Hyperbolic graphs, fractal boundaries, and graph limits

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In a seminal paper [5] Gromov introduced the notion of a hyperbolic graph and defined a finitely generated group to be hyperbolic if its Cayley graphs are hyperbolic. This notion, and the related construction of the *hyperbolic boundary*, has had a tremendous impact on group theory and other fields, starting with the work of Gromov [5] and developed further by many researchers [7]; see also [2]. Here we will concentrate on hyperbolic graphs from the point of view of graph theory. We will discuss how a sequence of finite graphs can give rise to an infinite hyperbolic graph, whose *boundary* can be thought of as a 'limit' of the sequence.

Let G be an infinite, locally finite graph. A *geodetic triangle* in G is a subgraph consisting of three vertices and a shortest path between each two of these vertices; these paths are the *sides* of the geodetic triangle. A geodetic triangle T is δ*-thin* if for every side S of T and every vertex v of S, the distance, in G, between v and the union of the other two sides of T is at most δ. We say that G is δ*-hyperbolic* if every geodetic triangle of G is δ-thin, and we say that G is *hyperbolic* if there is a $\delta \in \mathbb{N}$ such that G is δ-hyperbolic. See [8] for some equivalent definitions.

For example, every tree is 0–hyperbolic. Other examples of hyperbolic graphs include all tessellations of the hyperbolic plane.

Although hyperbolicity of a graph is a simple and rather local property, it implies deeper and more global properties. One of the most striking ones is the behaviour of geodesics: given a hyperbolic graph G and a vertex $v \in V(G)$, it is possible to fix an upper bound $M \in \mathbb{N}$ such that for every two 1-way infinite geodesics R, L starting at v one of the two following possibilities must hold. Either R, L are *parallel* to each other, that is, R is contained in the cylinder $\{u \in V(G) \mid d(u, L) < M\}$ of radius M around L (and vice versa), or R, L *diverge exponentially*; see [8] for details and a proof.

Hyperbolic graphs yield much of their importance from the *hyperbolic compactification*: this is a natural way to compactify a hyperbolic graph by adding a boundary to which the geodesics of the graph converge. This *hyperbolic boundary* is defined as the set of equivalence classes of 1-way infinite geodesics starting at a fixed vertex v where two such geodesics are equivalent if they are parallel. This set is endowed with a metric in which, intuitively, two classes of geodesics are close if they have representatives with long common initial subpaths. In the (hyperbolic) graph of Figure 1 for example, the boundary is homeomorphic to the real unit interval. A different approach for defining the hyperbolic boundary and its metric, based on an assignment of lengths to the edges of the graph, is explained in [3].

The variety of spaces that can be obtained as the boundary of some hyperbolic graph is impressive:

Theorem 1 ([5])**.** *Every compact metric space is isometric to the hyperbolic boundary of some hyperbolic graph.*

In order to prove this assertion, one starts with a sequence $(G_i)_{i \in \mathbb{N}}$ of finite graphs, which we will call the *horizontal levels*, that approximate the compact

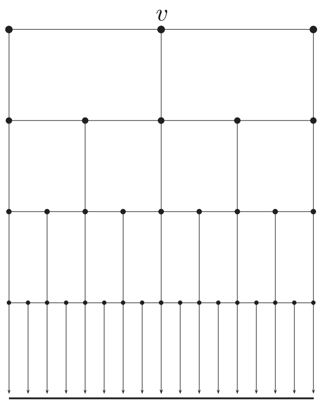

FIGURE 1. A hyperbolic graph and its boundary, which is in this case homeomorphic to the real unit interval.

metric space X, and joins all horizontal levels together into a single hyperbolic graph G by adding *perpendicular edges* that form a depth-first spanning tree of G. One does so in a manner that guarantees that G is hyperbolic and its boundary is isometric to X. For example, in the graph of Figure 1 the horizontal paths can be thought of approximations of the real unit interval, the boundary of that graph. Similarly, Figure 2 shows how to construct a hyperbolic graph whose boundary is the Sierpinski gasket [6].

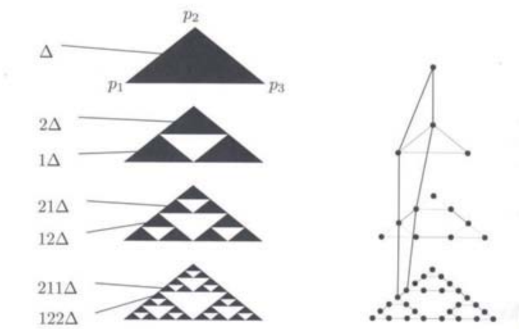

FIGURE 2. A hyperbolic graph whose boundary is the Sierpinski gasket. (Figure reproduced from [6].)

Hyperbolic graphs and their boundaries have been studied very thoroughly [7], but usually the graphs considered are Cayley graphs or otherwise closely related to some group. It was the main aim of my talk to argue that hyperbolic graphs can also be interesting in the absence of groups, and thus merit the attention of the graph theory community[1]. I mentioned two main reasons for this.

The first reason is that Theorem 1 provides a platform for proving topological results using graphs. Indeed, in [4] this theory is used in order to obtain a "graph-theoretical" characterization of path-connected continua, and this is applied to derive a graph-theoretical proof of the Hahn-Mazurkiewicz theorem. Interestingly, to achieve this, the problem of finding a 'space filling curve' in a locally connected metric space was reduced to finding well-behaved vertex-dominating walks in the finite graphs constituting the horizontal levels in the above construction. This suggests that (finite and infinite) graph theory might have applications in topology.

The second reason is that, starting with a sequence (G_i) of finite graphs, one could try to use constructions like the one of Figure 2 (see also [1] for further interesting examples) in order to obtain a hyperbolic boundary which can be thought of as the limit of the sequence (G_i); it would then be interesting to try to draw conclusions about the sequence by studying this limit. This approach seems to be more suited for sparse graphs (G_i).

Let me close with an informal problem.

Problem 1. *Find a way to construct infinite random hyperbolic graphs consisting of horizontal levels which are random finite graphs (of bounded degree?) joined by a perpendicular spanning tree (random or deterministic). What can you say about the random boundary of this graph? For example, is there a 'threshold' for its path-connectedness, and how does it relate to the threshold for connectedness of the horizontal levels?*

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[1] This idea was triggered by Wolfgang Woess (personal communication).