Electrical network reduction with a probabilistic interpretation of effective conductance

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Abstract

We introduce an elegant probabilistic approach for solving electrical network problems and use it to prove a basic, but apparently unknown, fact concerning the reduction of electrical networks with multiple external nodes.

1 Introduction

Based on the classical construction of random walks on graphs, we obtain elegant proofs of the solvability of the Dirichlet problem for electrical networks with an arbitrary number of 'voltage' or 'current' sources. Using our approach we prove a basic fact concerning the reduction of electrical networks with multiple external nodes that has apparently escaped the attention of both mathematicians and electrical engineers.

If a network N has only two *external nodes*, i.e. nodes attaching it to a power source or a larger network, then it is well known that it can be replaced by an *equivalent* network N' that consists of a unique edge between these two nodes and electrically behaves in the same way as N. It is not hard to prove, using elementary algebraic arguments, that a network with more than two external nodes can also be reduced to an equivalent network, consisting of a complete graph on the set of external nodes and having no internal nodes. A well-known special case of this fact is the Y–Δ transformation.

In this paper we prove the existence of a reduced equivalent network, independently of the number of external nodes, using our probabilistic arguments. More importantly, we show that the conductances of the edges of the reduced network admit a simple expression in terms of the behaviour of the corresponding random walk. More precisely, if N is any network and B is its set of external nodes, then for $a, b \in B$ we let $p^{a \to b}_{\dagger B}$ denote the probability that random walk in N (see Section 2 for its definition) starting at a will reach b before any other external node (the random walk is *killed* upon reaching B). For $a \in B$ we let c_a denote the sum of the conductances of the edges of N incident with a. Then our main result can be summarised as follows.

Theorem 1.1. *Let N be an electrical network*[1] *and let B be the set of its*

[1] In electrical engineers' terminology, N is a passive, resistive network, as defined in Section 2. Extending Theorem 1.1 to active networks, i.e. networks containing sources, is easily achieved by adding the endpoints of any sources to the set of terminal vertices.

external nodes. Then there is an equivalent network $N_{/B}$ with the same external nodes B that has no internal nodes, and this network is essentially unique. For every pair $a, b \in B$, this network $N_{/B}$ contains a single a-b edge of conductance $C^{ab} := c_a p_{\dagger B}^{a \to b} = c_b p_{\dagger B}^{b \to a}$.

We stated Theorem 1.1 rather informally here. The precise statements are given by Lemmas 4.1, 5.1 and 7.1 below. Some examples can be found at the end of Section 4.

In the case where $|B| = 2$, i.e. when there is only one source a and one sink z, we obtain that the effective conductance of the network equals c_a times the probability that random walk starting at a will visit z before returning to a; this fact can also be found in [1, §19] and is equivalent to [5, Theorem 1].

2 Definitions

We will use the terminology of [2] for graph theoretical terms and [6] for probabilistic ones.

A network is a tuple $N = (G, c, B)$, where $G = (V, E)$ is an (undirected) (multi–) graph, c is a mapping assigning a *conductance* $c(e) \in \mathbb{R}^+$ to each edge e of G, and B is a set of vertices of G, called the *external vertices*. The other vertices, those in $V(G) \backslash B$, will be called *internal nodes*. The reader will lose nothing by assuming that every edge has unit conductance.

Given a network N and a function $\hat{u} : B \to \mathbb{R}$, which we think of as an assignment of voltages imposed by power sources to the external vertices, the (discrete) *Dirichlet problem* consists in extending $\hat{u}$ into a function $u : V(G) \to \mathbb{R}$ so that u is *harmonic* on $V(G) \backslash B$, that is, for every $x \in V(G) \backslash B$ it satisfies

$$c_x u(x) = \sum_{xy \in E} c(xy) u(y) \tag{1}$$

where

$$c_x := \sum_{\{y | xy \in E\}} c(xy). \tag{2}$$

It is well-known that if G is finite then the Dirichlet problem always has a unique solution.

Any function $u : V(G) \to \mathbb{R}$ induces a function $i : \vec{E} \to \mathbb{R}$ on the *directed edges* of G, i.e. the set $\vec{E} := \{\langle x, y \rangle \in V(G)^2 \mid x, y \text{ are joined by an edge of } G\}$ by letting $i(xy) = c(xy)(u(x) - u(y))$, to be thought of as a flow from x to y, where we use the notation $i(xy)$ as shorthand for $i(\langle x, y \rangle)$. Using this function i we can rewrite (1) as

$$\textstyle\sum_{\{y | xy \in E\}} i(xy) = \sum_{\{y | xy \in E\}} c(xy)(u(x) - u(y)) = 0 \text{ for } x \in V(G) \backslash B, \tag{3}$$

which is known as *Kirchhoff's node law*. The equation $i(xy) = c(xy)(u(x) - u(y))$, which we used to define i, is known in physics as *Ohm's law*.

The *energy* of a function $f : V(G) \to \mathbb{R}$ is defined by

$$E(f) := \sum_{xy \in E} c(xy)(f(x) - f(y))^2 = \sum_{xy \in E} (f(x) - f(y)) i(xy).$$

(Mathematicians call $E(f)$ the "energy" although physicist call it the "power".)

Given an electrical network N as above, one associates to each vertex $x \in V(G)$ a random walk as follows. A particle starts at x at time $t = 0$, and for $t = 1, 2, \ldots$ it takes a random step from its current position $y \in V(G)$ to one of the vertices adjacent with y according to the following law: the probability of going from y to w is $c(yw)/c_y$, where $c_y := \sum_{yz \in E} c(yz)$ (and so these probabilities add up to 1).

3 Solution of the Dirichlet problem

Consider an instance of the discrete Dirichlet problem, i.e. a finite network (G, c, B) and a 'voltage' function $\hat{u} : B \to \mathbb{R}$ imposed on the 'boundary' B. We will find the solution $u : V(G) \to \mathbb{R}$ to this Dirichlet problem using a probabilistic method that will allow us to describe the equivalent network mentioned in the introduction.

We define a random process consisting of a system of particles performing random walk on N independently from each other: suppose that for every $b \in B$ we start a particle of charge $\hat{u}(b)/c_b$ at b and let it perform random walk on N as discribed in Section 2, killing it when it reaches B.

For every vertex v of N, let $v(x) := D(x)/c_x$ where $D(x)$ the expected total amount of charge departing from x in the whole process.

Similarly, for a directed edge $\vec{e}$ let $j(\vec{e})$ be the expectation of the net total amount of charge flowing through e.

We now prove that the functions v, j we just defined give in fact the solution to our network problem.

Lemma 3.1. *The pair v, j satisfies Ohm's law and v is the solution to the Dirichlet problem with boundary values $\hat{u}(b)$, $b \in B$.*

Proof. Consider any edge $xy \in E$. Recall that the expected amount of charge leaving x is $c_x v(x)$. For each particles leaving x, the next step will go to y with probability c_{xy}/c_x. Similarly, we expect a total charge $c_y v(y)$ to leave y, going through xy with probability c_{xy}/c_y. By definition, $j(xy)$ is the diference of these two amount, so we have

$$j(xy) = c_x v(x) c_{xy}/c_x - c_y v(y) c_{xy}/c_y = c_{xy} \left((v(x) - v(y)) \right),$$

in agreement with Ohm's law.

For the second assertion, note that $v(b) = \hat{u}(b)$ for every $b \in B$ by the definition of $v(x)$. We claim that v is harmonic in $V(G) \backslash B$. Indeed, by (3) v is harmonic at a vertex x if and only if $\sum_{y \sim x} c_{xy} \left((v(x) - v(y)) \right) = 0$. Since we proved that Ohm's law applies to (v, j), this sum can be rewritten as $\sum_{y \sim x} j(xy)$. It now easily follows from the definition of j that the sum equals zero: each trajectory of a particle comes to x the same number of times as it leaves x, and so its net contribution to the sum is zero. In other words, we have just remarked that j satisfies Kirchhoff's node law.

This means that v is the solution to the Dirichlet problem with boundary values $\hat{u}(b)$, $b \in B$. □

Remark 1: we chose the amount of charge started at b to be a deterministic quantity equal to $\hat{u}(b)/c_b$, but we could also have let it be a random variable with that expectation. We could also let all particles have charge ± 1 and start a random number of particles with that expectation at b.

Remark 2: A similar approach for solving the Dirichlet problem is to start a random walk at each vertex x, and let $v(x)$ be the expectation of $\hat{u}(b)$ where b is the first vertex of B it visits. This approach is well-known [3, 4]. It is a consequence of the reversibility of our random walks that the two approaches yield the same solution. Our approach has the advantage that it immeadiately yields the equivalent network, and it is easy to adapt to the current sources regime, see below.

Let us define $j(b) := \sum_{bx \in E} j(bx)$ to be the expected net amount of charge flowing out of b in our random process. By Theorem 3.1 this coincides with the actual net current flowing out of b when sources of voltage $\hat{u}(b)$ are applied to the network.

4 The equivalent network

Let $p^{a \to b}_{\dagger B}$ denote the probability that our killed random walk starting at a with exit B at b. It is not hard to prove that in the case when $B = \{a, b\}$, i.e. when there is only one source and one sink, the effective conductance of the network is $C^{ab} = c_a p^{a \to b}_{\dagger B} = c_b p^{b \to a}_{\dagger B}$. We will generalise this fact to networks with arbitrarily many sources.

It follows easily from the reversibility (see [4]) of our random walk that $c_a p^{a \to b}_{\dagger B} = c_b p^{b \to a}_{\dagger B}$ for every $a, b \in B$; our results will provide an indirect proof of this fact. We define

$$C^{ab} := c_a p^{a \to b}_{\dagger B}. \tag{4}$$

Let $N_{/B}$ be the network with vertex set B in which each two vertices a, b are joined by an edge of conductance C^{ab}. Let $N^{\circ}_{/B}$ be the auxiliary network obtained from $N_{/B}$ by attaching, for every vertex b, a loop incident with b with conductance $C^{bb} = c_b p^{b \to b}_{\dagger B}$. We claim that $N_{/B}$ is equivalent to G in the sense that given any assignment of 'voltages' to the elements of B the current flowing out of each element of B coincide with the corresponding values for $N_{/B}$.

To see this, suppose we perform our random experiment of Section 3 once on $N^{\circ}_{/B}$ and once on N, and in the latter case observe the particles only when they are at the boundary B, then the two processes will follow the same law since, by the definition of $N^{\circ}_{/B}$, the transition probabilities between vertices of B as well as the parameters c_b are the same in the two networks. Thus, the values $j(b), b \in B$ defined at the end of Section 3 will be identical for the two networks. Note that the loops of $N^{\circ}_{/B}$ have no effect on these values. We just proved

Lemma 4.1. *Given a network N and boundary conditions $\hat{u} : B \to \mathbb{R}$, the flow out of each vertex in the solution of the discrete Dirichlet problem has the same value in N, $N^{\circ}_{/B}$ and $N_{/B}$.*

In the next section we will check that $N_{/B}$ is equivalent to N in terms of energy dissipation too.

Examples
If N only has two vertices a, b, and several a-b edges, then the equivalent network $N_{/B}$ consists of a single a-b edge with conductance equal to the sum of the conductances of those edges, since $p_{\dagger B}^{a \to b} = 1$ in this case. This agrees with the well-known reduction rule for networks connected in parallel.

As a further example, consider a network N with several external nodes one of which, called s, separates N into two pieces A, D. Then for every a in A and d in D we have $C^{ad} = 0$, since any particle trying to travel from a to d will have to visit s and will be killed there.

5 Energy

In order to show that $N_{/B}$ also dissipates the same amount of energy as N we will now recall that the energy is determined by the boundary conditions $\hat{u}(a)$ and the values $j(a)$ provided by (6): we claim that

$$E(v) = \sum_{a \in B} \hat{u}(a) j(a). \tag{5}$$

Indeed, letting $\binom{V}{2}$ denote the set of pairs of vertices of N, we have

$$E(v) = \sum_{\{x,y\} \in \binom{V}{2}} \left(v(x) - v(y)\right) j(xy) = \sum_{x \in V} \sum_{y \sim x} v(x) j(xy) = \sum_{x \in V} v(x) j(x),$$

where we used the fact that $j(xy) = -j(yx)$. Now note that $j(x) = 0$ for every $x \notin B$ by Kirchhoff's node law, and so $E(v) = \sum_{a \in B} \hat{u}(a) j(a)$ as claimed.

Lemma 5.1. *Given a network N and boundary conditions $\hat{u} : B \to \mathbb{R}$, the energy dissipated by each of N and $N_{/B}$ equals $E(v) = \sum_{\{a,b\} \in \binom{B}{2}} C^{ab} \left(v(b) - v(a)\right)^2$.*

Proof. Combining (5) with Lemma 4.1 we obtain that the energy dissipated by N equals that dissipated by $N_{/B}$. The latter is given by the above formula by definition. □

Lemma 5.1 has the following interesting corollary that provides a purely probabilistic formula for the energy dissipated by a network, and this formula is, in a sense, linear.

Corollary 5.2. *Let N be an electrical network and let B be the set of its external nodes. Suppose that at each $b \in B$ we start a random number of particles $P(b)$, with $\mathbb{E}\{P(b)\} = c_b \hat{u}(b)$, that perform random walk killed at B. Let $\mathcal{P}$ be the (random) set of all these particles. If v is the solution to the Dirichlet problem with boundary values $\hat{u}(b)$, then*

$$E(v) = \mathbb{E}\{\sum_{p \in \mathcal{P}} \left(v(p^{in}) - v(p^{ter})\right)\},$$

where $p^{in} \in B$ is the vertex at which p started its walk and $p^{ter} \in B$ is the vertex at which it was killed.

Proof. Recall the definition of $j(b)$ from the end of Section 3. We claim that

For every $a \in B$ we have $j(a) = \sum_{b \in B} C^{ab} (v(a) - v(b))$. (6)

To see this, note that the only particles that have an effect on $j(a)$ are those that start or finish their trajectory at a, and $j(a)$ is by definition the diference of the expected charge carried by the former minus the charge of the latter. Decomposing $j(a)$ according to the other endpoint b of such a trajectory, we can write this difference as follows

$$j(a) = \sum_{b \in B} v(a) c_a p_{\dagger B}^{a \to b} - v(b) c_b p_{\dagger B}^{b \to a}.$$

Using (4) and factoring now yields the desired formula (6)

Combining (5) with (6) yields $E(v) = \sum_{a \in B} v(a) \sum_{b \in B} C^{ab} (v(a) - v(b))$. The claim now follows from the definition $C^{ab} := c_a p_{\dagger B}^{a \to b}$ and linearity of expectation. □

6 Current sources at the boundary

In this section we adapt our approach to the case where the boundary conditions are currents $\hat{\imath}(b), b \in B$ rather than potentials. In this case the problem consists in finding a flow $i : \vec{E} \to \mathbb{R}$ satisfying Kirchhoff's node law (3) at every vertex in $V \backslash B$, Kirchhoff's cycle law, as well as the boundary conditions. *Kirchhoff's cycle law* demands that for every directed cycle $\vec{C}$ we have $\sum_{\vec{e} \in E(\vec{C})} i(\vec{e})/c(\vec{e}) = 0$. We say that i satisfies the boundary conditions if, for every $b \in B$, we have $\hat{\imath}(b) = \sum_{bx \in E} i(bx)$.

It is well known that this problem has a solution if and only if $\sum \hat{\imath}(b) = 0$, and the solution is then unique. We provide a new proof using a method similar to the one of the previous section, and show that we can replace N by $N_{/B}$ also in this setting.

Let B^+, B^- be the subsets of B consisting of the elements b for which $\hat{\imath}(b)$ is positive and negative respectively. We define a random process as follows. For every $b \in B^+$, we start a particle of charge $\hat{\imath}(b)$ at b and let it perform random walk on N. This time, particles are not automatically killed once they reach B. Instead, each time a particle p with charge $ch(p)$ visits a vertex $a \in B^-$, each of $ch(p), |\hat{\imath}(a)|$ is reduced by the amount $\min\{ch(p), -\hat{\imath}(a)\}$, after which p is left to continue its random walk unless we now have $ch(p) = 0$, in which case p is killed. (To be more formal, we should have defined the values $ch(p), \hat{\imath}(a)$ as functions of time, with initial values set at the boundary conditions.) Note that if several particles arrive a vertex in B^- at the same time, then the order in which we consider them for charge reduction is important to the particles, but it will turn out not be important for us.

Note that all particles get killed after finite time with probability 1 since, as $\sum \hat{\imath}(b) = 0$, the total initial 'capacity' $\sum_{a \in B^-} \hat{\imath}(a)$ of B^- equals the total initial charge of the particles, and random walk on a finite network visits all vertices in finite time with probability 1. Moreover, after all particles are killed, each $\hat{\imath}(a), a \in B^-$ has been set to zero.

Let us now define the function j as in the previous section, except that we now have our new system of particles, and check that it satisfies Kirchhoff's cycle law. Indeed, consider a directed cycle $\vec{C}$, and suppose that a particle p lies at some vertex x incident with edges yx, xz of $\vec{C}$. Then the expected contribution of p to the sum $\sum_{\vec{e}\in E(\vec{C})} j(\vec{e})/c(\vec{e})$ during the next step is $\frac{c(yx)}{c_x}\frac{-ch(p)}{c(yx)} + \frac{c(xz)}{c_x}\frac{ch(p)}{c(xz)} = 0$. Thus $\sum_{\vec{e}\in E(\vec{C})} j(\vec{e})/c(\vec{e}) = 0$ as desired, and j is the solution of our network problem.

A different way to prove that j is the sought solution is to define the function v as in the previous section and notice that the pair v, j satisfies Ohm's law by the proof of Lemma 3.1.

The arguments of Section 4 also imply the equivalence of the networks N and $N_{/B}$: if we perform our random experiment once on $N^\circ_{/B}$ and once on N, and in the latter case observe the particles only when they are at the boundary B, then the two processes will follow the same law by the definition of $N^\circ_{/B}$. Thus the two solutions for v will be identical at B. Since, by (5), the energy is determined by the boundary values, $N_{/B}$ yields the right expression for energy in this setting too.

7 Uniqueness of the equivalent network

We now show that the equivalent network $N_{/B}$ we constructed is unique subject to the requirement that it contains precisely one edge between each pair of distinct external nodes (of course, one could obtain further equivalent networks by replacing an edge with several parallel edges of the same total conductance, or by removing edges of conductance 0).

Lemma 7.1. *The constants C^{ab} are the unique family of parameters satisfying $E(f) = \sum_{a,b\in B} C^{ab}\left(f(b) - f(a)\right)^2$ for every boundary value assignment $f : B \to \mathbb{R}$.*

Proof. Suppose there are two families of parameters $\{C^{ab} \mid a, b \in B\}$ and $\{D^{ab} \mid a, b \in B\}$ satisfying $E(f) = \sum C^{ab}(f(a) - f(b))^2, E(f) = \sum D^{ab}(f(a) - f(b))^2$ for every potential f We claim that $C^{ab} = D^{ab}$ for every i, j, that is, the two representations are the same.

Consider $d_{ab} := C^{ab} - D^{ab}$. Then for every f we have $\sum_{a,b\in B} d_{ab}(f(a) - f(b))^2 = 0$. Moreover, for every subset Y of B we have $\sum_{i\in Y, j\in B\setminus Y} d_{ij} = 0$ as can be easily seen by letting $f(i) = 1$ for i in Y and $f(j) = 0$ for j in $B\setminus Y$.

Letting $Y = \{a\}$ we obtain $\sum_{ij\in E(a)} d_{ij} = 0$ where $E(a)$ is the set of pairs ("edges") (a, i) for $i \in B$, and similarly $\sum_{ij\in E(b)} d_{ij} = 0$ for any other vertex $b \in B$. Letting $Y = \{a, b\}$ we obtain $\sum_{E(ab)} d_{ij} = 0$ where $E(ab)$ is the set of "edges" from a, b to $B\setminus\{a, b\}$.

But $E(ab) = E(a)\cup E(b)\setminus\{ab\}$, thus $\sum_{ij\in E(ab)} d_{ij} = \sum_{ij\in E(a)} d_{ij} + \sum_{ij\in E(b)} d_{ij} - 2d_{ab}$. Since all terms except the last one have already been shown to be zero, we obtain $d_{ab} = 0$. Thus we have $C^{ab} = D^{ab}$ as claimed. ☐

It is not hard to check that our constants C^{ab} are the unique parameters satisfying Lemma 4.1 too.

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