16) a) $\dfrac{(1+2i)z+9}{(3+4i)z-(9+4i)} = 8-5i$

$\Leftrightarrow (1+2i)z+9 = (8-5i)\left[(3+4i)z-(9+4i)\right] =$

$= (44+17i)z-(92-13i)$

$\Leftrightarrow (43+15i)z = 101-13i \quad \Big| :(43+15i)$

$\Rightarrow z = \dfrac{101-13i}{(43+15i)} = \dfrac{(101-13i)\cdot(43-15i)}{43^2+15^2} =$

$= \dfrac{4343-195}{1849+225} - \dfrac{13\cdot 43+101\cdot 15}{1849+225}\, i =$

$= \dfrac{4148}{2074} - \dfrac{559+1515}{2074}\, i = \underline{\underline{2-i}}$

b) $z^2 = 3-4i \qquad a=3,\ b=-4$

$\Rightarrow$ für $z = u+iv$

gilt $u = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)}$ mit $2uv = b$

$v = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}$

$\Rightarrow u = \pm\sqrt{\frac{1}{2}(5+3)} = \pm\sqrt{4} = \pm 2 \qquad 2uv = -4$

$v = \pm\sqrt{\frac{1}{2}(5-3)} = \pm\sqrt{1} = \pm 1 \qquad \Rightarrow$ 1 positives, 1 negatives Vorzeichen

$\Rightarrow z_1 = 2-i$

$z_2 = -2+i$

c) $z^2 - 7z + (13 - i) = 0$

$\Rightarrow z_{1,2} = \frac{7}{2} \pm \frac{\sqrt{49 - 4(13-i)}}{2} =$

(b)

$= \frac{7}{2} \pm \frac{\sqrt{-3+4i}}{2} = \frac{7}{2} \pm \frac{i \cdot \sqrt{3-4i}}{2} =$

$= \frac{7}{2} \pm \frac{i \cdot (2-i)}{2} = \frac{7}{2} \pm \frac{1-2i}{2}$

$\Rightarrow z_1 = \underline{\underline{4 - i}}$

$\Rightarrow z_2 = \underline{\underline{3 + i}}$

17) a) Wähle $\delta(\varepsilon, x_0) := \min\{\frac{x_0}{2}, \frac{\varepsilon \cdot x_0^2}{2}\}$

$\Rightarrow$ Sei $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| =$

$= \left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x \cdot x_0}\right| < \frac{\varepsilon \cdot x_0^2}{2} \cdot \frac{1}{|x \cdot x_0|} <$

$\frac{\varepsilon \cdot x_0^2}{2} \cdot \frac{1}{|(x_0 - \delta)(x_0)|} < \frac{\varepsilon \cdot x_0^2}{2} \cdot \frac{1}{|\frac{x_0}{2} \cdot x_0|} = \varepsilon$ ✓

b) Wähle $\delta(\varepsilon, x_0) := \begin{cases} \min\{|x_0|, \frac{\varepsilon}{3|x_0|}\} & \text{für } x_0 \neq 0 \\ \sqrt{\varepsilon} & x_0 = 0 \end{cases}$

$\Rightarrow$ Für $x_0 \neq 0$ gilt: $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| =$

$= \left|\frac{1}{x^2+4} - \frac{1}{x_0^2+4}\right| = \left|\frac{x_0^2+4 - (x^2+4)}{(x^2+4)(x_0^2+4)}\right| =$

$= \frac{|x - x_0||x + x_0|}{\underbrace{|x^2+4||x_0^2+4|}_{>1}} < \frac{\varepsilon}{3|x_0|} \cdot |x + x_0| < \frac{\varepsilon}{3|x_0|} 3|x_0| = \varepsilon$ ✓

Für $x_0 = 0$: $|x| < \delta \Rightarrow |f(x) - f(x_0)| =$

$= \left|\frac{1}{x^2+4} - \frac{1}{4}\right| = \left|\frac{-x^2}{\underbrace{(x^2+4)4}_{>1}}\right| < |x^2| < \delta^2 = \varepsilon$ ✓

18) a) $\sum_{n=1}^{\infty} \frac{i^n}{n} = \sum_{n=1}^{\infty} \frac{Re(i^n)}{n} + i \cdot \sum_{n=1}^{\infty} \frac{Im(i^n)}{n} =$

$= \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}}_{S_n} + i \cdot \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}}_{S_n'}$

S_n, S_n' konvergieren wegen des Leibniz-Kriteriums

$\Rightarrow \sum_{n=1}^{\infty} \frac{i^n}{n}$ konvergiert

b) $\sum_{n=1}^{\infty} \underbrace{\frac{(1+i)^n}{n^3}}_{a_n}$ $\qquad (1+i)^2 = 1 + 2i - 1 = 2i$

$\Rightarrow a_{2n} = \frac{(2i)^n}{(2n)^3} \Rightarrow |a_{2n}| = \frac{2^n}{8n^3} \xrightarrow[n\to\infty]{} \infty$

$\Rightarrow (a_n)_{n\in\mathbb{N}}$ keine Nullfolge $\Rightarrow \sum_{n=1}^{\infty} \frac{(1+i)^n}{n^3}$ divergiert

c) $\sum_{n=0}^{\infty} \frac{(3-5i)^n}{n!}$

zeige absolute Konvergenz: $\sum_{n=0}^{\infty} \frac{|3-5i|^n}{n!} = e^{|3-5i|} < \infty$

$\Rightarrow \sum_{n=0}^{\infty} \frac{(3-5i)^n}{n!}$ konvergiert

19) $f : [a,b] \to [a,b]$ stetig, $a < b \in \mathbb{R}$

ZZ: $\exists x \in [a,b] : f(x) = x$

Beweis: 1. Fall $f(a) = a$ ✓

2. Fall $f(b) = b$ ✓

Ansonsten betrachte $g : [a,b] \to \mathbb{R}$

$x \mapsto x - f(x)$

$\Rightarrow g$ stetig, $g(a) = a - f(a) < 0$

$g(b) = b - f(b) > 0$

$\underset{\text{ZWS}}{\Rightarrow} \exists x \in [a,b] : g(x) = 0 \Leftrightarrow f(x) = x$ $\square$

20) a) $\sin(s) - \sin(t) = 2\cos\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right)$

b) $\sin(s)\cos(t) = \frac{1}{2}\left(\sin(s+t) + \sin(s-t)\right)$

a) $2\cos\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right) = 2\cdot\left[\left(\frac{e^{\frac{s+t}{2}} + e^{-\frac{s+t}{2}}}{2}\right)\left(\frac{e^{\frac{s-t}{2}} - e^{-\frac{s-t}{2}}}{2}\right)\right]$

$= \frac{1}{2}\left[e^{\frac{2s}{2}} - e^{\frac{2t}{2}} + e^{\frac{-2t}{2}} - e^{-\frac{2s}{2}}\right] =$

$= \frac{1}{2}\left[e^{s} - e^{-s}\right] - \frac{1}{2}\left[e^{t} - e^{-t}\right] = \sin(s) - \sin(t)$ ✓

b) $\frac{1}{2}\left[\sin(s+t) + \sin(s-t)\right] =$

$= \frac{1}{4}\left[e^{s+t} - e^{-(s+t)} + e^{s-t} - e^{-(s-t)}\right] =$

$= \frac{1}{4}\left[e^{s+t} - e^{-s-t} + e^{s-t} - e^{-s+t}\right] = (*)$

$\sin(s)\cos(t) = \frac{1}{4}\left[\left(e^{s} - e^{-s}\right)\left(e^{t} + e^{-t}\right)\right] =$

$= \frac{1}{4}\left[e^{s+t} + e^{s-t} - e^{-s+t} - e^{-s-t}\right] = (*)$ ✓