OPTIMAL BOUND FOR THE DISCREPANCIES OF LACUNARY SEQUENCES

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ABSTRACT. The law of the iterated logarithm for discrepancies of lacunary sequences is studies. An optimal bound is given under very mild Diophantine type condition.

1. INTRODUCTION

The discrepancies of a sequence $\{a_k\}$ of real numbers are defined by

$$D_N\{a_k\} = \sup_{0\le a<b<1} \left| \frac{1}{N}{}^{\#}\{k \le N \mid \langle a_k\rangle \in [\,a, b)\} - (b-a) \right|,$$

$$D_N^*\{a_k\} = \sup_{0\le a<1} \left| \frac{1}{N}{}^{\#}\{k \le N \mid \langle a_k\rangle \in [\,0, a)\} - a \right|,$$

where $\langle x\rangle$ denotes the fractional part $x - [\,x\,]$ of x. It is used to measure deviation of the distribution of the fractional parts of a_k from the uniform distribution. One can find detailed survey on the theory of uniform distribution in [12].

The celebrated Chung-Smirnov Theorem [11, 28] asserts the law of the iterated logarithm below for the uniformly distributed i.i.d. sequence $\{U_k\}$:

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{U_k\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{U_k\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad \text{a.s.}$$

For a sequence $\{n_k\}$ of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \ge q > 1, \tag{1.1}$$

Philipp [26] proved the bounded law of the iterated logarithm below by modifying the method due to Takahashi [30]: for almost every x,

$$\frac{1}{4\sqrt{2}} \le \varlimsup_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} \le \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} \le \frac{166}{\sqrt{2}} + \frac{664}{\sqrt{2}(q^{1/2}-1)}.$$

Aistleitner [1] improved the estimates and replaced the lower bound and the upper bound by $1/2 - 8/q^{1/4}$ and $1/2 + 6/q^{1/4}$ when $q \ge 2$.

Recently, it is proved in [13] that these limsups with respect to the sequence $\{\theta^k x\}$ are equal to a constant for almost every x if $\theta > 1$. The constant is equal to the Chung-Smirnov constant $1/2$ when θ is not a power

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root of rational number, and is greater than 1/2 otherwise (Cf. [16]). In the latter case, the constant can be concretely evaluated under some arithmetic condition. For example, when $\theta = q \geq 3$ is an odd integer the constant is equal to $\frac{1}{2}\sqrt{\frac{q+1}{q-1}}$. Other sequences for which limsups are concretely calculated can be found in [17, 18, 19, 23, 24, 25].

Aistleitner [1] gave a nearly optimal Diophantine condition on the sequence $\{n_k\}$ to have Chung-Smirnov type result below. For positive integers N and d, and for non-negative integer u, we denote the cardinality of

$$\big\{(j, j', k, k') \in [1, d]^2 \times [1, N]^2 \bigm| jn_k - j'n_{k'} = u\big\} \cap \big\{(j, j, k, k) \bigm| j, k \in \mathbf{N}\big\}^c$$

by $L_{N,d,u}$, and we put $L^*_{N,d} = \sup_{u\in\mathbf{N}} L_{N,d,u}$.

Theorem 1 (Aistleitner [1])**.** *Let $\{n_k\}$ be a sequence of positive integers satisfying the Hadamard gap condition (1.1). For any $d \in \mathbf{N}$, suppose that there exists an $\varepsilon > 0$ such that*

$$L_{N,d,0} \vee L^*_{N,d} = O\big(N/(\log N)^{1+\varepsilon}\big).$$

Then $\displaystyle\varlimsup_{N\to\infty} \frac{ND^*_N\{n_kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \frac{1}{2}$, *a.e.*

As Aistleitner [2, 3] constructed lacunary sequences for which the limsups are not constant a.e., and we can also find related examples in [15, 22], we are interested in giving a condition to have constant limsups. Since all limsups so far determined for lacunary sequences with (1.1) belong to $I_q = \Big[\frac{1}{2}, \frac{1}{2}\sqrt{\frac{q+1}{q-1}}\Big]$, it is natual to expect the same bound for all lacunary sequences. Now we state our result.

Theorem 2. *Let $\{n_k\}$ be a sequence of positive integers satisfying the Hadamard gap condition (1.1). For all $d \in \mathbf{N}$, suppose that there exists an $\varepsilon \in (0, 1)$ such that*

$$L^*_{N,d} = O\big(N/(\log N)^{1+\varepsilon}\big). \tag{1.2}$$

Then there exists a constant $\Sigma_{\{n_k\}}$ such that

$$\varlimsup_{N\to\infty} \frac{ND^*_N\{n_kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \Sigma_{\{n_k\}} \in I_q, \quad a.e. \tag{1.3}$$

Moreover, if we assume

$$L_{N,d,0} = o(N) \quad (N \to \infty) \tag{1.4}$$

together with (1.2) for all d, then we have

$$\Sigma_{\{n_k\}} = \frac{1}{2}. \tag{1.5}$$

The estimate $\Sigma_{\{n_k\}} \in I_q$ in (1.3) is best possible when $q \geq 3$ is odd, since $\Sigma_{\{q^k\}}$ attains its upper bound and $\Sigma_{\{q^{k(k+1)}\}}$ attains its lower bound, (See [13, 14]). It is also proved in [20] that the set of constants $\Sigma_{\{q^{m(k)}\}}$ for all subsequences $\{q^{m(k)}\}$ of $\{q^k\}$ coincides with I_q. Note that our condition to have (1.5) is weaker than that in the previous theorem.

At least $L_{N,d,u} = o(N)$ is necessary to have constant limsups, since limsup for star discrepancy is not constant for $\{2^k - 1\}$ and we have $N \ll L_{N,d,u}$ (See [22]). Our condition (1.2) is stronger than this, and it is open if it is necessary or not.

The condition (1.4) is necessary to have (1.5), since we have $\Sigma_{\{q^k\}} > 1/2$ and $L_{N,d,0} \gg N$ in this case.

Before closing introduction, we mention a result in [21]. Suppose that $\{n_k\}$ is a sequence of non-zero real numbers and suppose that $\{|n_k|\}$ satisfies the Hadamard gap condition (1.1). Then for any permutation ϖ of $\mathbf{N}$ (i.e. bijection $\mathbf{N} \to \mathbf{N}$.), we have the bounded law of the iterated logarithm for the discrepancies of $\{n_{\varpi(k)}x\}$ with upper bound constant $\frac{1}{2}\sqrt{\frac{q-1+4/\sqrt{3}}{q-1}}$, a constant slightly greater than $\frac{1}{2}\sqrt{\frac{q+1}{q-1}}$. For other recent development and studies on permuted sequences, see papers by Aistleitner, Berkes, and Tichy [4, 5, 6, 7, 8, 9].

2. Proof

Let $\mathbf{1}_{[a,b)}$ be the indicator function of $[\,a,b)$, put $\widetilde{\mathbf{1}}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(\langle x\rangle) - (b-a)$, and denote $\widetilde{\mathbf{1}}_{[a,b);d}$ the d-th subsum of the Fourier series of $\widetilde{\mathbf{1}}_{[a,b)}$. Put $\rho_{q,d}^2 = \frac{4}{d}(\log_q d + \frac{2q-1}{q-1})$, $\tau_{q,d}^2 = \frac{1}{4}\frac{q+1}{q-1} + \frac{1}{2}\rho_{q,d}^2$, and $\zeta_{q,d}^2 = \frac{1}{4} - \frac{1}{2}\rho_{q,d}^2$. We first prove the following key inequalities.

$$\Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\Big\|_2^2 \le \Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[0,b-a);d}(n_k\,\cdot\,)\Big\|_2^2 + \rho_{q,d}^2 N, \tag{2.1}$$

$$\Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\Big\|_2^2 \le \tau_{q,d}^2 N, \quad \Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[0,1/2);d}(n_k\,\cdot\,)\Big\|_2^2 \ge \zeta_{q,d}^2 N, \tag{2.2}$$

$$\Bigg|\Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\Big\|_2^2 - N\|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2\Bigg| \le L_{M+N,d,0} - L_{M,d,0}. \tag{2.3}$$

For $k \le k'$, by putting $P = n_k/\gcd(n_k, n_{k'})$ and $Q = n_{k'}/\gcd(n_k, n_{k'})$, we have $\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'}x)\,dx = \int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx$. For coprime integers P and Q, we have (Lemma 1 of [13])

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx = \frac{\widetilde{V}(\langle Pa\rangle, \langle Pb\rangle, \langle Qa\rangle, \langle Qb\rangle)}{PQ},$$
$$\widetilde{V}(\langle Pa\rangle, \langle Pb\rangle, \langle Qa\rangle, \langle Qb\rangle) \le \widetilde{V}(0, \langle P(a-b)\rangle, 0, \langle Q(a-b)\rangle),$$
$$0 \le \widetilde{V}(0, \langle P/2\rangle, 0, \langle Q/2\rangle),$$

where $\widetilde{V}(x,y,\xi,\eta) = V(x,\xi)+V(y,\eta)-V(x,\eta)-V(y,\xi) \le \frac{1}{4}$ and $V(x,\xi) = x \wedge \xi - x\xi$ for $0 \le x,\ y,\ \xi,\ \eta < 1$. Hence we have

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx \le \frac{1}{4PQ} \le \frac{P}{4Q} = \frac{n_k}{4n_{k'}} \le \frac{1}{4q^{k'-k}}, \tag{2.4}$$

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx \le \int_0^1 \widetilde{\mathbf{1}}_{[0,b-a)}(n_k x)\widetilde{\mathbf{1}}_{[0,b-a)}(n_{k'} x)\,dx, \tag{2.5}$$

$$\int_0^1 \widetilde{\mathbf{1}}_{[0,1/2)}(n_k x)\widetilde{\mathbf{1}}_{[0,1/2)}(n_{k'} x)\,dx \ge 0, \tag{2.6}$$

$$\int_0^1 \widetilde{\mathbf{1}}_{[0,1/2)}(n_k x)\widetilde{\mathbf{1}}_{[0,1/2)}(n_k x)\,dx = \widetilde{V}(0,1/2,0,1/2) = \frac{1}{4}. \tag{2.7}$$

Since $\left\|\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\right\|_2^2 = \sum^*(2-\delta_{k,k'})\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx$ where $\sum^*$ stands for the summation for k and k' satisfying $M+1 \le k \le k' \le M+N$, by applying (2.4) and $\sum^*(2-\delta_{k,k'})/4q^{k'-k} \le N\frac{1}{4}\frac{q+1}{q-1}$, we have the first inequality of

$$\left\|\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\right\|_2^2 \le N\frac{1}{4}\frac{q+1}{q-1}, \quad \left\|\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[0,1/2)}(n_k\,\cdot\,)\right\|_2^2 \ge \frac{N}{4}, \tag{2.8}$$

while the second inequality is proved by (2.6) and (2.7). By (2.5), we can verify

$$\left\|\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\right\|_2^2 \le \left\|\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[0,b-a)}(n_k\,\cdot\,)\right\|_2^2. \tag{2.9}$$

By $\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(Px)\widetilde{\mathbf{1}}_{[a,b);d}(Qx)\,dx = \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx$, we have

$$\begin{aligned}
h_{k,k'} &:= \left|\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx - \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\widetilde{\mathbf{1}}_{[a,b);d}(n_{k'} x)\,dx\right| \\
&= \left|\int_0^1 (\widetilde{\mathbf{1}}_{[a,b)} - \widetilde{\mathbf{1}}_{[a,b);d})(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx\right| \le \sum_{|\lambda|\ge d/Q} \left|\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(Q\lambda)\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(-P\lambda)\right| \\
&\le \frac{2}{\pi^2 PQ}\sum_{\lambda\ge d/Q}\frac{1}{\lambda^2} \le \frac{2}{\pi^2 PQ}\Big(2\wedge\frac{2Q}{d}\Big) \le \frac{P}{Q}\wedge\frac{1}{d} = \frac{n_k}{n_{k'}}\wedge\frac{1}{d} \le \frac{1}{q^{k'-k}}\wedge\frac{1}{d}.
\end{aligned}$$

Here we used $|\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j)| \le 1/\pi|j|$. Hence we have

$$\begin{aligned}
&\left|\left\|\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\right\|_2^2 - \left\|\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\right\|_2^2\right| \le 2\sum\nolimits^* h_{k,k'} \\
&\le 2\sum\nolimits^* \frac{1}{q^{k'-k}}\wedge\frac{1}{d} \le 2N\sum_{l=0}^{\infty}\frac{1}{q^l}\wedge\frac{1}{d} = 2N\Big(\frac{l_0+1}{d} + q^{-(l_0+1)}\frac{q}{q-1}\Big) \\
&\le 2N\Big(\frac{\log_q d+1}{d} + \frac{1}{d}\frac{q}{q-1}\Big) \le \frac{\rho_{q,d}^2}{2}N,
\end{aligned}$$

where l_0 is the largest integer satisfying $q^{-l_0} \geq \frac{1}{d}$. By combining this with (2.8), we have (2.2), and with (2.9), we obtain (2.1). By summing

$$\left| \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(n_k x) \widetilde{\mathbf{1}}_{[a,b);d}(n_{k'} x)\, dx \right| \leq \sum_{0<|j|\leq d} \sum_{0<|j'|\leq d} \left| \widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j) \widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j') \right| \delta_{jn_k + j' n_k, 0}$$

$$\leq \frac{2}{\pi^2} \sum_{j=1}^{d} \sum_{j'=1}^{d} \delta_{jn_k - j' n_k, 0}$$

for $M+1 \leq k' < k \leq M+N$, we see that the left hand side of (2.3) is bounded by ${}^{\#}\{(j,j',k,k') \in [1,d]^2 \times [M+1, M+N]^2 \mid jn_k - j' n_{k'} = 0, k < k'\} \leq L_{M+N,d,0} - L_{M,d,0}$.

Now we use a method of martingale approximation, which is a slight modification of the proof given in [1] and originated in Berkes-Philipp [10]. We regard $[0,1)$ equipped with the Borel field and the Lebesgue measure as a probability space. First we recall two lemmas. The proof can be found in Berkes-Philipp [10] and [13].

Lemma 3. *If g is a bounded measurable function with period 1 satisfying $\int_0^1 g = 0$, then for all $a < b$ and $\lambda > 0$, we have $\left|\int_a^b g(\lambda x)\, dx\right| \leq \|g\|_\infty / \lambda$.*

Lemma 4. *Let g be a trigonometric polynomial with period 1 and degree d satisfying $\int_0^1 g = 0$. There exists a constant C_q depending only on q such that, for any sequence $\{n_k\}$ of positive integers satisfying the Hadamard gap condition (1.1), $\int_0^1 \left(\sum_{k=M+1}^{M+N} g(n_k x)\right)^4 dx \leq C_q \left(\sum_{|\nu|\leq d} |\widehat{g}(\nu)|\right)^4 N^2$ holds.*

Let us divide $\mathbf{N}$ into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \ldots$ satisfying ${}^{\#}\Delta_i' = [1 + 9 \log_q i]$ and ${}^{\#}\Delta_i = i$. Denote $i^- = \min \Delta_i$, $i^+ = \max \Delta_i$, and $l_M = {}^{\#}\Delta_1 + \cdots + {}^{\#}\Delta_M$. We have $M^- \sim M^+ \sim l_M = M(M+1)/2 \ll M^2$ and $n_{i^-}/n_{(i-1)^+} \geq q^{9 \log_q i} = i^9$. Put $\mu(i) = [\log_2 i^4 n_{i^+}] + 1$ and $\mathcal{F}_i = \sigma\{[j 2^{-\mu(i)}, (j+1) 2^{-\mu(i)}) \mid j = 0, \ldots, 2^{\mu(i)} - 1\}$. Note that $i^4 n_{i^+} \leq 2^{\mu(i)} \leq 2 i^4 n_{i^+}$. Denote $\widetilde{\mathbf{1}}_{[a,b);d}$ by f and put

$$T_i(x) = \sum_{k \in \Delta_i} f(n_k x), \quad T_i'(x) = \sum_{k \in \Delta_i'} f(n_k x), \quad Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1}).$$

We also denote T_i and Y_i by $T_{[a,b);d;i}$ and $Y_{[a,b);d;i}$ to specify the parameters $[a,b)$ and d. Clearly $\{Y_i, \mathcal{F}_i\}$ forms a martingale difference sequence. Here let us prove

$$\|Y_i - T_i\|_\infty \ll 1/i^3, \tag{2.10}$$

$$\|Y_i^2 - T_i^2\|_\infty \ll 1/i^2, \tag{2.11}$$

$$\|Y_i^4 - T_i^4\|_\infty \ll 1. \tag{2.12}$$

Here and later, the constant implied by $\ll$ and O depend only on a, b, and d.

Suppose that $k \in \Delta_i$ and $x \in I = [j 2^{-\mu(i)}, (j+1) 2^{-\mu(i)}) \in \mathcal{F}_i$. In this case we have $|f(n_k x) - E(f(n_k \cdot\,) \mid \mathcal{F}_i)| = |I|^{-1} \left| \int_I (f(n_k x) - f(n_k y))\, dy \right| \leq \max_{y \in I} |f(n_k x) - f(n_k y))| \leq \|f'\|_\infty n_k 2^{-\mu(i)} \leq \|f'\|_\infty n_k / i^4 n_{i^+} \leq \|f'\|_\infty / i^4$. Hence we obtain $|T_i - E(T_i \mid \mathcal{F}_i)| \leq \|f'\|_\infty {}^{\#}\Delta_i / i^4 = \|f'\|_\infty / i^3$. Take $J =$

$[j2^{-\mu(i-1)},(j+1)2^{-\mu(i-1)}) \in \mathcal{F}_{i-1}$. Then by applying Lemma 3, we have $|E(f(n_k\,\cdot\,)\mid \mathcal{F}_{i-1})| = |J|^{-1}\bigl|\int_J f(n_k y)\,dy\bigr| \le \|f\|_\infty 2^{\mu(i-1)}/n_k \le \|f\|_\infty 2(i-1)^4 n_{(i-1)^+}/n_{i^-} \le 2\|f\|_\infty/i^5$. Therefore $|E(T_i \mid \mathcal{F}_{i-1})| \le 2\|f\|_\infty {}^\#\Delta_i/i^5 = 2\|f\|_\infty/i^4$, and (2.10) is proved.

By $\|T_i\|_\infty \le i\|f\|_\infty$, we have $\|E(T_i \mid \mathcal{F}_i)\|_\infty$, $\|E(T_i \mid \mathcal{F}_{i-1})\|_\infty \le i\|f\|_\infty$. Hence we have $\|Y_i\|_\infty \le 2i\|f\|_\infty$, $\|Y_i+T_i\|_\infty \le 3i\|f\|_\infty$, $\|Y_i^2+T_i^2\|_\infty \le 5i^2\|f\|_\infty^2$. Because of $\|Y_i^2-T_i^2\|_\infty \le \|Y_i-T_i\|_\infty\|Y_i+T_i\|_\infty$ and $\|Y_i^4-T_i^4\|_\infty \le \|Y_i^2-T_i^2\|_\infty\|Y_i^2+T_i^2\|_\infty$, we have (2.11) and (2.12).

Put $\widetilde{\mathbf{1}}_{[a,b);d} = \sum_{j=1}^d (a_j\cos 2\pi jx + b_j \sin 2\pi jx)$, $v_i = v_{[a,b);d;i} = \int_0^1 T_{[a,b);d;i}^2$, $\beta_M = \beta_{[a,b);d;M} = v_{[a,b);d;1} + \cdots + v_{[a,b);d;M}$, and $V_M = \sum_{i=1}^M E(Y_i^2 \mid \mathcal{F}_{i-1})$. Set

$$\begin{aligned}
\Phi_i &= \{(k,k',j,j',\varsigma) \mid k,k' \in \Delta_i,\, j,j' = 1,\dots,d,\, \varsigma = +1,-1\},\\
\Phi_i^v &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid jn_k + \varsigma j' n_{k'} = 0\}\\
\Phi_i^U &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid 0 < |jn_k + \varsigma j' n_{k'}| < n_{(i-1)^+}\}\\
\Phi_i^W &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid n_{(i-1)^+} \le |jn_k + \varsigma j' n_{k'}| < n_{i^-}\}\\
\Phi_i^R &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid n_{i^-} \le |jn_k + \varsigma j' n_{k'}|\}.
\end{aligned}$$

For $\Psi \subset \Phi_i$, denote $\chi(\Psi) = \sum_{(k,k',j,j',\varsigma)\in\Psi} A_{k,k',j,j',\varsigma}$, where $2A_{k,k',j,j',\varsigma}(x) = (a_j a_{j'} - \varsigma b_j b_{j'})\cos 2\pi(jn_k + \varsigma j'n_{k'})x + (\varsigma a_j b_{j'} + b_j a_{j'})\sin 2\pi(jn_k + \varsigma j' n_{k'})x$.

We see $T_{[a,b);d;i}^2(x) = \chi(\Phi_i)$ and $v_{[a,b);d;i} = \chi(\Phi_i^v)$. Let $U_i = \chi(\Phi_i^U)$, $W_i = \chi(\Phi_i^W)$, and $R_i = \chi(\Phi_i^R)$. We can express Φ_i as a disjoint union $\Phi_i^v \cup \Phi_i^U \cup \Phi_i^W \cup \Phi_i^R$ and hence $T_i^2 = v_i + U_i + W_i + R_i$. We prove

(2.13)

$$\begin{aligned}
\|V_M - \beta_M\|_2 &\le \Bigl\|\sum_{i=1}^M E(Y_i^2 - T_i^2 \mid \mathcal{F}_{i-1})\Bigr\|_2 + \Bigl\|\sum_{i=1}^M E(U_i \mid \mathcal{F}_{i-1})\Bigr\|_2\\
&\quad + \Bigl\|\sum_{i=1}^M E(W_i \mid \mathcal{F}_{i-1})\Bigr\|_2 + \Bigl\|\sum_{i=1}^M E(R_i \mid \mathcal{F}_{i-1})\Bigr\|_2 \ll M^2(\log M)^{-(1+\varepsilon)/2},
\end{aligned}$$

where the first inequality is due to $Y_i^2 - v_i = (Y_i^2 - T_i^2) + U_i + W_i + R_i$.

By (2.11) we see $\bigl\|\sum_{i=1}^M E(Y_i^2 - T_i^2 \mid \mathcal{F}_{i-1})\bigr\|_2 = O(1)$.

By ${}^\#\Phi_i^R \le {}^\#\Phi_i \le 2d^2 i^2$, $|a_j a_{j'} - \varsigma b_j b_{j'}|/2 \le 1$, and $|\varsigma a_j b_{j'} + b_j a_{j'}|/2 \le 1$, we see $|E(R_i \mid \mathcal{F}_{i-1})| \le 4d^2 i^2 2^{\mu(i-1)}/n_{i^-} \le 8d^2/i^3$ and $\bigl\|\sum_{i=1}^M E(R_i \mid \mathcal{F}_{i-1})\bigr\|_2 = O(1)$.

Let k, $k' \in \Delta_i$, j, $j' = 1, \dots, d$. By $jn_k + j'n_{k'} \ge 2n_{i^-}$, we have $(k,k',j,j',+1) \notin \Phi_i^U \cup \Phi_i^W$. If $k \le k'$ and $n_{k'} > (d+1)n_k$, then $jn_k - j'n_{k'} \le dn_k - (d+1)n_k \le -n_{i^-}$. Hence $|jn_k - j'n_{k'}| < n_{i^-}$ implies $q^{k'-k} \le n_{k'}/n_k \le d+1$ or $k'-k \le \log_q(d+1)$. Therefore, if we fix k, j and j', then the number of k' such that $k \le k'$ and $|jn_k - j'n_{k'}| < n_{i^-}$ is at most $\log_q(d+1)+1$. Thereby we have ${}^\#(\Phi_i^U \cup \Phi_i^W) \le 2d^2(\log_q(d+1)+1)i$ and

$$\|U_i\|_\infty \ll i, \qquad \|W_i\|_\infty \ll i. \tag{2.14}$$

Hence we have $|E(W_i \mid \mathcal{F}_{i-1})| \le \|W_i\|_\infty \ll i$ and $\|\sum_{i=1}^M E(W_i \mid \mathcal{F}_{i-1})^2\|_\infty \ll M^3$. If $i < i'$, then $E\bigl(E(W_i \mid \mathcal{F}_{i-1})E(W_{i'} \mid \mathcal{F}_{i'-1}) \bigm| \mathcal{F}_{i-1}\bigr) = E(W_i \mid$

$\mathcal{F}_{i-1})E(W_{i'} \mid \mathcal{F}_{i-1}) = O(i)E(W_{i'} \mid \mathcal{F}_{i-1})$ and $\big|E\big(E(W_i \mid \mathcal{F}_{i-1})E(W_{i'} \mid \mathcal{F}_{i'-1})\big)\big| \ll iE\big|E(W_{i'} \mid \mathcal{F}_{i-1})\big|$.

Since we can write

$$W_{i'}(x) = \sum_{u=n_{(i'-1)^+}}^{n_{(i')^-}} (c_u \cos 2\pi ux + d_u \sin 2\pi ux) \text{ with } \sum_{u=n_{(i'-1)^+}}^{n_{(i')^-}} (|c_u|+|d_u|) \ll i',$$

by Lemma 3, we can verify $\big|E(W_{i'} \mid \mathcal{F}_{i-1})\big| \le \sum_u (|c_u| + |d_u|)2^{\mu(i-1)}/u \ll i' i^4 n_{(i-1)^+}/n_{(i'-1)^+} \ll i'^5 q^{(i-1)^+-(i'-1)^+} \ll i'^5 q^{-i'}$. Hence we have the estimate $\sum_{i<i'} \big|E\big(E(W_i \mid \mathcal{F}_{i-1})E(W_{i'} \mid \mathcal{F}_{i'-1})\big)\big| \ll \sum_{i<i'} ii'^5 q^{-i'} \ll \sum_{i'} i'^7 q^{-i'} \ll 1$. These imply $E\big(\sum_{i=1}^M E(W_i \mid \mathcal{F}_{i-1})\big)^2 \ll M^3$.

Since we can write

$$U_i(x) = \sum_{u=1}^{n_{(i-1)^+}} (c'_u \cos 2\pi ux + d'_u \sin 2\pi ux) \quad \text{with} \quad \sum_{u=1}^{n_{(i-1)^+}} (|c'_u| + |d'_u|) \ll i,$$

by $|E(\cos 2\pi u\cdot \mid \mathcal{F}_{i-1}) - \cos 2\pi ux| \le 2\pi u 2^{-\mu(i-1)} \ll n_{(i-1)^+}/i^4 n_{(i-1)^+} \ll 1/i^4$ and $|E(\sin 2\pi u\cdot \mid \mathcal{F}_{i-1}) - \sin 2\pi ux| \ll 1/i^4$, we have

$$\left|\sum_{i=1}^M E(U_i \mid \mathcal{F}_{i-1}) - \sum_{i=1}^M U_i\right| \ll \sum_{i=1}^M \sum_{u=1}^{n_{(i-1)^+}} (|c'_u| + |d'_u|)/i^4 \ll \sum_{i=1}^M \frac{1}{i^3} \ll 1.$$

We can write

$$\sum_{i=1}^M U_i(x) = \sum_{u=1}^{n_{(M-1)^+}} (c''_u \cos 2\pi ux + d''_u \sin 2\pi ux) \text{ with } \sum_{u=1}^{n_{(M-1)^+}} (|c''_u|+|d''_u|) \ll M^2,$$

and by (1.2) we have $|c''_u|, |d''_u| \le L_{M^+,d,u} \le L^*_{M^+,d} \ll M^2/(\log M)^{1+\varepsilon}$. Hence we have

$$\begin{aligned}\left\|\sum_{i=1}^M U_i\right\|_2^2 &= \sum_{u=1}^{n_{(M-1)^+}} \frac{(c''_u)^2 + (d''_u)^2}{2} \ll \frac{M^2}{(\log M)^{1+\varepsilon}} \sum_{u=1}^{n_{(M-1)^+}} (|c''_u| + |d''_u|) \\ &\ll \frac{M^4}{(\log M)^{1+\varepsilon}},\end{aligned}$$

and $\big\|\sum_{i=1}^M E(U_i \mid \mathcal{F}_{i-1})\big\|_2 \ll M^2(\log M)^{-(1+\varepsilon)/2}$. Hence we have (2.13).

We prepare another probability space on which a sequence $\{U, \xi_1, \xi_2, \dots\}$ of independent random variables satisfying $P(\xi_k = 1) = P(\xi_k = -1) = 1/2$ and $P(U \in A) = |A \cap [0,1]|$ is defined. We make the product of $[0,1)$ on which $\{Y_i\}$ is defined and this new probability space, and regard Y_i, U, and $\Xi_i = \sum_{k\in\Delta_i} \xi_k$ as random variables on this product probability space. Take $m \in \mathbf{N}$ arbitrarily and we define a martingale difference sequence $\{\widehat{Y}_i, \widehat{\mathcal{F}}_i\}$ on this space by putting $\widehat{\mathcal{F}}_i = \mathcal{F}_i \otimes \sigma\{\Xi_1, \dots, \Xi_i\}$,

$$\widehat{Y}_i = \widehat{Y}_{[a,b);d;m;i} = Y_{[a,b);d;m;i} + \frac{1}{m}\Xi_i, \text{ and } \widehat{\beta}_M = \widehat{\beta}_{[a,b);d;m;M} = \beta_{[a,b);d;M} + \frac{1}{m^2} l_M.$$

By Lemma 4 and (2.12), we have $\|\widehat{Y}_i\|_4 \le \|Y_i\|_4 + \|\Xi_i\|_4 = \|T_i\|_4 + \|\Xi_i\|_4 + O(1) \ll i^{1/2}$ or $E\widehat{Y}_i^4 \ll i^2$. We have $E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) = E(Y_i^2 \mid \mathcal{F}_{i-1}) + \frac{1}{m^2} i$ and hence $\widehat{V}_M := \sum_{i=1}^M E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) = V_M + \frac{1}{m^2} l_M \geq \frac{1}{m^2} l_M$ and $\|\widehat{V}_M - \widehat{\beta}_M\|_2 \ll M^2 (\log M)^{-(1+\varepsilon)/2}$. We prove

$$\widehat{V}_M = \widehat{\beta}_M + o\big(\widehat{\beta}_M (\log \widehat{\beta}_M)^{-\varepsilon/4}\big), \quad \text{a.s.} \tag{2.15}$$

Note that we have $v_i \ll i$ by (2.2), so $\beta_M \ll M^2$, and thereby $M^2 \ll \widehat{\beta}_M \ll M^2$. We also have $\beta_{M'} - \beta_M = \sum_{i=M+1}^{M'} v_i \ll M'(M'-M)$ and $\widehat{\beta}_{M'} - \widehat{\beta}_M \ll M'(M'-M)$. Put $\alpha = 1 - \varepsilon/2 + \varepsilon^2/4 < 1$ and $M_l = [\,2^{l^\alpha}\,]$. We have $(1+\varepsilon/2)\alpha > 1$, $(\alpha-1)/\alpha < \alpha - 1 < -\varepsilon/4$, and $M_{l+1}/M_l \sim 2^{\alpha l^{\alpha-1}} = 1 + O(l^{\alpha-1}) = 1 + O((\log M_l)^{(\alpha-1)/\alpha}) = 1 + o((\log M_l)^{-\varepsilon/4})$ or $M_{l+1} - M_l = o(M_l (\log M_l)^{-\varepsilon/4})$. Hence $0 \leq \widehat{\beta}_{M_{l+1}} - \widehat{\beta}_{M_l} \ll M_{l+1}(M_{l+1} - M_l) = o(M_l^2 (\log M_l)^{-\varepsilon/4}) = o(\widehat{\beta}_{M_l} (\log \widehat{\beta}_{M_l})^{-\varepsilon/4})$ or $\widehat{\beta}_{M_{l+1}}/\widehat{\beta}_{M_l} = 1 + o((\log M_l)^{-\varepsilon/4})$. Therefore we have

$$\sum_{l=1}^\infty E\bigg(\frac{\widehat{V}_{M_l} - \widehat{\beta}_{M_l}}{\widehat{\beta}_{M_l}(\log \widehat{\beta}_{M_l})^{-\varepsilon/4}}\bigg)^2 \ll \sum_{l=1}^\infty (\log M_l)^{-1-\varepsilon/2} \ll \sum_{l=1}^\infty l^{-\alpha(1+\varepsilon/2)} < \infty.$$

By applying Beppo-Levi's theorem, we have $(\widehat{V}_{M_l} - \widehat{\beta}_{M_l})/\widehat{\beta}_{M_l}(\log \widehat{\beta}_{M_l})^{-\varepsilon/4} \to 0$, a.s., or $\widehat{V}_{M_l} - \widehat{\beta}_{M_l} = o(\widehat{\beta}_{M_l}(\log \widehat{\beta}_{M_l})^{-\varepsilon/4})$, a.s.

If $M_l \leq M < M_{l+1}$, then we have $(\widehat{V}_{M_l} - \widehat{\beta}_{M_l}) + (\widehat{\beta}_{M_l} - \widehat{\beta}_{M_{l+1}}) \leq \widehat{V}_M - \widehat{\beta}_M \leq (\widehat{V}_{M_{l+1}} - \widehat{\beta}_{M_{l+1}}) + (\widehat{\beta}_{M_{l+1}} - \widehat{\beta}_{M_l})$ and hence we have (2.15).

Now we use the following theorem by Monrad-Philipp [27] which is a modification of Strassen's theorem [29].

Theorem 5. *Let $\{\widehat{Y}_i, \widehat{\mathcal{F}}_i\}$ be a square integrable martingale difference satisfying*

$$\widehat{V}_M = \sum_{i=1}^M E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) \to \infty \text{ a.s. and } \sum_{i=1}^\infty E\big(\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2 \geq \psi(\widehat{V}_i)\}} / \psi(\widehat{V}_i)\big) < \infty$$

for some non-decreasing ψ such that $\psi(\infty) = \infty$ and $\psi(x)(\log x)^\alpha/x$ is non-increasing for some $\alpha > 50$. If there exists a uniformly distributed random variable U which is independent of $\{\widehat{Y}_n\}$, there exists a standard normal i.i.d. $\{Z_i\}$ such that

$$\sum_{i \geq 1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i \leq t\}} = \sum_{i \leq t} Z_i + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big), \quad (t \to \infty) \quad a.s.$$

Put $\psi(x) = x/(\log x)^{51}$. We can verify $\widehat{V}_M \geq \frac{1}{m^2} l_M \to \infty$, and

$$\sum E\big(\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2 \geq \psi(\widehat{V}_i)\}} / \psi(\widehat{V}_i)\big) \leq \sum \frac{E\widehat{Y}_i^4}{\psi^2(\frac{1}{m^2} l_i)} \ll \sum i^2 (\log l_i)^{102}/l_i^2 < \infty.$$

Hence we have $\sum_{i=1}^M \widehat{Y}_i = \sum_{i \leq \widehat{V}_M} Z_i + o\big(\widehat{V}_M^{1/2}(\log \widehat{V}_M)^{-51/50}\big)$, a.s. By (2.15) and $\sup_{0 \leq |s| \leq t(\log t)^{-\varepsilon/4}} |W_{t+s} - W_t| = O\big(t^{1/2}(\log t)^{-\varepsilon/8}(\log\log t)^{1/2}\big)$, where $\{W_t\}$ is the Wiener process, we have

$$\sum_{i=1}^M \widehat{Y}_i = \sum_{i \leq \widehat{\beta}_M} Z_i + O\big(\widehat{\beta}_M^{1/2}(\log \widehat{\beta}_M)^{-\varepsilon/9}\big), \quad \text{a.s.}$$

Hence by denoting $\phi(x) = \sqrt{2x \log\log x}$ and by applying the 0-1 law, we see that there exists a constant $C_{[a,b);d;m}$ such that

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big| \sum_{i=1}^{M} \widehat{Y}_{[a,b);d;m;i} \Big| = \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big| \sum_{i \le \widehat{\beta}_{[a,b);d;m;M}} Z_i \Big| = C_{[a,b);d;m}, \tag{2.16}$$

almost surely. Now we apply the following lemma by putting $\bar{v}_i = v_{[a,b);d;i} + \frac{i}{m^2}$, $\bar{v}'_i = v_{[0,b-a);d;i} + \frac{i}{m^2}$, $\bar{\beta}_M = \widehat{\beta}_{[a,b);d;M}$ and $\bar{\beta}'_M = \widehat{\beta}_{[0,b-a);d;M}$.

Lemma 6. *Let $\{Z_k\}$ and $\{Z'_k\}$ be standard normal i.i.d. Suppose that $\{\bar{v}_k\}$ and $\{\bar{v}'_k\}$ are sequence of positive numbers satisfying $c_1 i \le \bar{v}_i \le c_2 i$, $d_1 i \le \bar{v}'_i \le d_2 i$, and $\bar{v}_i \le \bar{v}'_i + \gamma i$ for some $0 < c_1 < c_2 < \infty$, $0 < d_1 < d_2 < \infty$, and $0 < \gamma < \infty$. Then by putting $\bar{\beta}_M = \bar{v}_1 + \cdots + \bar{v}_M$, and $\bar{\beta}'_M = \bar{v}'_1 + \cdots + \bar{v}'_M$, we have*

$$\sqrt{c_1} \le \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big| \sum_{k \le \bar{\beta}_M} Z_k \Big| \le \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big| \sum_{k \le \bar{\beta}'_M} Z'_k \Big| + \sqrt{\gamma} \le \sqrt{d_2} + \sqrt{\gamma}, \quad \textit{a.s.}$$

By using conditions (2.1) and (2.2), we can verify the conditions of lemma for $c_1 = d_1 = \frac{1}{m^2}$, $c_2 = d_2 = \tau_{q,d}^2 + \frac{1}{m^2}$, and $\gamma = \rho_{q,d}^2$, and have

$$C_{[a,b);d;m} \le C_{[0,b-a);d;m} + \rho_{q,d} \le \left(\tau_{q,d}^2 + 1/m^2\right)^{1/2} + \rho_{q,d}.$$

Putting $\bar{v}_i = \bar{v}'_i = v_{[0,1/2);d;i} + \frac{i}{m^2}$ and $c_1 = c_2 = \zeta_{q,d}^2$, and d_1, d_2 as before, we have

$$C_{[0,1/2);d;m} \ge \zeta_{q.d}.$$

By $\Big| \frac{1}{\phi(l_M)} \big| \sum_{i=1}^{M} Y_{[a,b);d;i} \big| - \frac{1}{\phi(l_M)} \big| \sum_{i=1}^{M} \widehat{Y}_{[a,b);d;m;i} \big| \Big| \le \frac{1}{m\phi(l_M)} \big| \sum_{i=1}^{M} \Xi_i \big|$, we have

$$\Big| \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big| \sum_{i=1}^{M} Y_{[a,b);d;i} \Big| - C_{[a,b);d;m} \Big| \le \frac{1}{m}, \qquad \text{a.s.}$$

Hence $C_{[a,b);d} = \lim_{m\to\infty} C_{[a,b);d;m}$ is a constant satisfying

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big| \sum_{i=1}^{M} Y_{[a,b);d;i} \Big| = C_{[a,b);d}, \quad \text{a.s.}, \tag{2.17}$$

$$C_{[a,b);d} \le C_{[0,b-a);d} + \rho_{q,d} \le \tau_{q,d} + \rho_{q,d}, \quad \text{and} \quad C_{[0,1/2);d} \ge \zeta_{q,d}.$$

Since Y_i is a function with respect to x, by applying Fubini's theorem, we see that equality in (2.17) holds on $[0,1)$ and we can replace a.s. in (2.17) by a.e. By (2.10), we have $\big| \sum_{i=1}^{M} Y_{[a,b);d;i} \big| = \big| \sum_{i=1}^{M} T_{[a,b);d;i} \big| + O(1)$ and

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big| \sum_{i=1}^{M} T_{[a,b);d;i} \Big| = C_{[a,b);d}, \qquad \text{a.e.}$$

Because of ${}^{\#}\Delta'_1 + \cdots + {}^{\#}\Delta'_M \ll M \log M$ and $l_M \sim M^+$, by applying the law of the iterated logarithm for lacunary trigonometric series, we have

$\left|\sum_{i=1}^M T'_{[a,b);d;i}\right| \ll \sqrt{M \log M \log\log(M \log M)} = o(\phi(M^+))$. Therefore, we have

$$\varlimsup_{M\to\infty} \frac{1}{\phi(M^+)}\left|\sum_{i=1}^{M}(T_{[a,b);d;i} + T'_{[a,b);d;i})\right| = C_{[a,b);d}, \qquad \text{a.e.}$$

By noting $(M-1)^+ \sim M^+$ and $\max_{j=(M-1)^++1}^{M^+}\left|\sum_{k=(M-1)^++1}^{j} \widetilde{\mathbf{1}}_{[a,b);d}\right| \ll M = o(\phi(M^+))$, we have

$$\varlimsup_{N\to\infty} \frac{1}{\phi(N)}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\right| = C_{[a,b);d}, \quad \text{a.e.}$$

Now we apply the next proposition. It is essentially proved in [13]. The proof of the first part can be found in [16], and the full proof in [21].

Proposition 7. *Let $\{n_k\}$ be a sequence of positive numbers satisfying the Hadamard gap condition. Then for any dense countable set $S \subset [0,1)$, we have*

$$\begin{aligned}
&\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \sup_{S\ni a<b\in S} \varlimsup_{N\to\infty} \frac{1}{\phi(N)}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\right|, \\
&\varlimsup_{N\to\infty} \frac{ND^*_N\{n_k x\}}{\sqrt{2N\log\log N}} = \sup_{a\in S} \varlimsup_{N\to\infty} \frac{1}{\phi(N)}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)}(n_k x)\right|,
\end{aligned} \tag{2.18}$$

and

$$\varlimsup_{N\to\infty} \frac{1}{\phi(N)}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\right| = \lim_{d\to\infty} \varlimsup_{N\to\infty} \frac{1}{\phi(N)}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\right|, \tag{2.19}$$

for almost every $x \in \mathbf{R}$.

Put $S = [0,1) \cap \mathbf{Q}$. By applying (2.19), we have

$$\varlimsup_{N\to\infty} \frac{1}{\phi(N)}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\right| = C_{[a,b)} := \lim_{d\to\infty} C_{[a,b);d}, \quad \text{a.e.},$$

$C_{[a,b)} \le C_{[0,b-a)} \le \frac{1}{2}\sqrt{\frac{q+1}{q-1}}$, and $\frac{1}{2} \le C_{[0,1/2)}$. By (2.18), we have (1.3).

Suppose that the condition (1.4) is assumed. By (2.3) we have $\beta_{[a,b);d;M} = \sum_{i=1}^M ET^2_{[a,b);d;i} = \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 l_M + O(L_{M^+,d,0}) = \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 l_M + o(l_M)$, and thereby $\widehat{\beta}_{[a,b);d;m;M} \sim (\|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 + \frac{1}{m^2}) l_M$. Hence, by (2.16) we directly have

$$\begin{aligned}
C_{[a,b);d;m} &= \varlimsup_{M\to\infty} \frac{\phi(\widehat{\beta}_{[a,b);d;m;M})}{\phi(l_M)} \frac{1}{\phi(\widehat{\beta}_{[a,b);d;m;M})}\left|\sum_{i=1}^{[\widehat{\beta}_{[a,b);d;m;M}]} Z_i\right| \\
&= \sqrt{\|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 + \tfrac{1}{m^2}}.
\end{aligned}$$

Therefore $C_{[a,b);d} = \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2$ and $C_{[a,b)} = \|\widetilde{\mathbf{1}}_{[a,b)}\|_2 \le \|\widetilde{\mathbf{1}}_{[0,1/2)}\|_2 = \frac{1}{2} = C_{[0,1/2)}$.

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