Probability and metric discrepancy theory

Christoph Aistleitner*, István Berkes†

Abstract

We give an overview of probabilistic phenomena in metric discrepancy theory and present several new results concerning the asymptotic behavior of discrepancies $D_N(n_k x)$ and sums $\sum c_k f(n_k x)$ for sequences $(n_k)_{k \geq 1}$ of integers.

1 Uniform distribution

We say that a sequence of real numbers $(x_k)_{k \geq 1}$ is uniformly distributed (u.d.) mod 1 if for every pair a, b of real numbers with $0 \leq a < b \leq 1$

$$\lim_{N \to \infty} \frac{\sum_{k=1}^N \mathbb{1}_{[a,b)}(x_k)}{N} = b - a.$$

Here $\mathbb{1}_{[a,b)}$ denotes the indicator function of the interval $[a, b)$, extended with period 1. By a classical criterion by Weyl [78], a sequence $(x_k)_{k \geq 1}$ is u.d. mod 1 if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N e^{2\pi i h x_k} = 0 \quad \text{for all integers } h \neq 0.$$

Using this criterion it follows immediately that the sequence $(kx)_{k \geq 1}$ is u.d. mod 1 for all irrational x, a result playing an important role in many areas of mathematics. It also follows that $(n_k x)_{k \geq 1}$ is u.d. mod 1 for all irrational x in the following cases:

- $n_k = k^b \log^c k$, where $0 < b < 1,\ c \in \mathbb{R}$ (Fejér)
- $n_k = \log^c k$, where $c > 1$ (Fejér)
- $n_k = a_0 + a_1 k + \cdots + a_n k^n$, where $a_0, \ldots, a_n \in \mathbb{Z}, n \geq 1$ (Van der Corput)
- $n_k = p_k$, the kth prime (Vinogradov).

*MU Leoben, Lehrstuhl für Mathematatik und Statistik. e-mail: `aistleitner@finanz.math.tugraz.at`.
Research supported by FWF Grant S9610-N23.

†TU Graz, Institut für Statistik. e-mail: `berkes@tugraz.at`.
Research supported by FWF grant S9603-N23 and OTKA grants K 61052 and K 67986.

Mathematics Subject Classification: Primary 11K38, 42A55, 60F15

Keywords: lacunary series, uniform distribution, discrepancy, central limit theorem, law of the iterated logarithm, Diophantine equation, trigonometric series

For further examples of uniformly distributed sequences of this type, see Kuipers and Niederreiter [50]. On the other hand, note that $(n_k x)_{k\geq 1}$ is not uniformly distributed mod 1 for $n_k = k!$ and $x = e$. However, Weyl [78] proved the following result:

Theorem 1.1 *Let $(n_k)_{k\geq 1}$ be a sequence of distinct integers. Then the sequence $(n_k x)_{k\geq 1}$ is u.d. mod 1 for all real numbers x with the exception of a set with Lebesgue measure 0.*

To decide if an explicitly given x belongs to the exceptional set can be a very difficult problem. For example, the uniform distribution of $(2^k x)_{k\geq 1}$ mod 1 is equivalent to the normality of x in base 2, and it is still an open problem if simple irrational numbers like $x = \sqrt{2}$, e, π are normal or not.

Natural measures of the uniformity of a finite sequence $(x_1, \dots x_N)$ are the *discrepancy*, resp. *star discrepancy* defined by

$$D_N = D_N(x_1, \dots, x_N) := \sup_{0\leq a<b\leq 1} \left| \frac{\sum_{k=1}^N \mathbb{1}_{[a,b)}(x_k)}{N} - (b-a) \right|,$$

$$D_N^* = D_N^*(x_1, \dots, x_N) := \sup_{0<a\leq 1} \left| \frac{\sum_{k=1}^N \mathbb{1}_{[0,a)}(x_k)}{N} - a \right|,$$

respectively. It is easy to see that always $D_N^* \leq D_N \leq 2D_N^*$ and an infinite sequence $(x_k)_{k\geq 1}$ is u.d. mod 1 if and only if $D_N = D_N(x_1, \dots x_N) \to 0$ as $N \to \infty$. Easy calculations show that always $1/N \leq D_N \leq 1$. For infinite sequences $(x_k)_{k\geq 1}$ the lower bound can be improved to

$$N D_N \geq c \ \log N \quad \text{for infinitely many } N,$$

where $c > 0$ is an absolute constant (Schmidt [64]). Moreover, the last result is best possible: there exist sequences $(x_k)_{k\geq 1}$ such that $N D_N \leq C \log N$ for all N. A simple example is the sequence $x_k = k\alpha$ where α is an irrational number with bounded partial quotients, see [50]. Important tools to get upper bounds for the discrepancy of a sequence $(x_k)_{k\geq 1}$ are the Erdős-Turán inequality

$$D_N(x_k) \leq 6 \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i h x_k} \right| \right), \qquad (N \geq 1, m \geq 1) \tag{1.1}$$

and LeVeque's inequality

$$D_N(x_k) \leq \left(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i h x_k} \right|^2 \right)^{1/3} \qquad (N \geq 1) \tag{1.2}$$

(we write $D_N(x_k)$ for $D_N(x_1, \dots, x_N)$). A lower bound, also expressed in terms of exponential sums, is

$$D_N(x_k) \geq C \frac{1}{N} \left| \sum_{k=1}^{N} e^{2\pi i x_k} \right|, \tag{1.3}$$

which is a special case of Koksma's inequality (see Kuipers and Niederreiter [50] or Drmota and Tichy [23]).

2 Asymptotics of $\langle n_k x\rangle$: a historical overview

Let $\langle x\rangle = x - [x]$ denote the fractional part of $x \in \mathbb{R}$. To simplify the formulations, in the sequel we consider only sequences $(x_n)_{n\geq 1}$ in $(0,1)$ and accordingly drop the terminology "mod 1". By Theorem 1.1 the sequence $(n_k x)_{k\geq 1}$ is uniformly distributed for every sequence $(n_k)_{k\geq 1}$ of distinct integers and for almost all x and and thus its discrepancy tends to 0. The first estimate for the convergence speed was given independently by Erdős and Koksma [25] and Cassels [20], who showed that for any sequence $(n_k)_{k\geq 1}$ of distinct integers and any $\varepsilon > 0$ we have

$$D_N(n_k x) = O\left(\frac{(\log N)^{5/2+\varepsilon}}{\sqrt{N}}\right) \qquad \text{a.e.} \tag{2.1}$$

Baker [10] improved the exponent 5/2 to 3/2 and Berkes and Philipp [15] constructed a sequence $(n_k)_{k\geq 1}$ such that for almost every x the inequality

$$D_N(n_k x) \geq C\frac{(\log N)^{1/2}}{\sqrt{N}} \tag{2.2}$$

holds for infinitely many N. These results describe fairly precisely how "large" $D_N(n_k x)$ can be, but in general the order of magnitude of $D_N(n_k x)$ is different from the bounds in (2.1) and (2.2) and to determine its exact asymptotics is a difficult problem which has been solved only in a few cases. In the case $n_k = k$ Khinchin [48] proved that for any nondecreasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ the relation

$$ND_N(n_k x) = O\left((\log N) g(\log\log N)\right) \qquad \text{a.e.} \tag{2.3}$$

holds if and only if $\sum_{n=1}^\infty g(n)^{-1}$ converges. In particular,

$$D_N(n_k x) = O\left(\frac{(\log N)(\log\log N)^{1+\varepsilon}}{N}\right) \qquad \text{a.e.} \tag{2.4}$$

for every $\varepsilon > 0$ and this fails for $\varepsilon = 0$. In the same case $n_k = k$, Kesten [46] showed that

$$D_N(n_k x) \sim \frac{2}{\pi^2}\frac{\log N \log\log N}{N} \qquad \text{in measure.} \tag{2.5}$$

For further refinements, see Schoissengeier [65]. Note that, in view of Khinchin's results, relation (2.5) does not hold if convergence in measure is replaced by almost everywhere convergence. This behavior is typical for the partial sums of independent random variables with infinite means, see e.g. Theorem C of the Appendix. The exact link with probability theory is provided by an observation of Ostrowski [54] who showed that the discrepancy of $(kx)_{k\geq 1}$ is closely related to the sum $\sum_{k=1}^N a_k(x)$ of continued fraction digits of x, which is a sum of nearly independent random variables with infinite expectation. Another case when the order of magnitude of $D_N(n_k x)$ is known precisely is the lacunary case, i.e. when the sequence $(n_k)_{k\geq 1}$ grows very rapidly. Philipp [56], [57] proved that if $(n_k)_{k\geq 1}$ satisfies the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1 \qquad (k = 1, 2, \dots) \tag{2.6}$$

then

$$\frac{1}{4\sqrt{2}} \le \limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} \le C(q) \qquad \text{a.e.} \tag{2.7}$$

where $C(q)$ is a constant depending only on q. Note that for a sequence $(\xi_k)_{k\ge 1}$ of independent, identically distributed (i.i.d.) nondegenerate random variables in $(0,1)$ we have by the Chung-Smirnov law of the iterated logarithm

$$\limsup_{N\to\infty} \frac{ND_N(\xi_k)}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad \text{a.s.} \tag{2.8}$$

(see the Appendix). Thus Philipp's theorem shows that for Hadamard lacunary $(n_k)_{k\ge 1}$ the sequence $(\langle n_k x\rangle)_{k\ge 1}$ behaves like a sequence of independent random variables. The analogy with probability theory, however, is not complete: the limsup in (2.7) is generally different from the constant $1/2$ in (2.8) and, as Fukuyama [31] showed, it depends sensitively on the number theoretic properties of n_k. For example, for $n_k = 2^k$ the limsup in (2.7) equals $\sqrt{42}/9$, while for $n_k = 4^k$ it is $\sqrt{10/27}$. If $n_k = \theta^k$ with a transcendental $\theta > 1$, the limsup in (2.7) equals $1/2$. For general $(n_k)_{k\ge 1}$ the limsup is unknown even today, see Section 3 for more information on this point. Fukuyama [32] also showed that the LIL for $D_N(n_k x)$ is not permutation-invariant: the limsup in (2.7) can change by permuting of the sequence $(n_k)_{k\ge 1}$. This means a strong deviation from i.i.d. behavior characterized by (2.8), since asymptotic properties of i.i.d. sequences are permutation-invariant. These remarks show that the behavior of $D_N(n_k x)$ is only partly explained by the probabilistic picture; as we will see, the properties of this sequence are determined by a complicated interplay of probabilistic, analytic and number theoretic factors which are not completely understood even today.

Note that

$$D_N(n_k x) = \sup_{f\in\mathcal{F}} \frac{1}{N} \left| \sum_{k=1}^{N} f(n_k x) \right|,$$

where $\mathcal{F}$ denotes the class of class of centered indicator functions $f = \mathbb{1}_{[a,b)} - (b-a)$, $0 \le a < b \le 1$. Thus an important prerequisite for understanding the behavior of $D_N(n_k x)$ in the lacunary case is to describe the behavior of sums $\sum_{k=1}^{N} f(n_k x)$ for periodic measurable functions f and it is natural to study first the simplest case of lacunary trigonometric series. Starting with the paper of Kolmogorov [49], the study of lacunary trigonometric series played a prominent role in the early years of modern probability theory. Salem and Zygmund [60] proved that if $(n_k)_{k\ge 1}$ satisfies the Hadamard gap condition (2.6) and (a_k) is a sequence of real numbers satisfying

$$a_N = o(A_N) \quad \text{with} \quad A_N = \frac{1}{2}\left(\sum_{k=1}^{N} a_k^2\right)^{1/2}, \tag{2.9}$$

then $(\cos 2\pi n_k x)_{k\ge 1}$ obeys the central limit theorem

$$\lim_{N\to\infty} \lambda\{x \in (0,1) : A_N^{-1} \sum_{k=1}^{N} a_k \cos 2\pi n_k x \le t\} = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2} du, \tag{2.10}$$

where λ denotes the Lebesgue measure. Under the slightly stronger coefficient condition

$$a_N = o(A_N/(\log\log A_N)^{1/2}) \tag{2.11}$$

Weiss [77] proved (cf. also Salem and Zygmund [61], Erdős and Gál [26]) that $(\cos 2\pi n_k x)_{k\geq 1}$ obeys the law of the iterated logarithm

$$\limsup_{N\to\infty} (2A_N^2 \log\log A_N)^{-1/2} \sum_{k=1}^{N} a_k \cos 2\pi n_k x = 1 \qquad \text{a.e.} \tag{2.12}$$

Comparing these results with the classical forms of the central limit theorem and law of the iterated logarithm (see Theorem A of the Appendix), we see that under the Hadamard gap condition the functions $\cos 2\pi n_k x$ behave exactly like independent random variables. Using deeper probabilistic tools, Philipp and Stout [59] proved that if for the coefficients (a_k) we assume the stronger condition $a_N = o(A_N^{1-\delta})$ for some $\delta > 0$, then on the probability space $([0,1], \mathcal{B}, \lambda)$ one can construct a Brownian motion process $\{W(t),\, t \geq 0\}$ such that

$$\sum_{k\leq N} \cos 2\pi n_k x = W(A_N) + O(A_N^{1/2-\rho}) \qquad \text{a.e.} \tag{2.13}$$

for some $\rho > 0$. The last relation implies not only the CLT and LIL for $(\cos 2\pi n_k x)_{k\geq 1}$, but a whole class of further limit theorems for independent random variables; for examples and discussion we refer to [59].

As was noted first by Erdős [24], the previous results become generally false if we weaken the Hadamard gap condition (2.6). However, Erdős showed that the CLT (2.10) remains valid with coefficients $a_k = 1$ under the subexponential gap condition

$$n_{k+1}/n_k \geq 1 + c_k/\sqrt{k}, \qquad c_k \to \infty \tag{2.14}$$

and this becomes false in the case $c_k = c$, $k = 1, 2, \ldots$. More generally, Takahashi [70], [71] proved that if a sequence $(n_k)_{k\geq 1}$ of integers satisfies

$$n_{k+1}/n_k \geq 1 + ck^{-\alpha}, \qquad 0 \leq \alpha \leq 1/2 \tag{2.15}$$

then for any sequence (a_k) satisfying

$$a_N = o(A_N N^{-\alpha}) \quad \text{with} \quad A_N = \frac{1}{2}\left(\sum_{k=1}^{N} a_k^2\right)^{1/2} \tag{2.16}$$

we have the CLT (2.10) and this becomes false if we replace the o by O in (2.16). He also proved the corresponding LIL (2.12) under a slightly stronger coefficient condition, see [72], [73]. For $\alpha = 0$ condition (2.16) reduces to (2.9) and for any $0 \leq \alpha < 1/2$ it is still satisfied by the sequence $a_k = 1$, $k = 1, 2, \ldots$, but as α approaches $1/2$, the class of sequences (a_k) satisfying (2.16) becomes gradually smaller and for $\alpha = 1/2$ even the unweighted CLT becomes false. This shows that under the subexponential gap condition (2.15) the behavior of the functions $\cos 2\pi n_k x$ still resembles that of independent random variables, but with increasing α the independence becomes gradually weaker and at $\alpha = 1/2$ it disappears completely.

For general periodic f, the asymptotic theory of series $\sum c_k f(n_k x)$ is very different (and much more difficult) than the theory of trigonometric series. For example, it is easy to see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \cos 2\pi k x = 0$$

for all $x \in (0,1)$, but the question for which periodic measurable f the relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(kx) = 0 \qquad \text{a.e.} \tag{2.17}$$

holds, is still open. Khinchin [47] conjectured that (2.17) holds for each f satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\, dx = 0, \tag{2.18}$$

a conjecture remaining open for almost 50 years until Marstrand [51] solved it in the negative. Bourgain [19] found sufficient conditions for (2.17) in terms of metric entropy, but even today we have no characterization of functions f satisfying (2.17). Similarly, there is no satisfactory convergence theory for series $\sum_{k=1}^{\infty} c_k f(kx)$; in particular it is unknown for which f the analogue of the Carleson convergence theorem holds. Even the lacunary case exhibits a number of surprising phenomena. Kac [43] proved that for a function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\, dx = 0, \qquad \mathrm{Var}_{[0,1]} f < \infty, \tag{2.19}$$

the central limit theorem

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} f(2^k x) \le t\sigma\sqrt{N} \right\} = \Phi(t), \qquad t \in \mathbb{R},$$

holds provided

$$\sigma^2 = \int_0^1 f^2(x) dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(x) f(2^k x) dx \neq 0.$$

The corresponding law of the iterated logarithm

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} f(2^k x)}{\sqrt{2N \log\log N}} = \sigma \quad \text{a.e.}$$

was proved by Maruyama [52] and Izumi [42]. However, Erdős and Fortet showed (see [44], p. 646) that if

$$f(x) = \cos 2\pi x + \cos 4\pi x, \;\; n_k = 2^k - 1 \tag{2.20}$$

then

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} f(n_k x) \le t\sqrt{N} \right\} = \frac{1}{\sqrt{2\pi}} \int_0^1 \int_{-\infty}^{t/\sqrt{2}|\cos \pi s|} e^{-u^2/2}\, du\, ds$$

and

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} f(n_k x)}{\sqrt{2N \log\log N}} = \sqrt{2} \cos \pi x \qquad \text{a.e.}$$

In other words, replacing 2^k by $2^k - 1$, the CLT and the LIL become false. Gaposhkin [37] showed that the CLT and LIL hold for $f(n_k x)$ if n_{k+1}/n_k is an integer for all k or if n_{k+1}/n_k converges to a number α such that α^r is irrational for $r = 1, 2, \ldots$. These results show that the asymptotic behavior of sums $\sum_{k=1}^{N} f(n_k x)$ depends not only on the growth speed of $(n_k)_{k\ge1}$, but also on the number theoretic properties of n_k. An important step to clear up this phenomenon was made by by Gaposhkin [39], who proved that under mild technical conditions on f, the sequence $(n_k)_{k\ge1}$ satisfies the central limit theorem

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} f(n_k x) \le t\sigma_N \right\} = \Phi(t), \qquad t \in \mathbb{R}.$$

with a suitable norming sequence σ_N provided the number of solutions of the Diophantine equation

$$an_k \pm bn_\ell = c \tag{2.21}$$

is bounded by a constant $K(a,b)$ for every fixed nonzero integers a, b, c. Note that in the case $n_k = 2^k - 1$ the equation

$$2n_k - n_\ell = 1 \tag{2.22}$$

has infinitely many solutions, and thus the Diophantine condition of Gaposhkin clearly fails. Also, the validity of Gaposhkin's criterion is easily checked in the examples above, and in a number of further cases as well, e.g. if $n_{k+1}/n_k \to \infty$, a case settled earlier by Takahashi [67], [69]. However, will see in Section 3 that Gaposhkin's condition is far from necessary and the central limit problem for $f(n_k x)$ remains open.

Just like in the trigonometric case, the probabilistic theory of $f(n_k x)$ extends partly to subexponentially growing $(n_k)_{k\ge1}$; for example, $f(n_k x)$ will satisfy the CLT and LIL for arithmetically "nice" sequences $(n_k)_{k\ge1}$. Let $(n_k)_{k\ge1}$ be a so-called Hardy-Littlewood-Pólya sequence, i.e. let $(n_k)_{k\ge1}$ consist of the elements of the multiplicative semigroup generated by a finite set $(q_1, \ldots, q_r)$ of coprime integers, arranged in increasing order. Philipp [58] proved that for such sequences we have

$$\frac{1}{4\sqrt{2}} \le \limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} \le C \qquad \text{a.e.} \tag{2.23}$$

where C is a positive constant, depending only on the number τ of primes occurring in the prime factorization of $q_1, \ldots, q_r$. From this it follows easily that under (2.19) we have

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} f(n_k x)\right|}{\sqrt{2N \log\log N}} \le C \qquad \text{a.e.} \tag{2.24}$$

for some constant $C > 0$; the exact value of the limsup was determined by Fukuyama and Nakata [36]. Another case when the asymptotic behavior of $f(n_k x)$ is known precisely is when f is sufficiently smooth. Dhompongsa [22] and Takahashi [74] proved that if f satisfies (2.18) and $f \in \text{Lip}(\frac{1}{2} + \varepsilon)$ for some $\varepsilon > 0$, then the LIL (2.24) remains valid under the subexponential gap condition

$$n_{k+1}/n_k \geq 1 + ck^{-\alpha}, \qquad 0 < \alpha < 1/2.$$

Gaposhkin [38] proved that under $f \in \text{Lip}(\frac{1}{2} + \varepsilon)$, Carleson's theorem holds for the system $f(nx)$, i.e. $\sum_{k=1}^{\infty} c_k f(kx)$ converges a.e. provided $\sum_{k=1}^{\infty} c_k^2 < \infty$. Berkes [14] proved that both results become false for $f \in \text{Lip}\,(\frac{1}{2})$ and the behavior becomes even more pathological for $f \in \text{Lip}\,(\frac{1}{2} - \varepsilon)$: there exist sequences $(n_k)_{k \geq 1}$ arbitrary close to the exponential speed such that

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} a_k f(n_k x)}{\sqrt{N} (\log N)^{\rho}} = +\infty \qquad \text{a.e.}$$

for some constant $\rho > 0$ and a sequence $a_k = \pm 1$. These results show the extreme sensibility of the asymptotic behavior of $f(n_k x)$ on the smoothness properties of f and the critical role of the Lip $(\frac{1}{2})$ class. Note that the structure of the Fourier series of f plays also a crucial role: while the convergence behavior of $\sum_{k=1}^{\infty} c_k f(kx)$ remains unknown in general, precise convergence criteria can be given if all frequencies m_k in the Fourier series

$$f \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi m_k x + b_k \sin 2\pi m_k x)$$

are lacunary (Gaposhkin [38]) or if they are coprime (Berkes [14]).

3 Some new results

As we have seen above, for rapidly increasing $(n_k)_{k \geq 1}$ the behavior of the sequence $(\langle n_k x \rangle)_{k \geq 1}$ resembles that of independent random variables, but the probabilistic analogy is not complete, and the finer properties of sums $\sum c_k f(n_k x)$ and the discrepancy $D_N(n_k x)$ are determined by a combination of probabilistic, analytic and number theoretic factors. In recent years there has been a considerable interest in this field, leading to the solution of a number of old classical problems in the field, and also leading to considerable new information and a better insight to the nature of lacunary sequences. The purpose of this section is to review some recent results in this field, obtained in Aistleitner [1]–[5], Aistleitner and Berkes [6], [7], Aistleitner, Berkes and Tichy [8], [9] and Berkes, Philipp and Tichy [16].

As we discussed in Section 2, the asymptotic behavior of sums $\sum_{k=1}^{N} f(n_k x)$ for periodic measurable functions f depends sensitively on the number theoretic properties of the sequence $(n_k)_{k \geq 1}$. For example, the central limit theorem holds for such sums if $n_k = 2^k$, or more generally, if the fractions n_{k+1}/n_k are integers for all k and also if $n_{k+1}/n_k \to \alpha$, where α^r is irrational for $r = 1, 2, \ldots$, but it fails if $n_k = 2^k - 1$. Gaposhkin [37] proved that $f(n_k x)$ obeys the CLT if for any fixed nonzero integers a, b, c the number of solutions (k, ℓ) of the Diophantine equation

$$an_k - bn_\ell = c$$

is bounded by a constant $K(a, b)$, independent of c. This criterion contains all the above examples and many other interesting cases as well, but as the following example shows, it is not necessary.

Example. Let $m_k = k^2$ and let the sequence $(n_k)_{k\geq 1}$ consist of the numbers $2^{m_k} - 1$, $k = 1, 2, \ldots$, plus the numbers $2^{m_k+1} - 1$ for the indices k of the form $k = [n^\alpha]$, $\alpha > 2$. Let f be a periodic Lipschitz function with mean 0 and $\|f\|_2 = 1$. By a result of Takahashi [67], the central limit theorem holds for $f(m_j x)$ with the norming sequence $\sqrt{N}$. Clearly $\sum_{k=1}^N f(n_k x) = \sum_{j=1}^M f(m_j x) + O(N^{1/\alpha})$ where $N - 2N^{1/\alpha} \leq M \leq N$ for $N \geq N_0$, which implies that $f(n_k x)$ also satisfies the CLT. On the other hand, for infinitely many ℓ we have $n_\ell = 2^{m_k} - 1$, $n_{\ell+1} = 2^{m_k+1} - 1$ for some k and thus $n_{\ell+1} - 2n_\ell = 1$. The number of such ℓ's up to N is $\sim N^{1/\alpha}$ and thus the equation $2n_i - n_j = 1$ has at least $cN^{1/\alpha}$ solutions for the indices $1 \leq i, j \leq N$.

Let

$$L(N, d, \nu) = \#\{1 \leq a, b \leq d,\ 1 \leq k, \ell \leq N : an_k - bn_\ell = \nu\} \tag{3.1}$$

and

$$L(N, d) = \sup_{\nu > 0} L(N, d, \nu),$$

where d is a positive integer. The following theorem solves the central limit problem for $f(n_k x)$ completely, thereby closing a long line of research starting with the classical paper of Kac [43].

Theorem 3.1 (see [7]) *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition (2.6) and let $f(x)$ be a function satisfying (2.19). Assume that*

$$\sigma_N^2 := \int_0^1 \left(\sum_{k=1}^N f(n_k x) \right)^2 dx \geq CN \tag{3.2}$$

for a positive constant $C > 0$ and

$$L(N, d) = o(N) \qquad \text{as } N \to \infty \tag{3.3}$$

for any fixed $d \geq 1$. Then

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^N f(n_k x) \leq t\sigma_N \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du. \tag{3.4}$$

If f is a trigonometric polynomial of order r, it suffices to assume the condition $L(N, d) = o(N)$ for $d = r$.

Here the Diophantine condition (3.3) is best possible: for any $\varepsilon > 0$ there exists an integer $d \geq 1$, a trigonometric polynomial f of order d and a lacunary sequence $(n_k)_{k\geq 1}$ such that $L(N,d) \leq \varepsilon N$ for sufficiently large N and the central limit theorem fails for $f(n_k x)$. Condition (3.2) cannot be omitted, as is shown by the example $f(x) = \cos 2\pi x - \cos 4\pi x,\ n_k = 2^k$, for which the Diophantine condition of Theorem 3.1 is satisfied, but the CLT is not.

Of special interest is the following case, where we can calculate the asymptotic variance σ_N^2 explicitly. Slightly modifying the definition of $L(N,d)$ in (3.1), let

$$L^*(N,d) = \sup_{\nu \geq 0} L(N,d,\nu).$$

For $\nu = 0$ we exclude the trivial solutions $a = b,\ k = \ell$ from $L(N,d,\nu)$. Put also $\|f\|_2 = (\int_0^1 f^2(x)\,dx)^{1/2}$.

Theorem 3.2 (see [7]) *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition (2.6) and let $f(x)$ be a function of bounded variation satisfying (2.19) and $\|f\|_2 > 0$. Assume that for any fixed $d \geq 1$, $L^*(N,d) = o(N)$ as $N \to \infty$. Then*

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} f(n_k x) \leq t\|f\|_2\sqrt{N} \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du, \tag{3.5}$$

i.e. the central limit theorem (3.4) holds with $\sigma_N = \|f\|_2\sqrt{N}$.

A corresponding LIL is given by the following result.

Theorem 3.3 (see [4]) *Assume the conditions of Theorem 3.2 with the Diophantine condition $L^*(N,d) = o(N)$ replaced by*

$$L^*(N,d) = O(N/(\log N)^{1+\varepsilon})$$

for some $\varepsilon > 0$. Then we have

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} f(n_k x)}{\sqrt{2N \log\log N}} = \|f\|_2 \qquad a.e.$$

One paradoxical feature of lacunary behavior is the lack of permutational invariance. As the previous results show, for a periodic measurable function f and rapidly growing $(n_k)_{k\geq 1}$, the sequence $f(n_k x)$ behaves like a sequence of i.i.d. random variables, in fact it satisfies many classical asymptotic properties of such sequences. However, while an i.i.d. sequence remains i.i.d. after any permutation of its terms, and thus its asymptotic properties are also permutation-invariant, the properties of lacunary sequences $f(n_k x)$ change by permutation, as was first noted by Fukuyama [32]. As the following theorem shows, permutation-invariance is also closely connected with number theoretical factors, in fact, we give a necessary and sufficient Diophantine condition for the permutation-invariance of the CLT and LIL for $f(n_k x)$.

Theorem 3.4 (see [9]) *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition (2.6), let f satisfy (2.19) and let σ be a permutation of $\mathbb{N}$. Assume the Diophantine condition*

$$L(N,d) = O(1) \quad \text{as } N \to \infty. \tag{3.6}$$

Then $N^{-1/2}\sum_{k=1}^N f(n_{\sigma(k)}x)$ has a limit distribution iff

$$\gamma = \lim_{N\to\infty} N^{-1}\int_0^1 \left(\sum_{k=1}^n f(n_{\sigma(k)}x)\right)^2 dx$$

exists, and then

$$N^{-1/2}\sum_{k=1}^N f(n_{\sigma(k)}x) \to_d N(0,\gamma). \tag{3.7}$$

If condition (3.6) fails, there exists a permutation σ such that the normed partial sums in (3.7) have a nongaussian limit distribution.

Theorem 3.5 (see [9]) *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition (2.6) and the Diophantine condition (3.6). Let f be a measurable function satisfying (2.19), let σ be a permutation of $\mathbb{N}$ and assume that*

$$\gamma = \lim_{N\to\infty} N^{-1}\int_0^1 \left(\sum_{k=1}^n f(n_{\sigma(k)}x)\right)^2 dx \tag{3.8}$$

for some $\gamma \geq 0$. Then we have

$$\limsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}}\sum_{k=1}^N f(n_{\sigma(k)}x) = \gamma^{1/2} \quad \text{a.e.} \tag{3.9}$$

For analogous results for Hardy-Littlewood-Pólya sequences, see [8].

By the classical LIL of Philipp [56], [57], under the Hadamard gap condition the discrepancy $D_N(n_kx)$ satisfies

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N\to\infty} \frac{ND_N(n_kx)}{\sqrt{2N\log\log N}} \leq C(q) \qquad \text{a.e.} \tag{3.10}$$

where $C(q)$ is a constant depending only on q. However, the value of the limsup in (3.10) remained unknown until very recently, when Fukuyama [31] has been able to compute it for $n_k = \theta^k$, $\theta > 1$. He proved that

$$\limsup_{N\to\infty} \frac{ND_N(\theta^kx)}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty} \frac{ND_N^*(\theta^kx)}{\sqrt{2N\log\log N}} = \Sigma_\theta \qquad \text{a.e.,} \tag{3.11}$$

where

$$\begin{aligned}
\Sigma_\theta &= 1/2 \quad \text{if } \theta^r \text{is irrational for all } r \in \mathbb{N},\\
\Sigma_\theta &= \sqrt{42}/9, \quad \text{if } \theta = 2\\
\Sigma_\theta &= \frac{\sqrt{(\theta+1)\theta(\theta-2)}}{2\sqrt{(\theta-1)^3}} \quad \text{if } \theta \geq 4 \text{ is an even integer,}\\
\Sigma_\theta &= \frac{\sqrt{\theta+1}}{2\sqrt{\theta-1}} \quad \text{if } \theta \geq 3 \text{ is an odd integer.}
\end{aligned}$$

Note that in the case when θ^r is irrational for all $r \in \mathbb{N}$, the value of the $\limsup$ is exactly the same as for i.i.d. random variables. Fukuyama and Nakata [36] also calculated the exact value of the $\limsup$ for a large class of Hardy-Littlewood-Pólya-sequences.

The next theorem determines the value of the limsup for a large class of lacunary sequences $(n_k)_{k\geq 1}$ of integers, and also shows that under a condition only slightly stronger than the condition $L^*(N,d) = o(N)$ in Theorem 3.2, the limsup in (3.10) equals 1/2, i.e. the constant in the Chung-Smirnov LIL for the discrepancy of i.i.d. random variables.

Theorem 3.6 (see [4]) *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition, and assume that for any fixed $d \geq 1$ there exists an $\varepsilon > 0$ such that*

$$L^*(N,d) = \mathcal{O}\left(\frac{N}{(\log N)^{1+\varepsilon}}\right) \quad \textit{as} \quad N \to \infty. \tag{3.12}$$

Then

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} = \limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N \log\log N}} = \frac{1}{2} \qquad \text{a.e.} \tag{3.13}$$

It is natural to expect that the Diophantine condition (3.12) in Theorem 3.6 can be replaced by $L^*(N,d) = o(N)$, but this remains open. However, the log factor causes no problems in applications; in particular, Theorem 3.6 covers all classical examples where the CLT has been established for $f(n_k x)$ with the norming $\|f\|_2\sqrt{N}$. For example, it is easy to see that condition (3.12) is satisfied if $n_{k+1}/n_k \to \theta$, where θ^r is irrational for all $r \in \mathbb{N}$, or if $\lim_{k\to\infty} n_{k+1}/n_k = \infty$. Note also that in (3.12) $L^*(N,d)$ cannot be replaced by $L(N,d)$: for example, in the case $n_k = 2^k$ we have $L(N,d) = O(1)$ (and thus the CLT holds for $f(n_k x)$), but $L^*(N,d) \geq N-1$ and the limsup in (3.13) equals $\sqrt{42}/9$ by Fukuyama's theorem.

In all the cases covered by the above results, the $\limsup$ is constant a.e., and it is the same for D_N as for D_N^*. Answering an old question of Philipp [56], Aistleitner [2], [3] showed that in general this is not the case. Let $(n_k)_{k\geq 1}$ be defined by

$$\left.\begin{array}{ll} n_{2k-1} & = 2^{k^2} \\ n_{2k} & = 2^{k^2+1} - 1 \end{array}\right\} \quad k \geq 1. \tag{3.14}$$

Then $(n_k) = (2, 3, 16, 31, 512, 1023, \dots)$. Obviously this sequence is lacunary, since

$$n_{k+1}/n_k \geq 3/2, \quad k \geq 1.$$

For this sequence we have

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} \quad = \quad \frac{3}{4\sqrt{2}} \quad \text{a.e.}$$

and

$$\limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N \log\log N}} \quad = \quad \Psi^*(x) \quad \text{a.e.,}$$

where

$$\Psi^*(x) = \begin{cases} \dfrac{3}{4\sqrt{2}}, & 0 \le x \le 3/8,\ 5/8 \le x \le 1 \\ \dfrac{\sqrt{(4x(1-x)-x)}}{\sqrt{2}}, & 3/8 \le x \le 1/2 \\ \dfrac{\sqrt{(4x(1-x)-(1-x))}}{\sqrt{2}}, & 1/2 \le x \le 5/8, \end{cases}$$

see [2]. Note in particular that the limsup for $D_N^*(n_k x)$ is not a constant a.e. If the sequence $(n_k)_{k\ge1}$ in (3.14) is replaced by

$$n_k = \begin{cases} 2^{k^2} & \text{for} \quad k \equiv 1 \mod 4 \\ 2^{(k-1)^2+1} - 1 & \text{for} \quad k \equiv 2 \mod 4 \\ 2^{k^2+k} & \text{for} \quad k \equiv 3 \mod 4 \\ 2^{(k-1)^2+(k-1)+1} - 2 & \text{for} \quad k \equiv 0 \mod 4, \end{cases}$$

then also the value of the lim sup in the LIL for $D_N(n_k x)$ is not equal to a constant a.e. We note that independently and at the same time, Fukuyama [35] constructed a sequence $(n_k)_{k\ge1}$ with bounded gaps $n_{k+1} - n_k$ exhibiting the same pathological behavior.

Most results in the theory of uniform distribution and discrepancy extend for sequences with values in $\mathbb{R}^d$, although usually there is a price in accuracy to pay for the high dimensional result. In contrast, there are very few results on the discrepancy of sequences with multidimensional indices, even though the corresponding problem, namely the uniform asymptotic behavior of random fields, has been extensively studied in probability theory. In view of this fact, it is of considerable interest to study the multiparameter version of Philipp's theorem, one of the basic results in metric discrepancy theory.

Let $\mathbb{N}^d$ denote the set of d-dimensional vectors with positive integer components and let $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ be a sequence of integers with d-dimensional indices. Letting $\mathbf{k} = (k_1, \dots, k_d)$ and $\mathbf{k}' = (k_1', \dots, k_d')$, we say that $\mathbf{k} \le \mathbf{k}'$ if $k_i \le k_i'$, $1 \le i \le d$ and $\mathbf{k} < \mathbf{k}'$ if $\mathbf{k} \le \mathbf{k}'$ and $\mathbf{k} \ne \mathbf{k}'$. We say that $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is nondecreasing if $n_{\mathbf{k}} \le n_{\mathbf{k}'}$ provided $\mathbf{k} \le \mathbf{k}'$. Let $\mathbf{1}$ denote the d-dimensional vector $(1, \dots, 1)$ and for $\mathbf{N} = (N_1, \dots, N_d)$ we set $|\mathbf{N}| = \prod_{i=1}^d N_i$. The discrepancy $D_{\mathbf{N}}(n_{\mathbf{k}} x)$ of the sequence $(n_{\mathbf{k}} x)_{\mathbf{k}\ge\mathbf{1}}$ is defined, similarly to the one-parameter case, as

$$D_{\mathbf{N}}(n_{\mathbf{k}} x) = \sup_{0\le a<b\le1} \left| \frac{\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} \mathbb{1}_{[a,b)}(n_{\mathbf{k}} x)}{|\mathbf{N}|} - (b-a) \right|,$$

where $\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} = \sum_{\mathbf{k}:\ \mathbf{1}\le\mathbf{k}\le\mathbf{N}}$.

Theorem 3.7 (see [5]) *Let $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ be a nondecreasing sequence of positive integers for which*

$$\#\{\mathbf{k} \in \mathbb{N}^d :\ 2^r \le n_{\mathbf{k}} < 2^{r+1}\} \le Q, \qquad r = 0, 1, 2, \dots \tag{3.15}$$

with a constant Q. Then

$$\limsup_{|\mathbf{N}|\to\infty} \frac{|\mathbf{N}|D_{\mathbf{N}}(n_{\mathbf{k}}x)}{\sqrt{|\mathbf{N}|\log\log|\mathbf{N}|}} \le C_{Q,d} \qquad \text{a.e.,} \tag{3.16}$$

where $C_{Q,d}$ is a positive number depending on Q and d.

Note that a one-dimensional sequence $(n_k)_{k\ge1}$ satisfies (3.15) if and only if it is a finite union of sequences satisfying the Hadamard gap condition (2.6), so in this case Theorem 3.7 yields Philipp's LIL (3.10). It would be tempting to define the Hadamard gap condition for $(n_k)_{k\in\mathbb{N}^d}$ by requiring that

$$n_{\mathbf{k}'}/n_{\mathbf{k}} > q > 1, \qquad \mathbf{k}' > \mathbf{k}. \tag{3.17}$$

However, with this definition Theorem 3.7 fails. In [5] it is shown that there exist sequences satisfying (3.17) such that for almost all $x \in (0,1)$

$$|\mathbf{N}|D_{\mathbf{N}}(n_{\mathbf{k}}x) \ge C|\mathbf{N}|^{3/4}$$

holds for infinitely many $\mathbf{N}$.

As we noted earlier, weakening the Hadamard gap condition (2.6), Philipp's LIL (3.10) for the discrepancy of $(n_k x)_{k\ge1}$ becomes generally false. The following theorem yields upper bounds for $D_N(n_k x)$ in the subexponential domain. Set

$$a_{N,r} = \#\{k \le N : \; n_k \in [2^r, 2^{r+1})\}, \qquad r \ge 0, \; N \ge 1$$

and

$$B_N = \left(\sum_{r=0}^{\infty} a_{N,r}^2\right)^{1/2}, \qquad N \ge 1. \tag{3.18}$$

For fast growing sequences, B_N will be "small": for example, for a lacunary sequence $(n_k)_{k\ge1}$ the value of B_N will be between $\sqrt{N}$ and $C\sqrt{N}$. On the other hand, for very slowly growing sequences the value of B_N will be close to N.

Theorem 3.8 (see [6]) *Let $(n_k)_{k\ge1}$ be a nondecreasing sequence of positive integers satisfying*

$$a_{N,r} = \mathcal{O}\left(\frac{B_N}{(\log N)^{\alpha}}\right) \tag{3.19}$$

for some constant $\alpha > 3$, uniformly for $r \in \mathbb{N}$. Then

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{B_N^2 \log\log N}} \le C \quad \text{a.e.,} \tag{3.20}$$

where C is a positive constant.

Condition (3.19) means that breaking the set $\{n_k, 1 \le k \le N\}$ into its parts in the intervals $[2^r, 2^{r+1})$, $r = 1, 2, \ldots$, their cardinalities $a_{N,r}$ are much smaller than $B_N = (\sum_{r=0}^{\infty} a_{N,r}^2)^{1/2}$. Such "uniform negligibility" conditions are typical in probability theory, in particular in the theory of the central limit theorem and the law of the iterated logarithm (see the Appendix). Specifically, writing

$$X_r = \sum_{n_k \in [2^r, 2^{r+1})} \mathbb{1}_{[a,b)}(n_k x),$$

condition (3.19) is a variant of the classical Kolmogorov condition for the LIL for independent random variables. Thus Theorem 3.8 generalizes the classical heuristics that for rapidly growing $(n_k)_{k\ge1}$, the functions $\langle n_k x\rangle$ "almost" behave like i.i.d. random variables.

Note that in Theorem 3.8 we made no number theoretic assumptions on $(n_k)_{k\ge1}$ and in this sense it is the exact sublacunary counterpart of Philipp's LIL (3.10). Similarly to (3.10), the limsup in (3.20) remains undetermined. In Theorem 3.6 we computed the precise constant in Philipp's LIL under the assumption that the number of solutions of the Diophantine equation

$$a n_k - b n_\ell = c, \qquad 1 \le k, \ell \le N$$

is "not too large" compared with N. The next theorem gives a similar result in the sub-lacunary case, although the Diophantine conditions have to be slightly stronger. Let $(n_k)_{k\ge1}$ be an increasing sequence of positive integers and set

$$a_r = \#\{j : n_j \in [2^r, 2^{r+1})\}, \qquad r = 0, 1, \ldots$$

We say that $(n_k)_{k\ge1}$ satisfies

Condition ($\mathbf{K}_\alpha$), $0 \le \alpha < 1$, if there exists a constant $C_\alpha > 0$ such that

$$a_N \le C_\alpha \left(\sum_{0 \le j < N} a_j \right)^\alpha, \qquad N \ge 1.$$

Condition ($\mathbf{D}_\delta$), $0 \le \delta < 1$, if there exists a constant C_δ such that for every $N \ge 1$ and for fixed integers a, b with $0 < |a|, |b| \le N^2$ the number of solutions (k, ℓ) of the Diophantine equation

$$a n_k - b n_\ell = c, \qquad 1 \le k, \ell \le N$$

does not exceed $C_\delta N^\delta$, uniformly for all $c \in \mathbb{Z},\ c \ne 0$.

Condition ($\mathbf{D}_\gamma^{\mathbf{0}}$), $0 \le \gamma < 1$, if there exists a constant C_γ such that for every $N \ge 1$ and for fixed integers a, b with $0 < |a|, |b| \le N^2$, the number of solutions (k, ℓ) of the Diophantine equation

$$a n_k - b n_\ell = 0, \qquad 1 \le k, \ell \le N, \quad (a, k) \ne (b, \ell)$$

does not exceed $C_\gamma N^\gamma$.

Condition $(\mathbf{K}_\alpha)$ is an asymptotic negligibility condition, comparable to (3.19) in Theorem 3.8. Condition $(\mathbf{D}_\delta)$ and condition $(\mathbf{D}^\mathbf{0}_\gamma)$ are Diophantine conditions, comparable to the conditions on $L(N,d)$ and $L^*(N,d)$ in Theorem 3.1, Theorem 3.2 and Theorem 3.6. The main difference between the conditions there and conditions $(\mathbf{D}_\delta)$ and $(\mathbf{D}^\mathbf{0}_\gamma)$ is that here we have numbers δ and γ in the exponent of N, and that the size of the coefficients a,b increases as N goes to infinity. Condition $(\mathbf{D}_\delta)$, similar to a bound on $L(N,d)$, ensures that the fluctuation behavior of the system is not too wild, and condition $(\mathbf{D}^\mathbf{0}_\gamma)$ controls the asymptotic variance σ_N^2.

Theorem 3.9 (see [1]) *Let f be a function of bounded variation satisfying (2.19) and assume that (3.2) holds. Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying conditions $(\mathbf{K}_\alpha)$ and $(\mathbf{D}_\delta)$with*

$$\alpha + \delta < 1. \tag{3.21}$$

Let $S_N = \sum_{k=1}^N f(n_k x)$. Then the sequence $(S_N)_{N\geq 1}$ can be redefined on a new probability space (without changing its distribution) together with a Wiener process $W(t)$ such that

$$S_N = W(\sigma_N^2) + o(N^{1/2-\lambda}) \qquad \text{a.s.},$$

where $\lambda > 0$ depends on α and δ.

Theorem 3.10 (see [1]) *Let f be a function of bounded variation satisfying (2.19) and $\|f\|_2 > 0$ and let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying conditions $(\mathbf{K}_\alpha)$ and $(\mathbf{D}_\delta)$with*

$$\alpha + \delta < 1.$$

Assume that $(n_k)_{k\geq 1}$ additionally satisfies condition $(\mathbf{D}^\mathbf{0}_\gamma)$ for $\gamma < 1$. Then, letting $S_N = \sum_{k=1}^N f(n_k x)$, the sequence $(S_N)_{N\geq 1}$ can be redefined on a new probability space (without changing its distribution) together with a Wiener process $W(t)$ such that

$$S_N = W(\tau N) + o(N^{1/2-\lambda}) \qquad \text{a.s.},$$

where $\tau = \|f\|_2^2$ and $\lambda > 0$ depends on α, δ and γ.

Theorem 3.11 (see [1]) *Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying conditions $(\mathbf{K}_\alpha)$ and $(\mathbf{D}_\delta)$, for $\alpha + \delta < 1$, and condition $(\mathbf{D}^\mathbf{0}_\gamma)$ for $\gamma < 1$. Then*

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad \text{a.e.}$$

In the language of probability theory, Theorem 3.9 and 3.10 are *almost sure invariance principles*, showing that under certain conditions $\sum_{k=1}^N f(n_k x)$ can be approximated by a Wiener process. Of course, these a.s. invariance principles imply the central limit theorem and the law of the iterated logarithm for $\sum_{k=1}^N f(n_k x)$. Condition (3.21) connects the density and the Diophantine behavior of $(n_k)_{k\geq 1}$: if the sequence is very dense, the Diophantine condition has to be strong. If the sequence is sparse (e.g. if it is growing very rapidly), then we need weaker Diophantine conditions, in accordance with the results for lacunary $(n_k)_{k\geq 1}$ in [4] and [7].

Finally we note that in Berkes, Philipp and Tichy [16] the asymptotic behavior of the function

$$G_N(t) = G_N(t,x) = N^{-1} \sum_{k=1}^{N} \left(\mathbb{1}_{[0,t)}(n_k x) - t \right)$$

was investigated. Under Diophantine conditions on $(n_k)_{k\geq 1}$, functional versions of the LIL for $D_N^*(n_k x)$ were proved.

4 Random frequencies

As we have seen, the strong independence properties of lacunary trigonometric series under the Hadamard gap condition (2.6) remain partially valid under the subexponential gap condition

$$n_{k+1}/n_k \geq 1 + ck^{-\alpha}, \qquad 0 < \alpha < 1/2$$

but they become gradually weaker as α approaches 1/2 and they disappear at $\alpha = 1/2$. Near the critical gap condition

$$n_{k+1}/n_k \geq 1 + ck^{-1/2} \tag{4.1}$$

the behavior of $(\cos 2\pi n_k x)_{k\geq 1}$, $(\sin 2\pi n_k x)_{k\geq 1}$ becomes very complicated and exhibits a number of unusual phenomena. For example, near (4.1) it can happen that $(\cos 2\pi n_k x)_{k\geq 1}$ satisfies the CLT, but fails the LIL, or that $(\cos 2\pi n_k x)_{k\geq 1}$ satisfies the LIL, but fails other classical asymptotic theorems for i.i.d. random variables (see Berkes [13]). For sequences $(n_k)_{k\geq 1}$ growing slower than the speed required by (4.1), the normed partial sums $N^{-1/2} \sum_{k=1}^N \cos 2\pi n_k x$ can have nongaussian limit distributions or no limit distribution at all and all general patters observed in the Hadamard lacunary case disappear: the asymptotic properties of the sums $\sum_{k=1}^N \cos 2\pi n_k x$, $\sum_{k=1}^N \sin 2\pi n_k x$ and $\sum_{k=1}^N e^{2\pi i n_k x}$ will depend strongly on the sequence $(n_k)_{k\geq 1}$, leading to difficult analytic problems for any individual $(n_k)_{k\geq 1}$. Noting that

$$\int_0^1 \left(\sum_{k=1}^N \cos 2\pi n_k x \right)^p dx = 2^{-(p-1)} A_{N,p}$$

where $A_{N,p}$ denotes the number of solutions $(k_1, \ldots, k_p)$ of the Diophantine equation

$$\pm n_{k_1} \pm \cdots \pm n_{k_p} = 0, \qquad (1 \leq k_1, \ldots, k_p \leq N), \tag{4.2}$$

we see that the behavior of $\sum_{k=1}^N \cos 2\pi n_k x$ is still related to Diophantine equations. However, for slowly increasing sequences $(n_k)_{k\geq 1}$ counting the number of solutions of (4.2) is a very difficult combinatorial problem which has been solved only for a few special sequences $(n_k)_{k\geq 1}$. In analytic number theory, one typically follows an inverse path: from analytic estimates for exponential sums $\sum_{k=1}^N e^{2\pi i n_k x}$ one draws conclusions for the number of solutions of (4.2). One completely solved case is $n_k = k^2$ when Walfisz [75] and Fiedler, Jurkat and Körner [29] proved, using elliptic function theory, that

$$\sum_{k=1}^{N} \cos 2\pi n_k x = O\left(\sqrt{N}(\log N)^{1/4+\varepsilon}\right) \qquad \text{a.e.} \tag{4.3}$$

and this becomes false for $\varepsilon = 0$. There are partial results for $n_k = k^r$, $r = 3, 4, \ldots$ and other special sequences $(n_k)_{k\geq 1}$ like the sequence of primes (see e.g. [41]) but no precise asymptotic results are known.

While dealing with the Diophantine equation (4.2) is extremely difficult for "concrete" sequences $(n_k)_{k\geq 1}$, sharp results exist for random sequences $(n_k)_{k\geq 1}$ (see e.g. Halberstam and Roth [40]) and choosing the frequencies n_k at random provides a remarkable insight into the behavior of lacunary trigonometric sums and sums of the form $\sum c_k f(n_k x)$ for slowly increasing $(n_k)_{k\geq 1}$. The simplest way to construct an infinite random subset H of the positive integers is using "head or tail", i.e. to decide for each k if k belongs to the set H by flipping a fair penny. Denoting the so obtained random sequence by $(n_k)_{k\geq 1}$, from the results of Salem and Zygmund [62] for randomly signed trigonometric series $\sum \pm \cos nx$ it follows that

$$\lim_{N\to\infty} \lambda\{x \in (0,1) : \sum_{k=1}^{N} \cos 2\pi n_k x \leq t\sqrt{N/2}\} = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2} du, \tag{4.4}$$

and

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k=1}^{N} \cos 2\pi n_k x = 1 \qquad \text{a.e.} \tag{4.5}$$

with probability one. By the strong law of large numbers we have $n_k \sim 2k$ and thus we see that even though the validity of the classical probabilistic theory of lacunary trigonometric series breaks down at the subexponential gap condition (4.1), there exist linearly growing sequences $(n_k)_{k\geq 1}$ satisfying the CLT and LIL (4.4) and (4.5). Concerning the gaps $n_{k+1} - n_k$ of the so constructed random $(n_k)_{k\geq 1}$, from the classical "pure heads" theorem of probability theory (see e.g. Erdős and Rényi [27]) it follows that $\limsup_{k\to\infty}(n_{k+1} - n_k)/\log_2 k = 1$ a.s., where $\log_2$ denotes logarithm with base 2. Using a different random construction, Berkes [12] proved that for any sequence $\omega(k) \to \infty$ there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that

$$n_{k+1} - n_k = O(\omega(k))$$

and still (4.4) and (4.5) are valid. This raises the question if there exists a sequence $(n_k)_{k\geq 1}$ with bounded gaps, i.e. a sequence with

$$n_{k+1} - n_k \leq L, \qquad (k = 1, 2, \ldots) \tag{4.6}$$

for some L such that the CLT and LIL holds. Bobkov and Götze [18] proved that if (4.6) holds, then

$$N^{-1/2} \sum_{k=1}^{N} \cos 2\pi n_k x \to_d \mathcal{N}(0, \sigma^2) \tag{4.7}$$

cannot hold with $\sigma^2 = 1/2$, and in fact with any $\sigma^2 > 1/2 - 1/(2L)$. However, using a delicate random construction, Fukuyama [33] showed that for any $\sigma^2 < 1/2$ there exists a sequence $(n_k)_{k\geq 1}$ with bounded gaps such that (4.7) holds, completing the CLT theory for small gaps. Fukuyama [34] also showed that there exist sequences $(n_k)_{k\geq 1}$ with bounded gaps such that the exact LIL

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad \text{a.e.} \tag{4.8}$$

holds, a result even more surprising since here, unlike in the case of the CLT, we get exactly the standard i.i.d. result for trigonometric sequences $(n_k)_{k\geq 1}$ with bounded gaps. The explanation of the nice probabilistic behavior of $f(n_k x)$ for small gaps is not the Diophantine behavior of $(n_k)_{k\geq 1}$ (since for such dense $(n_k)_{k\geq 1}$ the number of solutions of the Diophantine equation (4.2) is huge), but the randomness of the $(n_k)_{k\geq 1}$. It would be nice to give nonrandom constructions having the same effect or to exhibit a property of slowly growing sequences $(n_k)_{k\geq 1}$ implying the near independent behavior of $\cos 2\pi n_k x$ or $f(n_k x)$.

Random constructions give substantial new information also for rapidly increasing $(n_k)_{k\geq 1}$. As we noted, the CLT (4.4) holds under the subexponential gap condition

$$n_{k+1}/n_k \geq 1 + c_k/\sqrt{k}, \qquad c_k \to \infty$$

but in general not if $c_k = c$, $k = 1, 2, \ldots$. In particular, the CLT holds if $n_k = e^{\omega_k \sqrt{k}}$ with $\omega_k \to \infty$, but the case $n_k = e^{c\sqrt{k}}$ for fixed $c > 0$ remains open. Erdős [24] conjectured that CLT holds if $n_k = e^{ck^\alpha}$ for any $0 < \alpha \leq 1/2$. For α sufficiently close to 1/2 this was proved by Murai [53], but the general conjecture is still open. However, Kaufmann [45] proved that for any f satisfying (2.19), for $n_k = e^{ck^\alpha}$ for all $\alpha > 0$ and almost all $c > 0$, $f(n_k x)$ satisfies the CLT and LIL. In view of the intractable number theoretic problem the CLT leads to for nonrandom c (see Berkes [11]), this shows the power of random constructions.

Another random construction producing slowly increasing sequences $(n_k)_{k\geq 1}$ is when the gaps $n_{k+1} - n_k$ are positive, bounded i.i.d. random variables, i.e. when $(n_k)_{k\geq 1}$ is an increasing random walk. In the case when the common distribution function of the $n_{k+1} - n_k$ are absolutely continuous, Schatte [63] proved the discrepancy LIL and Weber [76] and Berkes and Weber [17] proved various further results for $D_N(n_k x)$ both when the random walk generating $(n_k)_{k\geq 1}$ is absolutely continuous or discrete. In the absolutely continuous case Berkes and Weber [17] proved that under (2.19) $f(n_k x)$ obeys almost surely the analogue of the Carleson convergence theorem, i.e. $\sum_{k=1}^\infty c_k f(n_k x)$ converges a.e. provided $\sum_{k=1}^\infty c_k^2 < \infty$. In particular, for such f and sequences $(n_k)_{k\geq 1}$ the Khinchin conjecture is true, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(n_k x) = 0 \qquad \text{a.e.} \tag{4.9}$$

Concerning the Khinchin conjecture, a stronger result has been proved by Fukuyama [30], who proved that under (2.18) relation (4.9) holds for $n_k = k^\alpha$ for almost every $\alpha > 0$ and if $\int_0^1 f^2(x)\,dx < \infty$ then for almost all $\alpha > 0$.

5 Appendix

In this section we formulate some classical probability limit theorems used in this paper.

Theorem A. *Let $X_1, X_2, \dots$ be a sequence of bounded independent random variables with mean 0. Let $S_N = \sum_{k=1}^N X_k$ and assume that $B_N^2 = \sum_{k=1}^N \mathbb{E}X_k^2 \to \infty$ and*

$$\|X_k\|_\infty = o(B_k) \quad as \quad k \to \infty, \tag{5.1}$$

then

$$B_N^{-1} \sum_{k=1}^N X_k \to_d \mathcal{N}(0,1).$$

If instead of (5.1) we assume

$$\|X_k\|_\infty = o(B_k/(\log\log B_k)^{1/2}) \quad as \quad k \to \infty, \tag{5.2}$$

then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{\sqrt{2B_N^2 \log\log B_N^2}} = 1 \quad \text{a.s.}$$

Theorem A is the usual formulation of the central limit theorem and law of the iterated logarithm, (5.1) is the classical "uniform asymptotic negligibility" condition and (5.2) is the Kolmogorov condition. Here $\|\cdot\|_\infty$ denotes the sup norm. Note that both conditions (5.1) and (5.2) are sharp. For a proof, see e.g. Petrov [55].

Theorem B. *Let $X_1, X_2, \dots$ be independent random variables having the uniform distribution over $(0,1)$, let $F_N(t) = N^{-1}\sum_{k=1}^N I\{X_k \le t\}$ denote the empirical distribution function of the sample $(X_1, \dots, X_N)$ and let $T_N = \sup_{0\le t\le 1} |F_N(t) - t|$. Then*

$$\lim_{N\to\infty} \mathbb{P}\{\sqrt{N} T_N \le x\} = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2x^2} \qquad (x > 0) \tag{5.3}$$

and

$$\limsup_{N\to\infty} \frac{N T_N}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad a.s. \tag{5.4}$$

For the proof, see e.g. Shorack and Wellner [66].

Theorem C. *Let $X, X_1, X_2, \dots$ be i.i.d. random variables with $\mathbb{P}(X = 2^k) = 2^{-k}$, $k = 1, 2, \dots$ and let $S_n = \sum_{k=1}^n X_k$. Then*

$$\lim_{n\to\infty} \frac{S_n}{n\log_2 n} = 1$$

in probability, but

$$\liminf_{n\to\infty} \frac{S_n}{n\log_2 n} = 1, \quad \limsup_{n\to\infty} \frac{S_n}{n\log_2 n} = \infty \qquad a.s.$$

The sequence $(X_k)_{k\geq 1}$ corresponds to the so called St. Petersburg game, a simple i.i.d. sequence with infinite means. For the proofs and further information, see Feller [28] and Csörgő and Simons [21].

References

[1] C. Aistleitner, Diophantine equations and the lil for the discrepancy of sublacunary sequences. *Illinois J. Math.*, to appear.

[2] C. Aistleitner, Irregular discrepancy behavior of lacunary series. *Monatsh. Math.*, to appear.

[3] C. Aistleitner, Irregular discrepancy behavior of lacunary series II. *Monatsh. Math.*, to appear.

[4] C. Aistleitner, On the law of the iterated logarithm for the discrepancy of lacunary sequences. *Trans. Amer. Math. Soc.*, to appear.

[5] C. Aistleitner, On the law of the iterated logarithm for the discrepancy of sequences $(n_k x)$ with multidimensional indices. *Unif. Distrib. Theory* **2**, 89–104, 2007.

[6] C. Aistleitner and I. Berkes, On the law of the iterated logarithm for the discrepancy of $\langle n_k x\rangle$. *Monatshefte Math.* **156**, 103–121, 2009.

[7] C. Aistleitner and I. Berkes, On the central limit theorem for $f(n_k x)$. *Prob. Theory Rel. Fields* **146**, 267–289, 2010.

[8] C. Aistleitner, I. Berkes and R. Tichy, On permutations of Hardy-Littlewood-Pólya sequences, submitted.

[9] C. Aistleitner, I. Berkes and R. Tichy, Lacunarity, symmetry and Diophantine equations. Preprint.

[10] R. C. Baker, Metric number theory and the large sieve. *J. London Math. Soc.* **24**, 34-40, 1981.

[11] I. Berkes, On the central limit theorem for lacunary trigonometric series. *Analysis Math.* **4**, 159-180, 1978.

[12] I. Berkes, A central limit theorem for trigonometric series with small gaps. *Z. Wahrsch. Verw. Gebiete* **47**, 157–161, 1979.

[13] I. Berkes, Critical LIL behavior of the trigonometric system. *Trans. Amer. Math. Soc.* **338**, 553-585, 1993.

[14] I. Berkes, On the convergence of $\sum c_n f(nx)$ and the Lip 1/2 class. *Trans. Amer. Math. Soc.* **349**, 4143–4158, 1997.

[15] I. Berkes and W. Philipp, The size of trigonometric and Walsh series and uniform distribution mod 1. *J. London Math. Soc.* **50**, 454-464, 1994.

[16] I. Berkes, W. Philipp and R. Tichy, Empirical processes in probabilistic number theory: the LIL for the discrepancy of $(n_k\omega)$ mod 1. *Illinois J. Math.* **50**, 107–145, 2006.

[17] I. Berkes and M. Weber, On the convergence of $\sum c_k f(n_k x)$. *Mem. Amer. Math. Soc.* 201, No. 943, 2009.

[18] S. Bobkov and F. Götze, Concentration inequalities and limit theorems for randomized sums. *Probab. Theory Related Fields* **137**, 49–81, 2007.

[19] J. Bourgain, Almost sure convergence and bounded entropy. *Israel J. Math.* **63**, 79–97, 1988.

[20] J. W. S. Cassels, Some metrical theorems in Diophantine approximation III. *Proc. Cambridge Philos. Soc.* **46**, 219-225, 1950.

[21] S. Csörgő and G. Simons, St. Petersburg games with the largest gains withheld, *Statist. Probab. Lett.* **77**, 1185–1189, 2007.

[22] S. Dhompongsa, Uniform laws of the iterated logarithm for Lipschitz classes of functions. *Acta Sci. Math. (Szeged)* **50**, 105–124, 1986.

[23] M. Drmota and R. F. Tichy, Sequences, Discrepancies and Applications. *Lecture Notes Math.* **1651**, Springer, Berlin, Heidelberg, New York, 1997.

[24] P. Erdős, On trigonometric sums with gaps. *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **7**, 37–42, 1962.

[25] P. Erdős and J. F. Koksma, On the uniform distribution modulo 1 of sequences $(f(n, \theta))$. *Proc. Kon. Nederl. Akad. Wetensch.* **52**, 851-854, 1949.

[26] P. Erdős and I.S. Gál, On the law of the iterated logarithm, *Proc. Kon. Nederl. Akad. Wetensch.* **58**, 65-84, 1955.

[27] P. Erdős and A. Rényi, On a new law of large numbers. *J. Analyse Math.* **23**, 103–111, 1970.

[28] W. Feller, Note on the law of large numbers and "fair" games, *Ann. Math. Statist.* **16**, 301-304, 1945.

[29] H. Fiedler, W. Jurkat and O. Körner, Asymptotic expansions of finite theta series. *Acta Arith.* **32**, 129–146, 1977.

[30] K. Fukuyama, A version of the law of large numbers. *Colloq. Math.* **90**, 295–298, 2001.

[31] K. Fukuyama, The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$. *Acta Math. Hung.*, **118**, 155–170, 2008.

[32] K. Fukuyama, The law of the iterated logarithm for the discrepancies of a permutation of $n_k x$. *Acta Math. Hungar.* **123**, 121–125, 2009.

[33] K. Fukuyama, Pure gaussian limit distributions of trigonometric series with bounded gaps. Preprint.

[34] K. Fukuyama, A metric discrepancy result for lacunary sequence with small gaps. Preprint.

[35] K. Fukuyama, A law of the iterated logarithm for discrepancies: nonconstant limsup. *Monatshefte Math.*, to appear.

[36] K. Fukuyama and K. Nakata, A metric discrepancy result for the Hardy-Littlewood-Pólya sequences. *Monatsh. Math.*, to appear.

[37] V. F. Gaposhkin, Lacunary series and independent functions (in Russian). *Uspehi Mat. Nauk* **21/6** (1966), 3-82.

[38] V. F. Gaposhkin, On convergence and divergence systems (In Russian). *Mat. Zametki* **4**, 253–260, 1968.

[39] V. F. Gaposhkin, On the central limit theorem for some weakly dependent sequences (in Russian), *Teor. Verojatn. Prim.* **15**, 666-684, 1970.

[40] H. Halberstam and K. F. Roth, Sequences. Springer-Verlag, New York-Berlin, 1983.

[41] L.K. Hua, Additive theory of prime numbers. Translations of Mathematical Monographs, Vol. 13 American Mathematical Society, Providence, R.I. 1965.

[42] S. Izumi, Notes on Fourier analysis XLIV: on the law of the iterated logarithm of some sequence of functions. *J. Math. (Tokyo)* **1**, 1-22, 1951.

[43] M. Kac, On the distribution of values of sums of the type $\sum f(2^k t)$. *Ann. of Math.* **47**, 33–49, 1946.

[44] M. Kac, Probability methods in some problems of analysis and number theory. *Bull. Amer. Math. Soc.* **55**, 641–665, 1949.

[45] R. Kaufman, On the approximation of lacunary series by Brownian motion. *Acta Math. Acad. Sci. Hungar.* **35**, 61–66, 1980.

[46] H. Kesten, The discrepancy of random sequences $\{kx\}$. *Acta Arith.* **10**, 183-213, 1964/65.

[47] A. Khinchin, Ein Satz über Kettenbrüche mit arithmetischen Anwendungen. *Math. Zeitschrift* **18**, 289–306, 1923.

[48] A. Khinchin, Einige Sätze über Kettenbrüche mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.* **92**, 115–125, 1924.

[49] A. Kolmogorov, Une contribution à l'étude de la convergence des séries de Fourier, *Fund. Math.* **5**, 96-97, 1925.

[50] L. Kuipers and H. Niederreiter, Uniform distribution of sequences. Wiley, New York, 1974.

[51] J. Marstrand, On Khinchin's conjecture about strong uniform distribution. *Proc. London Math. Soc.* **21**, 540–556, 1969.

[52] G. Maruyama, On an asymptotic property of a gap sequence. *Kôdai Math. Sem. Rep.* **2**, 31–32, 1950.

[53] T. Murai, The central limit theorem for trigonometric series, *Nagoya Math. J.* **87**, 79–94, 1982.

[54] A. Ostrowski, Bemerkungen zur Theorie der Diophantischen Approximationen, *Abh. Math. Sem. Univ. Hamburg* **1**, 77–98, 1922.

[55] V. V. Petrov, Limit theorems of probability theory. Sequences of independent random variables. Oxford Studies in Probability, 4. Oxford Science Publications. Clarendon Press, Oxford University Press, New York, 1995.

[56] W. Philipp, Limit theorems for lacunary series and uniform distribution mod 1. *Acta Arith.* **26**, 241-251, 1975.

[57] W. Philipp, A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables, *Ann. Probab.* **5**, 319-350, 1977.

[58] W. Philipp, Empirical distribution functions and strong approximation theorems for dependent random variables. A problem of Baker in probabilistic number theory. *Trans. Amer. Math. Soc.*, **345**, 705–727, 1994.

[59] W. Philipp and W. Stout, Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Amer. Math. Soc.* **161**, 1975.

[60] R. Salem and A. Zygmund, On lacunary trigonometric series. *Proc. Nat. Acad. Sci. USA* **33**, 333-338, 1947.

[61] R. Salem and A. Zygmund, La loi du logarithme itéré pour les séries trigonométriques lacunaires. *Bull. Sci. Math.* **74**, 209–224, 1950.

[62] R. Salem and A. Zygmund, Some properties of trigonometric series whose terms have random signs. *Acta Math.* **91**, 245–301, 1954.

[63] P. Schatte, On a uniform law of the iterated logarithm for sums mod 1 and Benford's law. *Lithuanian Math. J.* **31**, 133–142, 1991.

[64] W. Schmidt, Irregularities of distribution. VII. *Acta Arith.* **21**, 45–50, 1972.

[65] J. Schoissengeier, A metrical result on the discrepancy of $(n\alpha)$, *Glasgow Math. J.* **40**, 393-425, 1998.

[66] G. R. Shorack and J. A. Wellner, *Empirical processes with applications to statistics.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.

[67] S. Takahashi, A gap sequence with gaps bigger than the Hadamards. *Tôhoku Math. J.* **13**, 105–111, 1961.

[68] S. Takahashi, An asymptotic property of a gap sequence. *Proc. Japan Acad.* **38**, 101–104, 1962.

[69] S. Takahashi, The law of the iterated logarithm for a gap sequence with infinite gaps. *Tôhoku Math. J.* **15**, 281–288, 1963.

[70] S. Takahashi, On lacunary trigonometric series. *Proc. Japan Acad.* **41**, 503–506, 1965.

[71] S. Takahashi, On lacunary trigonometric series II. *Proc. Japan Acad.* **44**, 766–770, 1968.

[72] S. Takahashi, On the law of the iterated logarithm for lacunary trigonometric series. *Tôhoku Math. J.* **24**, 319-329, 1972.

[73] S. Takahashi, On the law of the iterated logarithm for lacunary trigonometric series II. *Tôhoku Math. J.*

[74] S. Takahashi, An asymptotic behavior of $f(n_k t)$. *Sci. Rep. Kanazawa Univ.* **33**, 27–36, 1988.

[75] A. Walfisz, Ein metrischer Satz über diophantische Approximationen, *Fund. Math.* **16**, 361–385, 1930.

[76] M. Weber, Discrepancy of randomly sampled sequences of reals. *Math. Nachr.* **271**, 105–110, 2004.

[77] M. Weiss, The law of the iterated logarithm for lacunary trigonometric series. *Trans. Amer. Math. Soc.* **91**, 444–469, 1959.

[78] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.* **77**, 313-352, 1916.