# Spectral theory of Schrödinger operators with infinitely many point interactions and radial positive definite functions

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Schrödinger operators on the Hilbert space $L^2(\mathbb{R}^d)$, $1 \le d \le 3$, with potentials supported on a discrete (finite or countable) set $X = \{x_j\}_1^m$ of points in $\mathbb{R}^d$ arise in different problems of quantum mechanics (see [1] and references therein). They are given by the formal differential expression

$$\mathfrak{L}_d := \mathfrak{L}_d(X,\alpha) := -\Delta + \sum_{j=1}^{m} \alpha_j \delta(\cdot - x_j), \qquad \alpha_j \in \mathbb{R}, \quad m \in \mathbb{N} \cup \{\infty\}. \quad (1)$$

where $\alpha = \{\alpha_j\}_1^m$ is a sequence of real numbers.

In the case $d = 1$ there are several natural ways to associate the self-adjoint operator (Hamiltonian) on $L^2(\mathbb{R}^1)$ with the differential expression (1). In the case $d = 2, 3$ there is no natural Hamiltonian associated with the differential expression (1) (see [1]).

F. Berezin and L. Faddeev proposed in their pioneering paper [2] to consider the expression (1) (with $m = 1$ and $d = 3$) in the framework of the extension theory of symmetric operators. Following F. Berezin and L. Faddeev one associates the minimal symmetric operator $H$ with the expression (1) as the following restriction of the Laplacian $-\Delta$,

$$H_d := -\Delta \upharpoonright \mathrm{dom} H, \quad \mathrm{dom}(H_d) = \{f \in W^{2,2}(\mathbb{R}^d) : f(x_j) = 0,\ j \in \mathbb{N}\} \quad (2)$$

and studies the spectral properties of *all its selfadjoint extensions.*

The case of $m < \infty$ is well studied. We will discuss the case $m = \infty$. We investigate self-adjoint extensions of the operator $H = H_3$ (realizations of $\mathfrak{L}_3$) in the framework of boundary triplets approach. Namely, we construct a "good" boundary triplet for $H_d^*$ (with $d = 2, 3$) assuming that

$$d_*(X) := \inf_{j \ne k} |x_k - x_j| > 0. \quad (3)$$

Using this boundary triplet $\Pi$ we parameterize the set of self-adjoint extensions of $H = H_d$, compute the corresponding Weyl function $M(\cdot)$ and investigate various spectral properties of self-adjoint extensions (semi-boundedness, non-negativity, negative spectrum, resolvent comparability, etc.)

Our main result on spectral properties of Hamiltonians with point interactions concerns the absolutely continuous spectrum (*ac*-spectrum). For instance, if

$$C := \sum_{|j-k|>0} \frac{1}{|x_j - x_k|^2} < \infty, \tag{4}$$

we prove that the part $\widetilde{H}E_{\widetilde{H}}(C,\infty)$ of *every self-adjoint extension* $\widetilde{H}$ of $H$ is absolutely continuous. Moreover, under additional assumptions on $X$, we show that the singular part of $\widetilde{H}_+ := \widetilde{H}E_{\widetilde{H}}(0,\infty)$ is trivial, i.e. $\widetilde{H}_+ = \widetilde{H}_+^{ac}$. In the proof we apply technique elaborated in [3] as well as some new results on the characterization of *ac*-spectrum in terms of the Weyl function.

An important feature of our approach is an appeарently new *connection between the spectral theory of operators* (1) *for* $d = 3$ *and the class* $\Phi_3$ *of radial positive definite functions on* $\mathbb{R}^3$. We exploit this connection in both directions.

Namely, we investigate some properties of radial positive definite functions on $\mathbb{R}^d$ that have been inspired by possible applications in the spectral theory of operators (1). If $f$ is such a function and $X = \{x_n\}_1^\infty$ is a sequence of points of $\mathbb{R}^d$, we say that $f$ is *strongly $X$-positive definite* if there exists a constant $c > 0$ such that for all $\xi_1, \ldots, \xi_m \in \mathbb{C}$,

$$\sum_{j,k=1}^{m} \xi_k \overline{\xi}_j f(x_k - x_j) \geq c \sum_{k=1}^{m} |\xi_k|^2, \qquad m \in \mathbb{N}.$$

Using Schoenberg's theorem we derive a number of results showing under certain assumptions on $X$ that $f$ is strongly $X$-positive definite and that the Gram matrix $Gr_X(f) := \big(f(|x_k - x_j|)\big)_{k,j\in\mathbb{N}}$ defines a bounded operator on $l^2(\mathbb{N})$. The latter results correlate with the properties of the sequence $\{e^{i(\cdot,x_k)}\}_{k\in\mathbb{N}}$ of exponential functions to form a Riesz-Fischer sequence or a Bessel sequence, respectively, in $L^2(S_r^n;\sigma_n)$ for some $r > 0$.

In particular, using extension theory of the operator $H$ we show that any completely monotone function (in the sense of S, Bernshtein) is $X$-positive definite provided that $X = \{x_n\}_1^\infty$ satisfy condition (3).

The talk is based on a joint work with Konrad Schmüdgen.

# References


[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, *Solvable Models in Quantum Mechanics*, Sec. Edition, (with an Appendix by P. Exner) AMS Chelsea Publ., 2005.

[2] F. A. Berezin, L. D. Faddeev, *Remark on the Schrödinger equation with singular potential*, Dokl. Acad. Sci. USSR **137** (1961), pp. 1011–1014.

[3] M.M. Malamud, H. Neidhardt, *On the unitary equivalence of absolutely continuous parts of self-adjoint extensions*, J. Funct. Anal., **260**, No 3 (2011), pp. 613-638.