Spectral theory of Schrödinger operators with infinitely many point interactions and radial positive definite functions

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Schrödinger operators on the Hilbert space $L^2(\mathbb{R}^d)$, $1 \leq d \leq 3$, with potentials supported on a discrete (finite or countable) set $X = \{x_j\}_{i=1}^{m}$ of points in $\mathbb{R}^d$ arise in different problems of quantum mechanics (see [1] and references therein). They are given by the formal differential expression

$$\mathfrak{L}_d := \mathfrak{L}_d(X, \alpha) := -\Delta + \sum_{j=1}^{m} \alpha_j \delta(\cdot - x_j), \quad \alpha_j \in \mathbb{R}, \quad m \in \mathbb{N} \cup \{\infty\}. \quad (1)$$

where $\alpha = \{\alpha_j\}_{i=1}^{m}$ is a sequence of real numbers.

In the case $d = 1$ there are several natural ways to associate the self-adjoint operator (Hamiltonian) on $L^2(\mathbb{R}^1)$ with the differential expression (1). In the case $d = 2, 3$ there is no natural Hamiltonian associated with the differential expression (1) (see [1]).

F. Berezin and L. Faddeev proposed in their pioneering paper [2] to consider the expression (1) (with $m = 1$ and $d = 3$) in the framework of the extension theory of symmetric operators. Following F. Berezin and L. Faddeev one associates the minimal symmetric operator $H$ with the expression (1) as the following restriction of the Laplacian $-\Delta$,

$$H_d := -\Delta \upharpoonright \text{dom}H, \quad \text{dom}(H_d) = \{f \in W^{2,2}(\mathbb{R}^d) : f(x_j) = 0, \; j \in \mathbb{N}\} \quad (2)$$

and studies the spectral properties of all its selfadjoint extensions.

The case of $m < \infty$ is well studied. We will discuss the case $m = \infty$. We investigate self-adjoint extensions of the operator $H = H_3$ (realizations of $\mathfrak{L}_3$) in the framework of boundary triplets approach. Namely, we construct a ”good” boundary triplet for $H_d^*$ (with $d = 2, 3$) assuming that

$$d_*(X) := \inf_{j \neq k} |x_k - x_j| > 0. \quad (3)$$
Using this boundary triplet II we parameterize the set of self-adjoint extensions of $H = H_d$, compute the corresponding Weyl function $M(\cdot)$ and investigate various spectral properties of self-adjoint extensions (semi-boundedness, non-negativity, negative spectrum, resolvent comparability, etc.)

Our main result on spectral properties of Hamiltonians with point interactions concerns the absolutely continuous spectrum ($ac$-spectrum). For instance, if

$$C := \sum_{|j-k|>0} \frac{1}{|x_j - x_k|^2} < \infty,$$  \hspace{1cm} (4)

we prove that the part $\tilde{H}E_{\tilde{H}}(C, \infty)$ of every self-adjoint extension $\tilde{H}$ of $H$ is absolutely continuous. Moreover, under additional assumptions on $X$, we show that the singular part of $\tilde{H}^+ := \tilde{H}E_{\tilde{H}}(0, \infty)$ is trivial, i.e. $\tilde{H}^+ = \tilde{H}^{ac}$.

In the proof we apply technique elaborated in [3] as well as some new results on the characterization of $ac$-spectrum in terms of the Weyl function.

An important feature of our approach is an apparently new connection between the spectral theory of operators (1) for $d = 3$ and the class $\Phi_3$ of radial positive definite functions on $\mathbb{R}^3$. We exploit this connection in both directions.

Namely, we investigate some properties of radial positive definite functions on $\mathbb{R}^d$ that have been inspired by possible applications in the spectral theory of operators (1). If $f$ is such a function and $X = \{x_n\}_{n=1}^\infty$ is a sequence of points of $\mathbb{R}^d$, we say that $f$ is strongly $X$-positive definite if there exists a constant $c > 0$ such that for all $\xi_1, \ldots, \xi_m \in \mathbb{C}$,

$$\sum_{j,k=1}^m \xi_k \bar{\xi}_j f(x_k - x_j) \geq c \sum_{k=1}^m |\xi_k|^2, \quad m \in \mathbb{N}.$$  

Using Schoenberg’s theorem we derive a number of results showing under certain assumptions on $X$ that $f$ is strongly $X$-positive definite and that the Gram matrix $Gr_X(f) := (f(|x_k - x_j|))_{k,j \in \mathbb{N}}$ defines a bounded operator on $l^2(\mathbb{N})$. The latter results correlate with the properties of the sequence $\{e^{it\cdot|x_k|}\}_{k \in \mathbb{N}}$ of exponential functions to form a Riesz-Fischer sequence or a Bessel sequence, respectively, in $L^2(S^n_r; \sigma_n)$ for some $r > 0$.

In particular, using extension theory of the operator $H$ we show that any completely monotone function (in the sense of S. Bernstein) is $X$-positive definite provided that $X = \{x_n\}_{n=1}^\infty$ satisfy condition (3).

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References

