On completeness of the root vector system of boundary value problem for first order system

A. Lunyov

We consider first order systems of ordinary differential equations

$$Ly := L(Q)y := -iB\frac{dy}{dx} + Q(x)y = \lambda y, \quad y = \mathrm{col}(y_1, ..., y_n), \tag{1}$$

subject to the following boundary conditions (BC)

$$Cy(0) + Dy(1) = 0, \qquad C = (c_{jk}), \quad D = (d_{jk}) \in \mathbb{C}^{n\times n}. \tag{2}$$

Here B is a non-singular diagonal $n\times n$ matrix, and $Q(\cdot)$ is a L^2-potential matrix. We denote by $L_{C,D} := L_{C,D}(Q)$ the operator associated in $L^2([0,1];\mathbb{C}^n)$ with the boundary value problem (BVP) (1)–(2).

BVP (1)–(2) has first been investigated by G. D. Birkhoff and R. E. Langer [1]. They introduced the concepts of *regular and strictly regular boundary conditions* (2) and investigated the asymptotic behavior of eigenvalues and eigenfunctions of the corresponding operator $L_{C,D}$. Moreover, they proved *a pointwise convergence result* on spectral decompositions of the operator $L_{C,D}$.

Recently [2] a wider class of boundary conditions, the so-called weakly B-regular BC, was introduced. Moreover, it was shown in [2] that the system of root vectors of BVP (1)–(2) subject to weakly B-regular BC is complete and minimal in $L^2([0,1];\mathbb{C}^n)$.

We obtain some results completing results from [2] for the case of $n\times n$ system. However, the most complete results are obtained for the case of 2×2 Dirac type system ($B = B^*$). Below we present one of the typical result regarding this system.

Let

$$B = \mathrm{diag}(b_1^{-1}, b_2^{-1}), \quad b_1 < 0 < b_2, \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix}. \tag{3}$$

It is convenient for us to write BC (2) in the following form

$$U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1, 2\}. \qquad (4)$$

Put

$$J_{jk} = \det \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}, \ j, k \in \{1, \dots, 4\}.$$

The following statement substantially generalizes Theorem 5.1 from [2].

Theorem. *Let* $Q_{12}(\cdot), Q_{21}(\cdot) \in W_2^{n+1}[0,1]$, *and let* $J_{32} = J_{14} = 0$. *Let also the following conditions be satisfied*

$$\begin{aligned} b_1 J_{13} Q_{12}^{(k)}(0) + (-1)^k b_2 J_{42} Q_{21}^{(k)}(1) &= 0, \quad k \in \{0, 1, \dots, n-1\}, \\ b_1 J_{13} Q_{12}^{(n)}(0) + (-1)^n b_2 J_{42} Q_{21}^{(n)}(1) &\neq 0, \end{aligned}$$

and

$$\begin{aligned} b_1 J_{13} Q_{12}^{(k)}(1) + (-1)^k b_2 J_{42} Q_{21}^{(k)}(0) &= 0, \quad k \in \{0, 1, \dots, n-1\}, \\ b_1 J_{13} Q_{12}^{(n)}(1) + (-1)^n b_2 J_{42} Q_{21}^{(n)}(0) &\neq 0. \end{aligned}$$

Then the system of root functions of the problem (1),(3),(4) *is complete and minimal in* $L^2([0,1];\mathbb{C}^2)$.

The talk is based on a joint paper with M. Malamud.

References

[1] G.D. Birkhoff and R.E. Langer, The boundary problems and developments associated with a system of ordinary differential equations of the first order, *Proc. Am. Acad. Arts Sci.* **58** (1923), pp. 49–128.

[2] M.M. Malamud and L.L. Oridoroga, On the completeness of the system of root vectors for first-order systems, *arXiv*, 1109.3683.