Analysis
UE-Blatt 9

$\boxed{41}$ a) $\int_0^\infty e^{-t}\sin t \; dt$

$$< \left|\int_0^\infty e^{-t}\sin t\,dt\right| < \int_0^\infty |e^{-t}\sin t|\,dt < \int_0^\infty e^{-t}\,dt = \lim_{T\to\infty}\left(-e^{-t}\Big|_0^T\right) = 0+1 < \infty$$

$\Rightarrow$ konvergiert

b) $$\int_0^\infty \frac{1}{\sqrt{t^4+1}}\,dt = \int_0^1 \frac{1}{\sqrt{t^4+1}}\,dt + \int_1^\infty \frac{1}{\sqrt{t^4+1}}\,dt <$$

$$< \int_0^1 \frac{1}{\sqrt{t^4+1}}\,dt + \int_1^\infty \frac{1}{t^2}\,dt$$

$$= \underbrace{\int_0^1 \frac{1}{\sqrt{t^4+1}}\,dt}_{<\infty} + \underbrace{\left(-\frac{1}{t}\Big|_1^\infty\right)}_{=0+1} < \infty$$

$\Rightarrow$ konvergiert

c) $$\int_1^2 \frac{1}{t\ln t}\,dt =$$

Substitution: $x = \ln t$, $e^x = t$, $dt = e^x dx$; $x_1 = 0$, $x_2 = \ln 2$

$$\overset{*}{=} \int_{x_1}^{x_2} \frac{1}{e^x\cdot x}\cdot e^x dx$$

$$= \int_0^{\ln 2} \frac{1}{x}\,dx = \ln x\Big|_0^{\ln 2} = \ln\ln 2 - \ln 0 \to \infty$$

$\Rightarrow$ divergiert

d) $$\int_0^1 \frac{1}{\sqrt[4]{1-t^4}}\,dt = \int_0^1 \frac{1}{\sqrt[4]{(1+t^2)(1-t^2)}}\,dt < \int_0^1 \frac{1}{\sqrt[4]{1-t^2}}\,dt =$$

$$= \int_0^1 \frac{1}{\sqrt[4]{(1-t)(1+t)}}\,dt < \int_0^1 \frac{1}{\sqrt[4]{1-t}}\,dt = \int_0^1 \frac{1}{t^{1/4}}$$

konvergiert ~~für~~ laut Bsp 105 S.140 (Wert: $\frac{1}{1-1/4} = \frac{4}{3}$)

$\Rightarrow$ konvergiert

$\boxed{42}$ a) $$\int_0^1 \sqrt{\frac{x}{1-x}}\,dx = \int_0^{\pi/2} \sqrt{\frac{\sin^2 u}{\cos^2 u}}\cdot 2\sin u\cos u\,du \overset{*}{=} \int_0^{\pi/2} 2\sin^2 u\,du$$

Subst. $\sin^2(u) = x$
$2\sin(u)\cdot\cos(u)\,du = dx$
$u_1 = \sin^{-1}\sqrt{0} = 0$
$u_2 = \sin^{-1}\sqrt{1} = \pi/2$

part. Integr.
$v = \sin u$, $v' = \cos u$
$w' = \sin u$, $w = -\cos u$

$-\sin u\cdot\cos u = \int \sin^2 u\,du + \int\cos^2 u\,du$

* siehe Fortsetzung $\rightarrow$

b) $$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$$

$u = \arcsin x$, $u' = \frac{1}{\sqrt{1-x^2}}$ (Skript S. 91)
$v' = \frac{1}{\sqrt{1-x^2}}$, $v = \arcsin x$

Partielle Integration

$$\Rightarrow u\cdot v = \int u'v\,dx + \int uv'\,dx \iff \arcsin^2 x = 2\cdot\int\frac{\arcsin x}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \lim_{T\to 0}\frac{\arcsin^2 x}{2}\Big|_T^1 = \frac{\arcsin^2(1)}{2} - 0 = \frac{\pi^2}{8} \approx 1{,}2337$$

c) Standard Substitution S. 118: Rationale Fkt von $\sqrt{x^2+1}$

$$t=\sqrt{x^2+1}+x; \quad x=\frac{1}{2}\left(t-\frac{1}{t}\right); \quad \sqrt{x^2+1}=\frac{1}{2}\left(t+\frac{1}{t}\right)$$

$$dx=\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt \qquad \text{Grenzen: } t_1=\sqrt{1+1}+1=1+\sqrt{2}$$

$$t_2=\lim_{x\to\infty}\sqrt{x^2+1}+x=\infty$$

$$\Rightarrow \int_1^\infty \frac{1}{x\sqrt{x^2+1}}\,dx =$$

$$=\int_{1+\sqrt{2}}^\infty \frac{2}{t-\frac{1}{t}}\cdot\frac{2}{t+\frac{1}{t}}\cdot\frac{1}{2}\cdot\left(1+\frac{1}{t^2}\right)dt=\int_1^\infty \frac{2\cdot\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)\cdot t\cdot\left(1+\frac{1}{t^2}\right)}\,dt=$$

$$=\int_{1+\sqrt{2}}^\infty \frac{2}{t\left(t-\frac{1}{t}\right)}\,dt=\int_{1+\sqrt{2}}^\infty \frac{2}{t^2-1}\,dt \cancel{= -2\cdot \operatorname{Arcoth}\Big|_{1+\sqrt{2}}^\infty}$$

S. 92 Skript

$$=\lim_{T\to\infty}\left(-2\operatorname{Arcoth}\Big|_{1+\sqrt{2}}^T\right)$$

$$=2\cdot\operatorname{Arcoth}(1+\sqrt{2})+0 \approx \underline{\underline{0{,}88137}}$$

Weil referenziert: Hier Bsp 105

$$\int_0^1 \frac{1}{x^\alpha}\,dx=?$$

$$\int\frac{1}{x^\alpha}\,dx=\int x^{-\alpha}\,dx=\frac{x^{1-\alpha}}{1-\alpha}+C \qquad \text{f. } \alpha\neq 1$$

$$\int x^{-1}\,dx=\ln|x|+C \qquad \leftarrow \alpha=1$$

$$\int_0^1\frac{1}{x^\alpha}\,dx=\lim_{A\to 0^+}\int_A^1\frac{1}{x^\alpha}\,dx=\lim_{A\to 0^+}\frac{x^{1-\alpha}}{1-\alpha}\Big|_A^1=\lim_{A\to 0^+}\left(\frac{1}{1-\alpha}-\frac{A^{1-\alpha}}{1-\alpha}\right)$$

$$\Rightarrow \int_0^1\frac{1}{x^\alpha}\,dx=\begin{cases}\frac{1}{1-\alpha} & \text{f. } \alpha<1\\ +\infty & \text{f. } \alpha\geq 1\end{cases}$$

Fortsetzung zu 42 a) Wir wissen: $\int_0^1\sqrt{\frac{x}{1-x}}\,dx=2\int_0^{\pi/2}\sin^2 u\,du$

und: $-\sin u\cos u=\int\sin^2 u\,du-\int\cos^2 u\,du$

$$=\int\sin^2 u\,du-\int 1\,du+\int\sin^2 u\,du$$

$$\Rightarrow 2\cdot\int\sin^2 u\,du=\int 1\,du-\sin u\cos u = u-\sin u\cos u$$

$$\Rightarrow \int_0^1\sqrt{\frac{x}{1-x}}\,dx=2\cdot\int_0^{\pi/2}\sin^2 u\,du=(u-\sin u\cos u)\Big|_0^{\pi/2}=\underline{\underline{\frac{\pi}{2}}}$$

Analysis T1

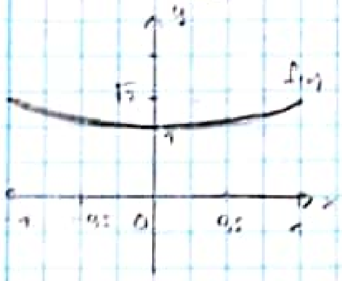
9. Übungsblatt
11.1.2019

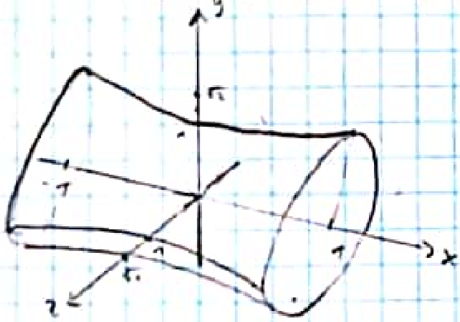
Aufgabe 43 Berechne das Volumen des Körpers, der durch Rotation der Kurve $y^2-x^2=1$ $(-1\le x\le 1,\ y>0)$ um die x-Achse entsteht.

Kurve nach y auflösen: $y^2-x^2=1 \Leftrightarrow y^2=1+x^2 \overset{\text{da } y>0}{\Leftrightarrow} y=\sqrt{1+x^2}$

$$\Rightarrow f(x)=\sqrt{1+x^2} \Rightarrow V=\pi\int_a^b f(x)^2\,dx = \pi\int_{-1}^{1}\left(\sqrt{1+x^2}\right)^2dx$$

$$=\pi\cdot\int_{-1}^{1} 1+x^2\,dx = \pi\left[x+\frac{x^3}{3}\right]_{-1}^{1} = \pi\left(1+\frac13-\left(-1-\frac13\right)\right) = \underline{\underline{\pi\cdot\frac83}}$$





Aufgabe 44 Oberfläche und Volumen des Körpers, der durch Rotation der Kettenlinie $y=\cosh(x)$ $(-1\le x\le 1)$ um die x-Achse entsteht.

$$y=f(x)=\cosh(x) \qquad O=2\pi\cdot\int_a^b\sqrt{1+f'(x)^2}\cdot f(x)\,dx = 2\pi\int_{-1}^{1}\sqrt{\underbrace{1+\sinh(x)^2}_{\cosh(x)^2}}\cdot\cosh(x)\,dx$$

$$=2\pi\int_{-1}^{1}\underbrace{\sqrt{\cosh(x)^2}}_{\cosh(x)>0}\cdot\cosh(x)\,dx = 2\pi\cdot\int_{-1}^{1}\cosh(x)^2\,dx$$

NR:

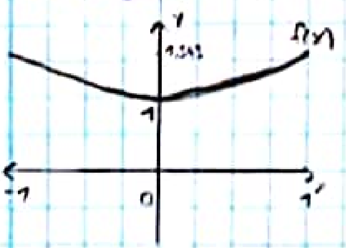
$$\int_{-1}^{1}\cosh(x)^2dx \overset{\text{Part. Int.}}{=} \sinh(x)\cdot\cosh(x)\Big|_{-1}^{1}-\int_{-1}^{1}\sinh(x)^2dx \overset{\cosh^2(x)-\sinh^2(x)=1}{=} \sinh(x)\cdot\cosh(x)\Big|_{-1}^{1}+\int_{-1}^{1}1-\cosh^2(x)\,dx$$

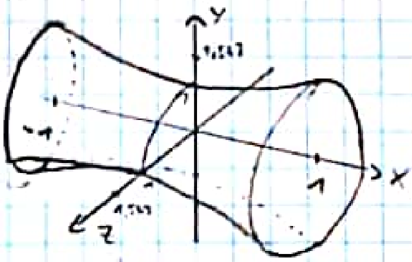
$$\Rightarrow 2\cdot\int_{-1}^{1}\cosh(x)^2dx = \sinh(x)\cdot\cosh(x)\Big|_{-1}^{1}+\int_{-1}^{1}1\,dx = \sinh(x)\cdot\cosh(x)+x\Big|_{-1}^{1}$$

$$\Rightarrow \int_{-1}^{1}\cosh(x)^2dx = \frac{\sinh(x)\cdot\cosh(x)+x}{2}\Big|_{-1}^{1} = \frac{\sinh(1)\cdot\cosh(1)+1}{2}-\frac{\sinh(-1)\cdot\cosh(-1)-1}{2}$$

$$=\underline{\sinh(1)\cdot\cosh(1)+1} \Rightarrow O=2\pi\cdot(\sinh(1)\cdot\cosh(1)+1)=2\pi\cdot(1{,}81343+1) = 5{,}62686\,\pi$$

$$V=\pi\int_a^b f(x)^2dx = \pi\int_{-1}^{1}\cosh(x)^2dx = \pi\cdot(\sinh(1)\cdot\cosh(1)+1) = 2{,}81343\,\pi$$





Aufgabe 45 Bogenlänge der Kurve $\vec x(t)=\begin{pmatrix}t\\ \frac23 t^{3/2}\\ \frac14 t^2\end{pmatrix}$ zwischen $A=(0,0,0)$, $B=(4,\frac{16}{3},4)$

$\vec x(t)=A \Rightarrow t=0$, Bogenlänge von $\vec x(t)$ im Intervall $t\in[0,4]$
$\vec x(t)=B \Rightarrow t=4$

$$L=\int_a^b\sqrt{\dot x_1(t)^2+\dot x_2(t)^2+\dot x_3(t)^2}\,dt = \int_0^4\sqrt{1+(t^{1/2})^2+\left(\tfrac t2\right)^2}\,dt = \int_0^4\sqrt{1+t+\tfrac{t^2}{4}}\,dt$$

$$\dot{\vec x}(t)=\begin{bmatrix}1\\ t^{1/2}\\ \frac t2\end{bmatrix} \quad =\int_0^4\sqrt{\left(1+\tfrac t2\right)^2}\,dt = \int_0^4\left(1+\tfrac t2\right)dt = t+\frac{t^2}{4}\Big|_0^4 = 4+\frac{16}{4}-0 = \underline{\underline{8}}$$