n but all C_i equal to 1
- J. Z. (2013): for $n = 4$ and parameters C_i
- P. Hegedűs, J. Z. (2016): for an arbitrary n and arbitrary parameters

Main results

If $C_i = 1$ for all i, then we call d a *Volterra derivation.*

Theorem (Hegedűs)

The ring of constants of Volterra derivation in n *variables is finitely generated over* k *with* $[(n+2)/2]$ *generators. In each case it is a polynomial ring.*

Theorem (Hegedűs and Z.)

The ring of constants of Lotka-Volterra derivation in $n \geqslant 4$ *variables is finitely generated over* k *with at most three generators if there exists* i *such that* $C_i \neq 1$*. In each case it is a polynomial ring.*

Generators

Let $f = \sum_{i=1}^{n} (\prod_{j=1}^{i-1} C_j) x_i = x_1 + C_1 x_2 + C_1 C_2 x_3 + \ldots + C_1 C_2 \cdots C_{n-1} x_n$.

Consider nonempty subsets $\mathcal{A} \subseteq \mathbb{Z}_n$ of integers mod n closed under $i \mapsto i+2$. If n is odd, then $\mathcal{A} = \mathbb{Z}_n$. If n is even, then we also have subsets $\mathcal{E} = \{2i;\ i \leqslant n/2\}$ and $\mathcal{O} = \{2i-1;\ i \leqslant n/2\}$. For a given $\mathcal{A}$ we define a polynomial $g_{\mathcal{A}}$ if $C_i \in \mathbb{Q}_+$ for all $i \in \mathcal{A}$ and $\prod_{i \in \mathcal{A}} C_i = 1$. Then there exist unique coprime $\theta_i \in \mathbb{N}_+$ for $i \in \mathcal{A}$ such that $\theta_{i+2} = C_i \theta_i$. Let then $g_{\mathcal{A}} = \prod_{i \in \mathcal{A}} x_i^{\theta_i}$.

Generators for $n > 4$

Theorem (Hegedűs and Z.)

Let $n > 4$ and there exists i such that $C_i \neq 1$. Then the number of generators of the Lotka-Volterra derivation with respect to the parameters $C_1, \dots, C_n$ is equal to:

- 0, *if* $\prod C_i \neq 1$ *and no* $g_{\mathcal{A}}$ *is defined,*
- 3, *if* n *is even with both* $g_{\mathcal{E}}$ *and* $g_{\mathcal{O}}$ *are defined,*
- 2, *if* n *is odd and* $g_{\mathbb{Z}_n}$ *is defined, or* n *is even and* $\prod C_i = 1$ *but only one of* $g_{\mathcal{E}}$ *and* $g_{\mathcal{O}}$ *is defined,*
- 1 *in every other case.*

In each case the generators are those polynomials $g_{\mathcal{A}}$ *that are defined together with* f *if* $\prod C_i = 1$.

$n = 4$

Let $n = 4$ and let $C_1 C_2 C_3 C_4 = -1$.

If there are two consecutive indices i such that $C_i = 1$, then there is a constant f_4. We define it for $C_1 = C_2 = 1$ and $C_4 = -1/C_3$, for the other possibilities one has to rotate the indices appropriately.

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2 x_4^2 + 2x_1x_2 - 2x_1x_3 - 2C_3x_1x_4 + 2x_2x_3 - 2C_3x_2x_4 + 2C_3x_3x_4.$$

Generators for $n = 4$

Theorem (Hegedűs and Z.)

Let $n = 4$ and there exists i such that $C_i \neq 1$. Then the number of generators of the Lotka-Volterra derivation with respect to the parameters $C_1, \dots, C_n$ equals:

- 0, *if* $\prod C_i \neq 1$ *and none of* $g_{\mathcal{O}}$, $g_{\mathcal{E}}$, f_4 *is defined,*
- 3, *if both* $g_{\mathcal{E}}$ *and* $g_{\mathcal{O}}$ *are defined,*
- 2, *if* $\prod C_i = 1$ *but only one of* $g_{\mathcal{E}}$ *and* $g_{\mathcal{O}}$ *is defined, or one of the parameters is* -1 *and the other three are* 1,
- 1 *in every other case.*

In each case the generators are those polynomials $g_{\mathcal{A}}$ *that are defined together with* f *if* $\prod C_i = 1$ *or together with* f_4 *if* $C_1 C_2 C_3 C_4 = -1$ *and two consecutive parameters equals* 1.

Further notation

Denote by $k[X]_{(m)}$ the homogeneous component of $k[X]$ of degree m.

Let $k[X]^d_{(m)} = k[X]_{(m)} \cap k[X]^d$.

Since d is homogeneous, we have $k[X]^d = \bigoplus_{m=0}^{\infty} k[X]^d_{(m)}$.

Exemplary lemmas

Let $C_n \neq 1$. Consider the standard lexicographic ordering on the monomials of $k[X]_{(m)}$. Suppose that h has lexicographically the smallest leading monomial among all homogenous polynomials that are counterexamples to the theorem. Let $m_1 = \prod_{i=1}^{n} x_i^{\alpha_i}$ be the leading monomial of h. Let $M(h)$ denote the set of monomials occurring in h with nonzero coefficient.

Lemma (Hegedűs and Z.)

Suppose $m = \prod_{i=1}^{n} x_i^{\gamma_i}$ is a monomial and r is a positive integer with the following properties:

1. $\gamma_n = \alpha_n$,
2. $\gamma_{2i-1} = \alpha_{2i-1}$ *for* $1 \leqslant i \leqslant r$,
3. $\gamma_{2i} = C_{2i-2}\gamma_{2i-2}$ *for* $1 \leqslant i \leqslant r-1$,
4. $\gamma_{2r} \neq C_{2r-2}\gamma_{2r-2}$.

Then $m \notin M(h)$.

Exemplary lemmas

Lemma (Hegedűs and Z.)

Suppose $m = \prod_{i=1}^{n} x_i^{\gamma_i} \in M(h)$ *is a monomial and* $r < n/2$ *is a positive integer with the following properties:*

1. $\gamma_n = \alpha_n$ *(or* $C_n = 0$*),*
2. $\gamma_{2i-1} = \alpha_{2i-1}$ *for* $1 \leqslant i \leqslant r$*,*
3. $\gamma_{2i} = \alpha_{2i}$ *for* $1 \leqslant i \leqslant r$*.*

Then there exists a nonnegative integer β_{2r-1} *such that* $C_{2r-1}(\gamma_{2r-1} - \beta_{2r-1}) = \gamma_{2r+1}$ *and* $m' = m(x_{2r}/x_{2r-1})^{\beta_{2r-1}} \in M(h)$*. In particular, there exist nonnegative integers* β'_{2i-1} *such that* $C_{2i-1}(\alpha_{2i-1} - \beta'_{2i-1}) = \alpha_{2i+1}$ *for* $1 \leqslant i < n/2$*.*

Exemplary lemmas

Lemma (Hegedűs and Z.)

Suppose $C_n \neq 0$ *and* $m = \prod x_i^{\gamma_i} \in M(h)$ *is such that* $\gamma_n = \alpha_n$, $\gamma_1 = \alpha_1$ *and* $\gamma_2 = \alpha_2 = C_n \alpha_n$. *Then* $l = \gamma_1 - C_{n-1}\gamma_{n-1}$ *is a nonnegative integer and* $m' = m(x_n/x_1)^l \in M(h)$. *In particular,* $\alpha_1 - C_{n-1}\alpha_{n-1}$ *is a nonnegative integer.*

Lemma (Hegedűs and Z.)

Let $h, g \in R^d$, *where* g *is a monomial. If the leading monomial of* h *is* $m_1 = x_1^k g$ *with* $k > 0$ *then* $C_1 C_2 \cdots C_n$ *is a* k*-th root of unity, in particular all* $C_i \neq 0$. *If further* $n > 4$ *then* $C_1 C_2 \cdots C_n = 1$.

Strict polynomials

We call a polynomial $g \in k[X]$ *strict* if it is homogeneous and not divisible by the variables $x_1, \ldots, x_n$.

Every nonzero homogeneous polynomial $f \in k[X]$ has the unique representation $f = X^{\alpha} g$, where X^{α} is a monomial and g is strict.

Darboux polynomials

A nonzero polynomial f is said to be a *Darboux polynomial* of a derivation $\delta : k[X] \to k[X]$ if $\delta(f) = \Lambda f$ for some $\Lambda \in k[X]$. We will call Λ a *cofactor* of f.

Lemma (Ossowski and Z.)

Let $n = 4$. If $g \in k[X]$ is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in $\mathbb{N}$.

Lemma (Ossowski and Z.)

Let $n = 4$. If $d(f) = 0$ and $f = X^\alpha g$, where g is strict, then $d(X^\alpha) = 0$ and $d(g) = 0$.

Restrictions of n-variable polynomials

Let $\varphi \in k[X]$. For every $A \subseteq \{1, \dots, n\}$ we denote by φ^A the sum of terms of φ that depend on variables x_i for $i \in A$, that is, $\varphi^A = \varphi|_{x_j=0 \text{ for } j \notin A}$.

Lemma (Ossowski and Z.)

If $A \subseteq \{1, \dots, n\}$, then for every $\varphi \in k[X]^d_{(m)}$ we have $d(\varphi^A)^A = 0$.

Lemma (Ossowski and Z.)

Let $\varphi \in k[X]_{(m)}$ and $A = \{i, i+1\} \subset \{1, \dots, n\}$. If $d(\varphi^A)^A = 0$, then $\varphi^A = a(x_i + C_i x_{i+1})^m$ for $a \in k$.

Lemma (Ossowski and Z.)

Let $n \geqslant 4$, $\varphi \in k[X]_{(m)}$ and $i \in \{1, \dots, n\}$. Let $C_i \notin \mathbb{Q}_+$ and $A = \{i, i+1, i+2\}$. If $d(\varphi^A)^A = 0$, then $\varphi^A \in k[x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2}]$.

Field of rational constants

For any derivation $\delta : k[X] \to k[X]$ there exists exactly one derivation $\bar{\delta} : k(X) \to k(X)$ such that $\bar{\delta}_{|k[X]} = \delta$.

By a *rational constant* of the derivation $\delta : k[X] \to k[X]$ we mean the constant of its corresponding derivation $\bar{\delta} : k(X) \to k(X)$.

The rational constants of δ form a field.

Theorem

If d is the four-variable Volterra derivation, then

$$k(X)^d = k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4).$$

Denote by D_0 the field of fractions of an integral domain D.

Corollary

If d is the four-variable Volterra derivation, then $k(X)^d = (k[X]^d)_0$.

Field of rational constants

Theorem

If d is a four-variable Lotka-Volterra derivation, then $k(X)^d$ contains a nontrivial rational constant if and only if at least one of the following conditions is fulfilled:

(1) $C_1 C_2 C_3 C_4 = 1$,

(2) $C_1 C_2 C_3 C_4 = -1$ *and* $C_i = 1$ *for two consecutive indices,*

(3) $C_1, C_3 \in \mathbb{Q}$ *and* $C_1 C_3 = 1$,

(4) $C_2, C_4 \in \mathbb{Q}$ *and* $C_2 C_4 = 1$.

Application to monomial derivations

Theorem

Let $s_1, \ldots, s_4 \in \mathbb{N}_+$, *where* $(s_1, s_3) \neq (1,1)$ *and* $(s_2, s_4) \neq (1,1)$. *Let* $D : k(X) \to k(X)$ *be a derivation of the form*

$$D(x_i) = x_{i-1}^{s_{i-1}+1} x_i^{s_i+1} x_{i+2}^{s_{i+2}}$$

for $i = 1, \ldots, 4$ *(in a cyclic sense). Then* $k(X)^D = k$.

Thank you very much :)
Danke schön :)