On the class of limits of lacunary trigonometric series

Christoph Aistleitner*

Abstract

Let $(n_k)_{k\geq 1}$ be a lacunary sequence of positive integers, i.e. a sequence satisfying $n_{k+1}/n_k > q > 1,\ k \geq 1$, and let f be a "nice" 1-periodic function with $\int_0^1 f(x)\ dx = 0$. Then the probabilistic behavior of the system $(f(n_k x))_{k\geq 1}$ is very similar to the behavior of sequences of i.i.d. random variables. For example, Erdős and Gál proved in 1955 the following law of the iterated logarithm (LIL) for $f(x) = \cos 2\pi x$ and lacunary $(n_k)_{k\geq 1}$:

$$\limsup_{N\to\infty} (2N \log\log N)^{-1/2} \sum_{k=1}^{N} f(n_k x) = \|f\|_2 \tag{1}$$

for almost all $x \in (0,1)$, where $\|f\|_2 = \left(\int_0^1 f(x)^2\ dx\right)^{1/2}$ is the standard deviation of the random variables $f(n_k x)$. If $(n_k)_{k\geq 1}$ has certain number-theoretic properties (e.g. $n_{k+1}/n_k \to \infty$), a similar LIL holds for a large class of functions f, and the constant on the right-hand side is always $\|f\|_2$. For general lacunary $(n_k)_{k\geq 1}$ this is not necessarily true: Erdős and Fortet constructed an example of a trigonometric polynomial f and a lacunary sequence $(n_k)_{k\geq 1}$, such that the lim sup in the LIL (1) is not equal to $\|f\|_2$ and not even a constant a.e. In this paper show that the class of possible functions on the right-hand side of (1) can be very large: we give an example of a trigonometric polynomial f, such that for any function $g(x)$ with sufficiently small Fourier coefficients there exists a lacunary sequence $(n_k)_{k\geq 1}$ such that (1) holds with $\sqrt{\|f\|_2^2 + g(x)}$ instead of $\|f\|_2$ on the right-hand side.

1 Introduction

An increasing sequence of positive integers is called a lacunary sequence if it satisfies the Hadamard gap condition

$$\frac{n_{k+1}}{n_k} > q > 1, \quad k \geq 1.$$

By a classical heuristics, systems of the form $(\cos 2\pi n_k x)_{k\geq 1}$ or $(f(n_k x))_{k\geq 1}$, where $(n_k)_{k\geq 1}$ is a lacunary sequence and f is a "nice" 1-periodic function, replicate many properties of

*Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: aistleitner@finanz.math.tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N23. This paper was written during a stay at the Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences, which was made possible by an MOEL scholarship of the *Österreichische Forschungsgemeinschaft* (ÖFG).

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systems of i.i.d. random variables. For example,

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} \cos 2\pi n_k x \le t\sqrt{N/2} \right\} = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2}\, du$$

(Salem and Zygmund [20]) and

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \cos 2\pi n_k x\right|}{\sqrt{2N \log\log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e.} \tag{2}$$

(Erdős and Gál [11]), which is in perfect accordance with the central limit theorem (CLT) and the law of the iterated logarithm (LIL) for systems of i.i.d. random variables (here, and in the sequel, λ will stand for the Lebesgue measure on $(0,1)$, and "a.e." will always refer to this measure). However, the analogy is not perfect. If f is a more general function satisfying

$$\int_0^1 f(x)\, dx = 0, \qquad f(x+1) = f(x), \qquad \mathrm{Var}_{[0,1]} f < \infty, \tag{3}$$

e.g. f is a trigonometric polynomial, the limiting distribution of

$$\frac{\sum_{k=1}^N f(n_k x)}{\sqrt{N}}$$

may be non-Gaussian and the value of

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} \tag{4}$$

does not always have to be $\|f\|_2 = \left(\int_0^1 f(x)^2\, dx\right)^{1/2}$ a.e. and not even have to be a constant a.e. Erdős and Fortet (cf. [17, p. 646]) showed that for

$$f(x) = \cos 2\pi x + \cos 4\pi x \qquad \text{and} \qquad n_k = 2^k - 1, \quad k \ge 1$$

we have

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} f(n_k x) \le t\sqrt{N} \right\} = \pi^{-1/2} \int_0^1 \int_{-\infty}^{t/2|\cos \pi s|} e^{-u^2}\, du\, ds$$

and

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} = \sqrt{2}|\cos \pi x| \qquad \text{a.e.} \tag{5}$$

Gaposhkin [16] observed that there is an intimate connection between the validity of the CLT for $(f(n_k x))_{k\ge1}$ and the number of solutions (k_1, k_2) of Diophantine equations of the type

$$u n_{k_1} \pm v n_{k_2} = w, \qquad u, v, w \in \mathbb{Z}, \tag{6}$$

and in terms of the number of solutions of such equations Aistleitner and Berkes [5] gave a full characterization of those lacunary sequences $(n_k)_{k\ge1}$ for which $(f(n_k x))_{k\ge1}$ satisfies the

CLT.
The situation is somehow similar in the case of the LIL, since here it is also possible to find sufficient conditions on the number of solutions of the Diophantine equations in (6) that guarantee the validity of the exact LIL

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} f(n_k x)\right|}{\sqrt{2N \log\log N}} = \|f\|_2 \quad \text{a.e.} \tag{7}$$

for $(f(n_k x))_{k\geq 1}$ for f satisfying (3) (cf. Aistleitner [4]; the problem to find a sufficient *and* necessary condition, like in the case of the CLT, is still open). If the number of the solutions of some Diophantine equations of type (6) is "too large", the value of the lim sup in (4) does not have to be equal to $\|f\|_2$ a.e. - this is what we can observe in the example (5) of Erdős and Fortet, where e.g. the Diophantine equation $n_{k_1} - 2n_{k_2} = 1$ has "many" solutions (cf. also Fukuyama [13] and Aistleitner [2],[3]; a similar phenomenon can also be observed in the case of sub-lacunary growing $(n_k)_{k\geq 1}$, see e.g. Aistleitner [1], Berkes [7], Berkes, Philipp and Tichy [9], [10], Fukuyama [12] and Fukuyama and Nakata [14]). A special case of sequences with "few" solutions of Diophantine equations of the form (6) are sequences satisfying the large gap condition

$$\frac{n_{k+1}}{n_k} \to \infty, \qquad k \to \infty,$$

for which Takahashi [23] proved already in 1963

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} f(n_k x)\right|}{\sqrt{2N \log\log N}} = \|f\|_2 \quad \text{a.e.} \tag{8}$$

For general $(n_k)_{k\geq 1}$ and f satisfying (3) the best possible result is

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} f(n_k x)\right|}{\sqrt{2N \log\log N}} \leq C_{f,q} \quad \text{a.e.,}$$

where $C_{f,q}$ is some number depending on f and q (Takahashi [22], Philipp [19]).

2 Statement of results and open problems

The example of Erdős and Fortet shows that functions different from the constant function $\|f\|_2$ (a.e.) are possible values of the lim sup in (4). Naturally the question arises *which* functions are possible values of this lim sup for appropriate f and $(n_k)_{k\geq 1}$. We show, that even if f is a trigonometric polynomial the class of possible values of the lim sup in (4) for different sequences $(n_k)_{k\geq 1}$ can be very large. More precisely, we will prove the following theorem:

Theorem 1 *Let*

$$f(x) = \cos 2\pi x + \cos 4\pi x - \cos 6\pi x + \sin 10\pi x. \tag{9}$$

Then for any function $g(x)$ satisfying

$$\|g\|_A \leq \frac{1}{2} \tag{10}$$

there exists a lacunary sequence $(n_k)_{k\geq 1}$ *of positive integers such that*

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} = \sqrt{\|f\|_2^2 + g(x)} \quad \text{a.e.} \tag{11}$$

In the statement of this theorem, $\|g\|_A$ is defined as

$$\|g\|_A = |a_0| + \sum_{j=1}^{\infty} \left(|a_j| + |b_j|\right),$$

where a_j and b_j are the coefficients from the Fourier series expansion of g in the form

$$g(x) \sim a_0 + \sum_{j=1}^{\infty} a_j \cos 2\pi jx + b_j \sin 2\pi jx.$$

Informally speaking, Theorem 1 says that for the function f in (9) and general lacunary $(n_k)_{k\geq 1}$ the value of the lim sup in (4) can be the square root of any function that has a Fourier series expansion which is "not too different" from the Fourier series expansion of the constant function $\|f\|_2^2$. It is not surprising that there has to be a connection between the (Fourier coefficients of the) function f and the possible lim sup's: if $f = \sum_{j=1}^d a_j \cos 2\pi jx + b_j \sin 2\pi jx$ denotes some trigonometric polynomial, then necessarily for any lacunary $(n_k)_{k\geq 1}$

$$\begin{aligned}
& \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} \qquad (12)\\
\leq\ & \sum_{j=1}^d |a_j| \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \cos 2\pi n_k x\right|}{\sqrt{2N \log\log N}} + \sum_{j=1}^d |b_j| \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \sin 2\pi n_k x\right|}{\sqrt{2N \log\log N}} \\
\leq\ & \frac{\sum_{j=1}^d |a_j| + |b_j|}{\sqrt{2}} \\
=\ & \frac{\|f\|_A}{\sqrt{2}} \qquad \text{a.e.,}
\end{aligned}$$

where the last inequality follows from (2) (and the corresponding result for sin instead of cos).

Several interesting questions remain unsolved. We mention three open problems:

Open problem 1: The function f in Theorem 1 is a trigonometric polynomial which consists of 4 terms. In view of (2) it is clear that for a simple trigonometric function ($\cos 2\pi x$ or $\sin 2\pi x$) a result like Theorem 1 is not possible, since the value of the lim sup equals $1/\sqrt{2}$ a.e. for any lacunary $(n_k)_{k\geq 1}$. Find a trigonometric polynomial which consists of only 3 (or even only 2) terms, such that a result like Theorem 1 holds, i.e. find a three-term (or even two-term) trigonometric polynomial f such that there exists a positive ε such that for any g with $\|g\|_A \leq \varepsilon$ there is a lacunary sequence $(n_k)_{k\geq 1}$ such that (11) holds (or prove that such a trigonometric polynomial does not exist).

Open problem 2: In our example, we require that $\|g\|_A \leq 1/2$, and since the function f in our example has $\|f\|_2 = \sqrt{2}$ and $\|f\|_A = 4$, this means that $\|g\|_A \leq \|f\|_A^2/2 - \|f\|_2^2 - 11/2$. On the other hand, the considerations in (12) show that necessarily for any trigonometric polynomial f always $g \leq \|f\|_A^2/2 - \|f\|_2^2$ a.e., or, in other words, that there does not exist any trigonometric polynomial f and lacunary $(n_k)_{k\geq 1}$ such that (11) holds with a function g which is larger than $\|f\|_A^2/2 - \|f\|_2^2$ on a set of positive measure. In fact, there is reason to believe that even $\|g\|_A > \|f\|_A^2/2 - \|f\|_2^2$ is impossible (which is a stronger assertion, since trivially always $g \leq \|g\|_A$). The bound $\|g\|_A = \|f\|_A^2/2 - \|f\|_2^2$ is reached for some functions f and certain sequences $(n_k)_{k\geq 1}$, e.g. in the example of Erdős and Fortet, but we have not been able to find a function f such that a general result like Theorem 1 holds with $\|g\|_A \leq \|f\|_A^2/2 - \|f\|_2^2$ instead of (10). Find an example of such a function f or prove that it does not exist, or, more generally, find the best possible upper bound for $\|g\|_A$ in a result like Theorem 1 (in terms of $\|f\|_2^2$ and $\|f\|_A^2$).

Open problem 3: We have mentioned above that if the number of solutions of Diophantine equations of the form (6) is "small" then $(f(n_k x))_{k\geq 1}$ satisfies the LIL with the same constant as in the case of i.i.d. random variables and (7) holds. On the other hand, Theorem 1 shows that without any Diophantine conditions the class of possible values of the lim sup in (4) is "large". Find a similar result for the central limit theorem, i.e. describe functions f such that the class of possible limiting distributions (possibly along subsequences) of $\sum_{k=1}^N f(n_k x)/\sqrt{N}$ (for lacunary sequences $(n_k)_{k\geq 1}$) is "large".
In this context we mention a recent result of Fukuyama and Takahashi [15] on sums of the form $\sum_{k=1}^N a_k \cos 2\pi k(x + \alpha_k)$. They showed that for any variance mixture Q of Gaussian distributions there exist sequences $(a_k)_{k\geq 1}$ and $(\alpha_k)_{k\geq 1}$ such that the aforementioned sum, multiplied with an appropriate norming factor, converges to Q.

3 Idea of the proof

In this section we present the main ideas of the proof of Theorem 1, in order to enhance the comprehensibility of our presentation.

Let a function $g(x)$ be given, and assume the Fourier coefficients of g are sufficiently small. We construct a lacunary sequence $(n_k)_{k\geq 1}$ for which the numbers

$$S_{N,j_1,j_2,c} = \#\{1 \leq k_1, k_2 \leq N : \ j_1 n_{k_1} - j_2 n_{k_2} = c\}, \quad j_1, j_2 \in \{1,2,3,5\}, \tag{13}$$

counting the solutions of certain Diophantine equations (for indices k_1, k_2 up to N), are of appropriate size. Then we show that there exist a Wiener process ξ and a sequence $(T_N)_{N\geq 1}$ of random variables such that the sums $\sum_{k=1}^N f(n_k x)$ can be approximated by $\xi(T_N)$. To estimate T_N we calculate

$$\left(\sum_{k=1}^N f(n_k x)\right)^2 = \left(\sum_{k=1}^N (\cos 2\pi n_k x + \cos 4\pi n_k x - \cos 6\pi n_k x + \sin 10\pi n_k x)\right)^2. \tag{14}$$

Since f is a trigonometric polynomial (with frequencies 1, 2, 3, 5) the function in (14) is also a trigonometric polynomial, and we show that T_N is essentially of the same size as the sum of those trigonometric functions in (14) that have "small" frequencies. The well-known formulas

$$\begin{aligned} \cos x \cos y &= \frac{1}{2}\left(\cos(x+y)+\cos(x-y)\right), \\ \sin x \sin y &= \frac{1}{2}\left(-\cos(x+y)+\cos(x-y)\right), \\ \sin x \cos y &= \frac{1}{2}\left(\sin(x+y)+\sin(x-y)\right) \end{aligned}$$

show that there is a direct connection between the number of summands in (14) that have the same "small" frequency c and the numbers $S_{j_1,j_2,c}$ in (13). Note that in particular (14) contains the constant function $\|f\|_2^2 N$. This means we have

$$T_N \approx \|f\|_2^2 N + \sum_j \left(c_j \cos 2\pi j x + d_j \sin 2\pi j x\right), \tag{15}$$

where the coefficients $c_j,\ d_j$ are in direct connection with the numbers $S_{N,j_1,j_2,c}$ in (13), and if these numbers we appropriately chosen, then the function of the right-hand side of (15) is close to $N\left(\|f\|_2^2 + g(x)\right)$. Using the law of the iterated logarithm for $\xi(T_N)$ and

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N f(n_k x)}{\sqrt{2T_N \log\log T_N}} \approx \limsup_{N\to\infty} \frac{\xi(T_N)}{\sqrt{2T_N \log\log T_N}} = 1 \quad \text{a.e.}$$

we get the desired result.

4 Preliminaries

Assume that g is fixed, and that $\|g\|_A \leq 1/2$. If $\|g\|_A = 0$, i.e. $g \equiv 0$, we can choose a sequence $(n_k)_{k\geq 1}$ satisfying the "large gap condition" $n_{k+1}/n_k \to \infty$ and get (11) as mentioned in (8). Thus in the sequel we will restrict ourselves to the case $\|g\|_A \neq 0$.

Write

$$g(x) = a_0 + \sum_{j=1}^{\infty} \left(a_j \cos 2\pi j x + b_j \sin 2\pi j x\right)$$

for the Fourier series of g.

We divide the set of positive integers into blocks $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_i, \ldots$ of lengths $2, 8, 18, \ldots, 2i^2, \ldots$, i.e.

$$\Delta_i = \left\{ k \in \mathbb{Z}^+ :\ \frac{2(i-1)i(2i-1)}{6} < k \leq \frac{2i(i+1)(2i+1)}{6} \right\}, \qquad i \geq 1.$$

We construct a sequence $(n_k)_{k\geq 1}$ that has the following properties (throughout the rest of this paper, $\log y$ should be read as $\max\{1, \log y\}$):

- (P1) $$\frac{n_{k+1}}{n_k} > \max\{7, k^3\}, \qquad k \geq 1, \quad \text{if } k \text{ is even}$$
  $$\frac{n_{k+1}}{n_k} \in \left\{ \left[\tfrac{19}{10}, \tfrac{21}{10}\right], \left[\tfrac{29}{10}, \tfrac{31}{10}\right], \left[\tfrac{49}{10}, \tfrac{51}{10}\right], \left[\tfrac{69}{10}, \tfrac{71}{10}\right] \right\}, \qquad k \geq 1, \quad \text{if } k \text{ is odd}$$

- (P2) for any $i \geq 1$ and any j satisfying $0 \leq j \leq \lceil \log \log i \rceil$ the number of solutions $(k_1, k_2), \ k_1, k_2 \in \Delta_i$ of the Diophantine equation

$$n_{k_1} - 2n_{k_2} = j$$

is $\lfloor \max\{2a_j, 0\} i^2 \rfloor$.

- (P3) for any $i \geq 1$ and any j satisfying $0 \leq j \leq \lceil \log \log i \rceil$ the number of solutions $(k_1, k_2), \ k_1, k_2 \in \Delta_i$ of the Diophantine equation

$$n_{k_1} - 3n_{k_2} = j$$

is $\lfloor \max\{-2a_j, 0\} i^2 \rfloor$.

- (P4) for any $i \geq 1$ and any j satisfying $1 \leq j \leq \lceil \log \log i \rceil$ the number of solutions $(k_1, k_2), \ k_1, k_2 \in \Delta_i$ of the Diophantine equation

$$n_{k_1} - 5n_{k_2} = -\operatorname{sgn}(b_j) j$$

is $\lfloor 2|b_j| i^2 \rfloor$ and the number of solutions of $n_{k_1} - 5n_{k_2} = \operatorname{sgn}(b_j) j$ is 0.

- (P5) for any $i \geq 1$, any $k_1, k_2 \in \Delta_i$ and any $u, v \in \{1, 2, 3, 5\}, \ u \leq v, \ (u, v) \notin \{(1,1), (1,2), (1,3), (1,5), (2,2), (3,3), (5,5)\}$ we have

$$|un_{k_1} - vn_{k_2}| \geq \frac{\min\{n_{k_1}, n_{k_2}\}}{10}. \tag{16}$$

If $(u, v) \in \{(1,1), (2,2), (3,3), (5,5)\}$ we have

$$|un_{k_1} - vn_{k_2}| \geq \frac{\min\{n_{k_1}, n_{k_2}\}}{10},$$

except in the trivial case $k_1 = k_2$, when $un_{k_1} - vn_{k_2} = 0$.
If $(u, v) \in \{(1,2), (1,3)\}$, we have either (16)

$$un_{k_1} - vn_{k_2} = j \qquad \text{for some} \qquad 0 \leq j \leq \lceil \log \log i \rceil.$$

If $(u, v) = (1, 5)$ we have either (16) or

$$un_{k_1} - vn_{k_2} = -\operatorname{sgn}(b_j) j \qquad \text{for some} \qquad 1 \leq j \leq \lceil \log \log i \rceil.$$

A sequence having properties (P1)-(P5) can be constructed recursively in the following way: assume that the values of n_k for $k \in \Delta_1 \cup \ldots\ldots \Delta_{i-1}$ are fixed. For the smallest integer k in Δ_i, choose an arbitrary, but sufficiently large value for n_k such that (P1) is satisfied. In case $i = 1$ we choose $n_1 = 10\,(\lceil \log \log 1 \rceil)$. Similarly, we choose all the values for n_k, where $k \in \Delta_i$ and k is odd, arbitrarily, but sufficiently large, such that (P1) holds. It remains to fix the values of n_k for $k \in \Delta$, where k is even. For the first

$$\lfloor \max\{2a_0, 0\} i^2 \rfloor \tag{17}$$

even integers k in Δ_i, we set

$$n_k = 2n_{k-1}$$

(the expression in (17) may also be zero, which is no problem - in this case we do not fix any values n_k). For the following

$$\left\lfloor \max\{2a_1, 0\} i^2 \right\rfloor$$

even integers we set

$$n_k = 2n_{k-1} + 1.$$

Repeating this scheme, we onwardly fix always the following

$$\left\lfloor \max\{2a_j, 0\} i^2 \right\rfloor$$

values of n_k for the even integers in Δ_i by setting them

$$n_k = 2n_{k-1} + j,$$

until we reach $j = \lceil \log \log i \rceil$. In total, we have now fixed all values of $n_k,\ k \in \Delta_i$ for odd k, and in total

$$\sum_{j=0}^{\lceil \log \log i \rceil} \left\lfloor \max\{2a_j, 0\} i^2 \right\rfloor$$

values for even k.
Now we fix the first

$$\left\lfloor \max\{-2a_0, 0\} i^2 \right\rfloor$$

values of n_k for the remaining even values $k \in \Delta_i$ by setting them

$$n_k = 3n_{k-1},$$

then the next

$$\left\lfloor \max\{-2a_1, 0\} i^2 \right\rfloor$$

values of n_k for the remaining even values $k \in \Delta_i$ by setting them

$$n_k = 3n_{k-1} + 1,$$

and again, onwardly, we always fix the next

$$\left\lfloor \max\{-2a_j, 0\} i^2 \right\rfloor$$

values of n_k by setting them

$$n_k = 3n_{k-1} + j,$$

where again j runs up to $\lceil \log \log i \rceil$. This means, in total we have fixed all the values of $n_k,\ k \in \Delta_i$ for odd k and the first

$$\sum_{j=0}^{\lceil \log \log i \rceil} \left\lfloor \max\{2a_j, 0\} i^2 \right\rfloor + \sum_{j=0}^{\lceil \log \log i \rceil} \left\lfloor \max\{-2a_j, 0\} i^2 \right\rfloor = \sum_{j=0}^{\lceil \log \log i \rceil} \left\lfloor 2|a_j| i^2 \right\rfloor$$

values for even k. Now we fix the values of n_k for the next

$$\left\lfloor 2|b_1| i^2 \right\rfloor$$

even integers in Δ_i but setting them

$$n_k = 5n_{k-1} - \operatorname{sgn}(b_1)1,$$

then the values of n_k for the next

$$\lfloor 2|b_2|i^2 \rfloor$$

even integers by setting them

$$n_k = 5n_{k-1} - \operatorname{sgn}(b_2)2,$$

and so on, always fixing the values of n_k for the following

$$\lfloor 2|b_j|i^2 \rfloor$$

even integers by setting

$$n_k = 5n_{k-1} - \operatorname{sgn}(b_j)j,$$

where we let j run up to $\lceil \log\log i \rceil$. Now we have fixed the values of n_k for all odd $k \in \Delta_i$ (which are exactly i^2 indices), and for the first

$$\sum_{j=0}^{\lceil \log\log i \rceil} \lfloor 2|a_j|i^2 \rfloor + \sum_{j=1}^{\lceil \log\log i \rceil} \lfloor 2|b_j|i^2 \rfloor$$

even indices $k \in \Delta_i$. Observe that by (10)

$$\begin{aligned} & i^2 + \sum_{j=0}^{\lceil \log\log i \rceil} \lfloor 2|a_j|i^2 \rfloor + \sum_{j=1}^{\lceil \log\log i \rceil} \lfloor 2|b_j|i^2 \rfloor \\ \leq \; & i^2 + 2i^2 \left(\sum_{j=1}^{\lceil \log\log i \rceil} |a_j| + |b_j| \right) \\ \leq \; & 2i^2 = |\Delta_i| \end{aligned}$$

(here and in the sequel, we write $|\Delta_i|$ for the number of elements of Δ_i). Thus we are still within Δ_i. It is possible (or even likely) that for some even indices k in Δ_i no value has been assigned to n_k so far. For these remaining indices k we set

$$n_k = 7n_{k-1}.$$

Thus all values of n_k, where $k \in \Delta_i$, are fixed, and we can continue and fix the values of n_k for $k \in \Delta_{i+1}$, etc.

Now we want to show that the sequence $(n_k)_{k\geq 1}$ constructed in this way really satisfies (P1)-(P5). The first part of (P1), concerning even k is trivial by construction. The second part of (P1) is also an easy consequence of our construction. We have

$$n_2 = un_1 + j$$

for some $u \in \{2,3,5,7\}$ and $j \in [-\lceil \log\log 1 \rceil, \lceil \log\log 1 \rceil]$. Thus, since we set $n_1 = 10(\lceil \log\log 1 \rceil)$ we have

$$\frac{n_2}{n_1} = u + \frac{j}{n_k} \in \left[u - \frac{1}{10}, u + \frac{1}{10} \right]$$

Similarly, for $i \geq 2$ and odd $k \in \Delta_i$ we have

$$n_{k+1} = un_k + j$$

for some $u \in \{2,3,5,7\}$ and $|j| \leq \lceil \log\log i \rceil$, and again this implies

$$\frac{n_{k+1}}{n_k} = u + \frac{j}{n_k} \in \left[u - \frac{1}{10}, u + \frac{1}{10} \right].$$

Here $j/n_k \in [-1/10, 1/10]$ is clear since n_k grows much faster than the range $[-\lceil \log\log 1 \rceil, \lceil \log\log i \rceil]$ of possible values of j. Thus $(n_k)_{k\geq 1}$ has property (P1), and in particular $(n_k)_{k\geq 1}$ is a lacunary sequence, since

$$\frac{n_{k+1}}{n_k} \geq \frac{19}{10}, \qquad k \geq 1.$$

Next we show (P5). Let $k_1, k_2 \in \Delta_i$, and let $u, v \in \{1,2,3,5\},\ u \leq v$. If $u = v$, then we either have the trivial case $k_1 = k_2$, or $k_1 \neq k_2$ and

$$|un_{k_1} - vn_{k_2}| \geq |n_{k_1} - n_{k_2}| \geq \left(\frac{19}{10} - 1 \right) \min\{n_{k_1}, n_{k_2}\}$$

and (16) holds. It remains to consider the case $u < v$. First assume that $k_2 > k_1$. Then

$$un_{k_1} - vn_{k_2} \leq n_{k_1} - n_{k_2} \leq \left(1 - \frac{19}{10}\right) n_{k_1} \leq -\frac{9}{10} n_{k_1},$$

and (16) holds. Now assume that $k_1 \geq k_2$. If $k_1 \geq k_2 - 2$, then

$$\frac{k_2}{k_1} \geq 7 \cdot \frac{19}{10}$$

and

$$|un_{k_1} - vn_{k_2}| \geq \left(\frac{7 \cdot 19}{10} - 5 \right) n_{k_2} \geq \frac{83}{10} n_{k_2}$$

and (16) holds. Thus the only remaining case is when $k_2 \leq k_1 \leq k_2 + 1$. Then by (P1)

$$\frac{n_{k_1}}{n_{k_2}} \in \left\{ \{1\}, \left[\frac{19}{10}, \frac{21}{10}\right], \left[\frac{29}{10}, \frac{31}{10}\right], \left[\frac{49}{10}, \frac{51}{10}\right] \right\} \qquad \text{or} \qquad \frac{n_{k_1}}{n_{k_2}} \geq \frac{69}{10}. \tag{18}$$

Assume that $u \geq 2$. If e.g. $(u,v) = (2,3)$, then

$$\frac{un_{k_1}}{vn_{k_2}} = \frac{2}{3} \qquad \text{or} \qquad \frac{n_{k_1}}{n_{k_2}} \geq \frac{38}{30},$$

and

$$|un_{k_1} - vn_{k_2}| \geq \frac{8}{30} n_{k_2}$$

and (16) holds. Similar arguments show that (16) holds for $(u,v) = (2,5)$ and $(u,v) = (3,5)$. It remains to consider the case $u = 1,\ u < v$. Let e.g. $u = 1,\ v = 2$. By (18)

$$\frac{n_{k_1}}{n_{k_2}} \in \left\{ \left\{\frac{1}{2}\right\}, \left[\frac{19}{20}, \frac{21}{20}\right] \right\} \text{or} \qquad \frac{n_{k_1}}{n_{k_2}} \geq \frac{29}{20},$$

and

$$\frac{n_{k_1}}{n_{k_2}} \in \left[\frac{19}{20}, \frac{21}{20}\right]$$

if and only if $k_2 = k_1 + 1$ for a pair (k_1, k_2) for which

$$n_{k_1} = 2n_{k_2} + j$$

for some $j \in \{0, \dots, \lceil \log \log i \rceil\}$. In all other cases (16) holds. This also shows, that the only solutions (k_1, k_2) of the Diophantine equation

$$n_{k_1} - 2n_{k_2} = j$$

are the $\lfloor \max\{2a_j, 0\} \rfloor$ or $\lfloor \max\{-2a_j, 0\} \rfloor$ pairs which are explicitely mentioned in the construction of $(n_k)_{k\geq 1}$, and therefore $(n_k)_{k\geq 1}$ really has property (P2). Similar arguments solve the remaining cases $(u, v) = (1, 3)$ and $(u, v) = (1, 5)$, which proves (P5), and as side results we get that $(n_k)_{k\geq 1}$ also satisfies (P3) and (P4).

5 Proof of Theorem 1

The proof of Theorem 1 uses standard tools from the theory of lacunary series, that have been developed by Philipp [19], Berkes and Philipp [8], Fukuyama [13], Aistleitner [4] and others. The main ingredient is an a.s. invariance principle of Strassen [21], that contains the following result:

Lemma 1 ([21], Corollary 4.5) *Let $(Y_i, \mathcal{F}_i, i \geq 1)$ be a martingale difference sequence with finite fourth moments, let $V_M = \sum_{i=1}^{M} \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1})$ and assume $V_1 = \mathbb{E}Y_1^2 > 0$ and $V_M \to \infty$. Assume additionally*

$$\liminf_{M\to\infty} \frac{V_M}{r_M} \geq 1 \qquad \text{a.s.}$$

with some sequence of positive real numbers $r_M \to \infty$ such that

$$\sum_{M=1}^{\infty} \frac{(\log r_M)^{10}}{r_M^2} \mathbb{E}Y_M^4 < +\infty.$$

Then

$$\limsup_{M\to\infty} \frac{|Y_1 + \cdots + Y_M|}{\sqrt{2V_M \log\log V_M}} = 1 \quad \text{a.s.}$$

(to see how this lemma follows from Strassens Corollary 4.5 see [2], Lemma 11 and Corollary 3).

We will also use the following 2 lemmas:

Lemma 2 ([2], Lemma 1) *For any real numbers $s < t$ and $\lambda > 0$,*

$$\left| \int_s^t \cos(2\pi\lambda x)\ dx \right| \leq \frac{2}{\lambda}, \qquad \left| \int_s^t \sin(2\pi\lambda x)\ dx \right| \leq \frac{2}{\lambda}.$$

Lemma 3 ([8], Lemma 2.2) *For any $N_2 > N_1$ and the function f and the sequence $(n_k)_{k\geq 1}$ from Theorem 1*

$$\int_0^1 \left(\sum_{k=N_1}^{N_2} f(n_k x) \right)^4 dx \leq c(N_2 - N_1)^2$$

for some constant c.

Proof of Theorem 1: For any $i \geq 1$, write $m(i)$ for the largest integer in Δ_i, and $\mathbb{F}_i$ for the σ-field generated by the intervals

$$\left[r 2^{-\lceil \log_2 n_{m(i)} \rceil - \lceil \log_2 m(i) \rceil}, (r+1) 2^{-\lceil \log_2 n_{m(i)} \rceil - \lceil \log_2 m(i) \rceil} \right),$$

where r runs from 0 to $\lceil \log_2 n_{m(i)} \rceil + \lceil \log_2 m(i) \rceil - 1$ (and, for notational convenience, we write $\mathbb{F}_0$ for the trivial σ-field that contains only the empty set and the unit interval). For any $i \geq 1$ and $k \in \Delta_i$ set

$$\varphi_k(x) = \mathbb{E}\left(f(n_k x)|\mathbb{F}_i\right) - \mathbb{E}\left(f(n_k x)|\mathbb{F}_{i-1}\right).$$

Then trivially φ_k is $\mathbb{F}_i$-measurable and $\mathbb{E}(\varphi_k|\mathbb{F}_{i-1}) = 0$. Since the fluctuation of $f(n_k x)$, $k \in \Delta_i$ on any atom of $\mathbb{F}_i$ is at most

$$\begin{aligned} \left| \max_{x\in[0,1]} f'(n_k x) \right| 2^{-\lceil \log_2 n_{m(i)} \rceil - \lceil \log_2 m(i) \rceil} &\leq (1+2+3+5)\, 2\pi n_k 2^{-\lceil \log_2 n_k \rceil - \lceil \log_2 k \rceil} \\ &\leq 70 \cdot \frac{2 n_k}{n_k} \cdot \frac{2}{k} \\ &\leq 280 k^{-1} \end{aligned} \tag{19}$$

and (since f is continuous) we have

$$\left|\mathbb{E}\left(f(n_k x)|\mathbb{F}_i\right) - f(n_k x)\right| \ll k^{-1}.$$

(here and in the sequel, the implied constant in expressions $\ll$ must not depend on k, i, N etc.) On the other hand, for $k \in \Delta_i$, by (P1) and Lemma 2 (and since every block Δ_i starts with an odd integer)

$$\begin{aligned} \left|\mathbb{E}\left(f(n_k x)|\mathbb{F}_{i-1}\right)\right| &\leq 2\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}\right) \frac{1}{n_k}\, 2^{-\lceil \log_2 n_{m(i-1)} \rceil - \lceil \log_2 m(i-1) \rceil} \\ &\ll \frac{n_{m(i-1)}}{n_k} \cdot m(i-1) \\ &\ll (m(i-1))^{-2} \\ &\ll k^{-2} \end{aligned} \tag{20}$$

Combining (19) and (20) gives

$$|\varphi_k(x) - f(n_k x)| \ll k^{-1},$$

and hence, since

$$\left| \sum_{k=1}^N \varphi_k(x) - \sum_{k=1}^N f(n_k x) \right| \ll \sum_{k=1}^N k^{-1} \ll \log N$$

it is sufficient to prove

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \varphi_k(x)\right|}{\sqrt{2N\log\log N}} = \sqrt{\|f\|_2^2 + g(x)} \quad \text{a.e.} \tag{21}$$

instead of (11).

Set

$$Y_i = \sum_{k\in\Delta_i} \varphi_k(x), \quad i \geq 1.$$

Then $(Y_i, \mathbb{F}_i, i \geq 1)$ is a martingale difference sequence, since Y_i is $\mathbb{F}_i$-measurable (it is a sum of $\mathbb{F}_i$-measurable functions)

$$\mathbb{E}(Y_i|\mathbb{F}_{i-1}) = \mathbb{E}\left(\sum_{k\in\Delta_i} \varphi_k|\mathbb{F}_{i-1}\right) = \sum_{k\in\Delta_i} \mathbb{E}\left(\varphi_k|\mathbb{F}_{i-1}\right) = 0,$$

and clearly the fourth moments of Y_i are bounded (since Y_i is a finite sum of bounded functions). Define, like in Lemma 1,

$$V_M = \sum_{i=1}^M \mathbb{E}(Y_i^2|\mathbb{F}_{i-1}), \qquad M \geq 1.$$

To calculate the value of V_M we consider $\mathbb{E}(Y_i^2|\mathbb{F}_{i-1})$ for some fixed $i \geq 1$. By Minkowski's inequality

$$\begin{aligned}
&\left|\left(\mathbb{E}(Y_i^2|\mathbb{F}_{i-1})\right)^{1/2} - \left(\mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \middle|\mathbb{F}_{i-1}\right)\right)^{1/2}\right| \\
\leq\ & \left(\mathbb{E}\left(\left(\sum_{k\in\Delta_i} \varphi_k(x) - f(n_k x)\right)^2 \middle|\mathbb{F}_{i-1}\right)\right)^{1/2} \\
\ll\ & \sum_{k\in\Delta_i} k^{-1} \\
\ll\ & |\Delta_i| i^{-3} \\
\ll\ & i^{-2},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left|V_M - \sum_{i=1}^M \mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \middle|\mathbb{F}_{i-1}\right)\right| \\
\leq\ & \sum_{i=1}^M \left|\left(\mathbb{E}(Y_i^2|\mathbb{F}_{i-1})\right)^{1/2} - \left(\mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \middle|\mathbb{F}_{i-1}\right)\right)^{1/2}\right|.
\end{aligned}$$

$$
\cdot \left| \left(\mathbb{E}(Y_i^2|\mathbb{F}_{i-1})\right)^{1/2} + \left(\mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \Big| \mathbb{F}_{i-1}\right)\right)^{1/2} \right|
$$

$$
\ll \sum_{i=1}^{M} i^{-2} \cdot i^2 \ll M. \tag{22}
$$

We have

$$
\begin{aligned}
& \left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \\
= & \left(\sum_{k\in\Delta_i} (\cos 2\pi n_k x + \cos 4\pi n_k x - \cos 6\pi n_k x + \sin 10\pi n_k x)\right)^2 \\
= & \sum_{k_1,k_2\in\Delta_i} \Big(\cos 2\pi n_{k_1} x \cdot \cos 2\pi n_{k_2} x + 2\cos 2\pi n_{k_1} x \cdot \cos 4\pi n_{k_2} x \\
& \quad -2\cos 2\pi n_{k_1} x \cdot \cos 6\pi n_{k_2} + 2\cos 2\pi n_{k_1} x \cdot \sin 10\pi n_{k_2} x \\
& \quad +\cos 4\pi n_{k_1} x \cdot \cos 4\pi n_{k_2} x - 2\cos 4\pi n_{k_1} x \cdot \cos 6\pi n_{k_2} x \\
& \quad +2\cos 4\pi n_{k_1} x \cdot \sin 10\pi n_{k_2} x + \cos 6\pi n_{k_1} x \cdot \cos 6\pi n_{k_2} x \\
& \quad -2\cos 6\pi n_{k_1} x \cdot \sin 10\pi n_{k_2} x + \sin 10\pi n_{k_1} x \cdot \sin 10\pi n_{k_2} x \Big) \\
= & \sum_{k_1,k_2\in\Delta_i} \sum_{(u,v)\in\{(1,1),(2,2),(3,3)\}} \frac{1}{2}\Big(\cos 2\pi(u n_{k_1} + v n_{k_2})x \\
& \qquad + \cos 2\pi(u n_{k_1} - v n_{k_2})x \Big) \\
& + \sum_{k_1,k_2\in\Delta_i} \sum_{(u,v)\in\{(1,2),(1,3),(2,3)\}} (-1)^{(\mathbb{1}(v=3))} \Big(\cos 2\pi(u n_{k_1} + v n_{k_2})x \\
& \qquad + \cos 2\pi(u n_{k_1} - v n_{k_2})x \Big) \\
& + \sum_{k_1,k_2\in\Delta_i} \sum_{(u,v)\in\{(1,5),(2,5),(3,5)\}} (-1)^{(\mathbb{1}(u=3))} \Big(\sin 2\pi(u n_{k_1} + v n_{k_2})x \\
& \qquad - \sin 2\pi(u n_{k_1} - v n_{k_2})x \Big) \\
& + \sum_{k_1,k_2\in\Delta_i} \sum_{(u,v)\in\{(5,5)\}} \frac{1}{2}\Big(-\cos 2\pi(u n_{k_1} + v n_{k_2})x \\
& \qquad + \cos 2\pi(u n_{k_1} - v n_{k_2})x \Big) \\
= & \sum_{k_1,k_2\in\Delta_i} \sum_{(u,v)\in\{(1,1),(2,2),(3,3)\}} \frac{1}{2} (\cos 2\pi(u n_{k_1} + v n_{k_2})x)
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
&+\sum_{k_1,k_2\in\Delta_i}\ \sum_{(u,v)\in\{(1,2),(1,3),(2,3)\}} (-1)^{(\mathbb{1}(v=3))}\left(\cos 2\pi(un_{k_1}+vn_{k_2})x\right) && (24)\\
&+\sum_{k_1,k_2\in\Delta_i}\ \sum_{(u,v)\in\{(1,5),(2,5),(3,5)\}} (-1)^{(\mathbb{1}(u=3))}\left(\sin 2\pi(un_{k_1}+vn_{k_2})x\right) && (25)\\
&+\sum_{k_1,k_2\in\Delta_i}\ \sum_{(u,v)\in\{(5,5)\}} \frac{1}{2}\left(-\cos 2\pi(un_{k_1}+vn_{k_2})x\right) && (26)\\
&+\sum_{k_1,k_2\in\Delta_i}\ \sum_{(u,v)\in\{(1,1),(2,2),(3,3)\}} \frac{1}{2}\left(\cos 2\pi(un_{k_1}-vn_{k_2})x\right) && (27)\\
&+\sum_{k_1,k_2\in\Delta_i}\ \sum_{(u,v)\in\{(1,2),(1,3),(2,3))\}} (-1)^{(\mathbb{1}(v=3))}\left(\cos 2\pi(un_{k_1}-vn_{k_2})x\right) && (28)\\
&+\sum_{k_1,k_2\in\Delta_i}\ \sum_{(u,v)\in\{(1,5),(2,5),(3,5)\}} (-1)^{(\mathbb{1}(u=3))}\left(-\sin 2\pi(un_{k_1}-vn_{k_2})x\right) && (29)\\
&+\sum_{k_1,k_2\in\Delta_i}\ \sum_{(u,v)\in\{(5,5)\}} \frac{1}{2}\left(\cos 2\pi(un_{k_1}-vn_{k_2})x\right) && (30)\\
=\ & U_i(x)+W_i(x),
\end{aligned}
$$

where U_i is the sum of all the trigonometric functions with frequencies at least $\min_{k\in\Delta_i} n_k/10$, and W_i is the sum of all trigonometric functions with frequencies smaller than $\min_{k\in\Delta_i}/10$ (in the above forumlas, $\mathbb{1}(\cdot)$ stands for the indicator function, i.e. $(-1)^{\mathbb{1}(u=3)}$ gives -1 if $u=3$ and 1 if $u\neq 3$). In particular, U_i contains all the trigonometric functions in (23), (24), (25) and (26). A trigonometric function from (27), (28), (29) or (30) is contained in W_i (i.e. it has a frequency less than $\min_{k\in\Delta_i} n_k/10$) if and only if the corresponding values of (k_1,k_2) and (u,v) satisfy

$$
|un_{k_1}-vn_{k_2}| < \frac{\min_{k\in\Delta_i} n_k}{10}. \qquad (31)
$$

The solutions of (31) for different values of (u,v) values is described in (P5), and the only solutions are those, which are also mentioned in (P2), (P3) and (P4), resp. Thus, writing W_i in the form

$$
W_i = c_0 + \sum_{j=1}^{\infty}\left(c_j\cos 2\pi jx + d_j \sin 2\pi jx\right),
$$

we have

$$
\begin{aligned}
c_j \ =\ & \frac{1}{2}\,\#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i,\ (u,v)\in\{(1,1),(2,2),(3,3),(5,5)\}\ :\ |un_{k_1}-vn_{k_2}|=j\right\}\\
& \#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i,\ (u,v)\in\{(1,2)\}\ :\ |un_{k_1}-vn_{k_2}|=j\right\}\\
& -\#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i,\ (u,v)\in\{(1,3),(2,3)\}\ :\ |un_{k_1}-vn_{k_2}|=j\right\}, \qquad j\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
d_j \ =\ & \#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i,\ (u,v)\in\{(1,5),(2,5)\}\ :\ un_{k_1}-vn_{k_2}=-j\right\}\\
& +\#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i,\ (u,v)\in\{(3,5)\}\ :\ un_{k_1}-vn_{k_2}=j\right\}\\
& -\#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i,\ (u,v)\in\{(1,5),(2,5)\}\ :\ un_{k_1}-vn_{k_2}=j\right\}\\
& -\#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i,\ (u,v)\in\{(3,5)\}\ :\ un_{k_1}-vn_{k_2}=j\right\}, \qquad j\geq 1,
\end{aligned}
$$

and, more precisely, in view of (P2), (P3), (P4) and (P5),

$$\begin{aligned}
c_j &= \#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i:\ n_{k_1}-2n_{k_2}=j\right\}\\
&\quad -\#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i:\ n_{k_1}-3n_{k_2}=j\right\}\\
&= \left\lfloor \max\{2a_j,0\}i^2\right\rfloor - \left\lfloor \max\{-2a_j,0\}i^2\right\rfloor\\
&= \operatorname{sgn}(a_j)\left\lfloor |2a_j| i^2\right\rfloor, \qquad 1\le j\le \lceil\log\log i\rceil,\\
c_0 &= \frac{4|\Delta_i|}{2}+\operatorname{sgn}(a_0)\left\lfloor 2|a_0|i^2\right\rfloor\\
&= 2i^2+\operatorname{sgn}(a_0)\left\lfloor 2|a_0|i^2\right\rfloor,\\
d_j &= \#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i:\ n_{k_1}-5n_{k_2}=-j\right\}\\
&\quad -\#\left\{(k_1,k_2)\in\Delta_i\times\Delta_i:\ n_{k_1}-5n_{k_2}=j\right\}\\
&= \operatorname{sgn}(b_j)\left\lfloor |2b_j| i^2\right\rfloor, \qquad 1\le j\le \lceil\log\log i\rceil,\\
c_j &= 0, \qquad j>\lceil\log\log i\rceil,\\
d_j &= 0, \qquad j>\lceil\log\log i\rceil.
\end{aligned}$$

Thus

$$\begin{aligned}
W_i &= 2i^2+\operatorname{sgn}(a_0)\left\lfloor 2|a_0|i^2\right\rfloor\\
&\quad +\sum_{j=1}^{\lceil\log\log i\rceil}\operatorname{sgn}(a_j)\left\lfloor |2a_j|i^2\right\rfloor\cos 2\pi jx\\
&\quad +\sum_{j=1}^{\lceil\log\log i\rceil}\operatorname{sgn}(b_j)\left\lfloor |2b_j|i^2\right\rfloor\sin 2\pi jx\\
&= 2i^2+2a_0\ i^2\\
&\quad +\sum_{j=1}^{\lceil\log\log i\rceil}2a_j\ i^2\cos 2\pi jx\\
&\quad +\sum_{j=1}^{\lceil\log\log i\rceil}2b_j\ i^2\sin 2\pi jx\\
&\quad +S_i,
\end{aligned}$$

where

$$|S_i|\le |a_0|+\sum_{j=1}^{\lceil\log\log i\rceil}2\ll\log\log i.$$

U_i is a sum of at most $\ll|\Delta_i|^2$ trigonometric functions with frequencies at least $\min_{k\in\Delta_i}n_k/10$, and thus by Lemma 2 and property (P1)

$$\begin{aligned}
\mathbb{E}(U_i|\mathbb{F}_{i-1}) &\ll |\Delta_i|^2\frac{2}{n_k}\,2^{-\lceil\log_2 n_{m(i-1)}\rceil-\lceil\log_2 m(i-1)\rceil}\\
&\ll i^4\cdot i^{-9}\cdot i\ll i^{-4}. \qquad (32)
\end{aligned}$$

On the other hand,

$$\mathbb{E}(W_i|\mathbb{F}_{i-1}) = W_i + R_i,$$

where by Lemma 2

$$|R_i| = |\mathbb{E}(W_i|\mathbb{F}_{i-1}) - W_i| \ll |\Delta_i|(\log\log i) 2^{-\lceil \log_2 n_{m(i)} \rceil - \lceil \log_2 m(i) \rceil} \ll i^{-4}. \tag{33}$$

Thus for $M \geq 1$ by (22)

$$\begin{aligned}
V_M &= \sum_{i=1}^{M} \mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \middle| \mathbb{F}_{i-1}\right) \\
&\quad + \left(V_M - \sum_{i=1}^{M} \mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \middle| \mathbb{F}_{i-1}\right)\right) \\
&= \sum_{i=1}^{M} \mathbb{E}(U_i + W_i|\mathbb{F}_{i-1}) + \left(V_M - \sum_{i=1}^{M} \mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \middle| \mathbb{F}_{i-1}\right)\right) \\
&= \sum_{i=1}^{M} W_i + \widetilde{R}_i,
\end{aligned}$$

where

$$\widetilde{R}_i = \mathbb{E}(U_i|\mathbb{F}_{i-1}) + R_i + \left(V_M - \sum_{i=1}^{M} \mathbb{E}\left(\left(\sum_{k\in\Delta_i} f(n_k x)\right)^2 \middle| \mathbb{F}_{i-1}\right)\right)$$

and by (22), (32), (33)

$$\left|\widetilde{R}_i\right| \ll M.$$

For arbitrary $M \geq 1$,

$$\begin{aligned}
&\sum_{i=1}^{M} (W_i - S_i) \\
&= \sum_{i=1}^{M} \left(2i^2 + 2a_0\ i^2 + \sum_{j=1}^{\lceil \log\log i \rceil} \left(2a_j\ i^2 \cos 2\pi j x + 2b_j\ i^2 \sin 2\pi j x\right)\right) \\
&= \sum_{i=1}^{M} \left(2i^2 + 2a_0\ i^2\right) \\
&\quad + \sum_{i\geq 1:\ \lceil \log\log i \rceil \geq \lceil \log\log M \rceil - 1} \ \sum_{j=1}^{\lceil \log\log M \rceil - 1} \left(2a_j\ i^2 \cos 2\pi j x + 2b_j\ i^2 \sin 2\pi j x\right) \\
&\quad + \sum_{i\geq 1:\ \lceil \log\log i \rceil < \lceil \log\log M \rceil - 1} \ \sum_{j=1}^{\lceil \log\log i \rceil} \left(2a_j\ i^2 \cos 2\pi j x + 2b_j\ i^2 \sin 2\pi j x\right) && (34) \\
&\quad + \sum_{i\geq 1:\ \lceil \log\log i \rceil = \lceil \log\log M \rceil - 1} \Big(2a_{(\lceil \log\log M \rceil)}\ i^2 \cos 2\pi j x && (35)
\end{aligned}$$

$$+2b_{(\lceil \log\log M \rceil)} \; i^2 \sin 2\pi j x \Big), \tag{36}$$

(here $a_{(\lceil \log\log M \rceil)}$ is the coefficient a_j for index $j = \lceil \log\log M \rceil$) where (34) is at most

$$\sum_{i \leq M^{1/e}} \sum_{j=1}^{\lceil \log\log i \rceil} \left(2a_j \; i^2 + 2b_j \; i^2\right) = o(M^3)$$

and the sum in (35), (36) is $o(M^3)$ since $a_j \to 0$, $b_j \to 0$ as $j \to \infty$. Define

$$\begin{aligned} & p_{(\lceil \log\log M \rceil - 1)}(x) \\ = \;\; & a_0 + \sum_{i \geq 1:\; \lceil \log\log i \rceil \geq \lceil \log\log M \rceil - 1} \; \sum_{j=1}^{\lceil \log\log M \rceil - 1} \left(a_j \cos 2\pi j x + b_j \sin 2\pi j x\right). \end{aligned}$$

Then

$$\begin{aligned} V_M &= \sum_{i=1}^{M} \left(W_i + \widetilde{R}_i\right) \\ &= \sum_{i=1}^{M} \left((W_i - S_i) + S_i + \widetilde{R}_i\right) \\ &= \left(\sum_{i=1}^{M} (W_i - S_i)\right) + o\left(M^3\right) \\ &= \sum_{i=1}^{M} \left(2i^2 + 2i^2 p_{(\lceil \log\log M \rceil - 1)}(x)\right) + o\left(M^3\right), \end{aligned}$$

and

$$\frac{V_M}{\sum_{i=1}^{M} 2i^2} = 2 + p_{(\lceil \log\log M \rceil - 1)}(x).$$

By Carleson's theorem (cf. e.g. Arias de Reyna [6] or Mozzochi [18])

$$p_{(\lceil \log\log M \rceil - 1)} \to g \qquad \text{a.e.} \quad \text{as} \quad M \to \infty,$$

and therefore

$$\frac{V_M}{\sum_{i=1}^{M} 2i^2} \to 2 + g(x) \qquad \text{a.e.} \tag{37}$$

By Minkowski's inequality, Hölder's inequality and Lemma 3

$$\begin{aligned} \left(\mathbb{E} Y_i^4\right)^{1/4} &= \left(\mathbb{E} \left(\sum_{k \in \Delta_i} \varphi_k(x)\right)^4\right)^{1/4} \\ &\leq 2 \left(\mathbb{E} \left(\sum_{k \in \Delta_i} f(n_k x)\right)^4\right)^{1/4} \ll |\Delta_i|^{1/2} \ll i. \end{aligned}$$

We apply Strassen's Lemma 1 with $r_M = \varepsilon M^3$ for some sufficiently small $\varepsilon > 0$, such that

$$\liminf_{M\to\infty} \frac{V_M}{\varepsilon M^3} \geq 1 \quad \text{a.e.}$$

(such an $\varepsilon > 0$ exists because of (37)), and since

$$\sum_{M=1}^{\infty} \frac{(\log r_M)^{10}}{r_M^2} \mathbb{E} Y_M^4 \ll \sum_{M=1}^{\infty} (\log M)^{10} M^{-6} M^4 < \infty$$

we get

$$\limsup_{M\to\infty} \frac{\left|\sum_{i=1}^{M} Y_i\right|}{\sqrt{2 V_M \log\log V_M}} = 1 \qquad \text{a.e.} \tag{38}$$

Since by the Carleson-Hunt inequality (cf. again [6],[18]) and Lemma 3

$$\begin{aligned} & \left\| \max_{N\geq 1} \left| \sum_{k\in\Delta_M,\ k\leq N} \varphi_k(x) \right| \right\|_4 \\ \ll & \left\| \max_{N\geq 1} \left| \sum_{k\in\Delta_M,\ k\leq N} f(n_k x) \right| \right\|_4 + |\Delta_M| M^{-3} \\ \ll & \left\| \sum_{k\in\Delta_M} f(n_k x) \right\|_4 + M^{-1} \\ \ll & \ M, \end{aligned}$$

the Borel-Cantelli lemma and the Markov inequality imply

$$\limsup_{M\to\infty} \frac{\max_{N\geq 1} \left|\sum_{k\in\Delta_M, k\leq N} \varphi_k(x)\right|}{\sqrt{2N\log\log N}} = 0 \quad \text{a.e.,}$$

which together with (37) and (38) implies

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} \varphi_k(x)\right|}{\sqrt{2N\log\log N}} = \sqrt{2 + g(x)} \quad \text{a.e.}$$

Thus we have shown (21), which is sufficient to prove Theorem 1.

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