On the law of the iterated logarithm for the discrepancy of lacunary sequences II

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Abstract

By a classical heuristics, lacunary function systems exhibit many asymptotic properties which are typical for systems of independent random variables. For example, for a large class of functions f the system $(f(n_k x))_{k \geq 1}$, where $(n_k)_{k \geq 1}$ is a lacunary sequence of integers, satisfies a law of the iterated logarithm (LIL) of the form

$$c_1 \leq \limsup_{N \to \infty} \frac{\sum_{k=1}^N f(n_k x)}{\sqrt{2N \log \log N}} \leq c_2 \qquad \text{a.e.,} \tag{1}$$

where c_1, c_2 are appropriate positive constants. In a previous paper we gave a criterion, formulated in terms of the number of solutions of certain linear Diophantine equations, which guarantees that the value of the lim sup in (1) equals the L^2-norm of f for a.e. x, which is exactly what one would also expect in the case of i.i.d. random variables. This result can be used to prove a precise LIL for the discrepancy of $(n_k x)_{k \geq 1}$, which corresponds to the Chung-Smirnov LIL for the Kolmogorov-Smirnov-statistic of i.i.d. random variables.

In the present paper we give a full solution of the problem in the case of "stationary" Diophantine behavior, by this means providing an unifying explanation of the aforementioned "regular" LIL behavior and the "irregular" LIL behavior which has been observed by Kac, Erdős, Fortet and others.

1 Introduction and statement of results

By a classical heuristics lacunary function systems fulfill many limit theorems for systems of independent, identically distributed (i.i.d.) random variables, such as the central limit theorem (CLT), the law of the iterated logarithm (LIL), convergence results, almost sure invariance principles etc. We have investigated the problem concerning the LIL in an earlier paper [4]; for a general introduction to the topic we refer the reader to our survey article [6].

Let $(n_k)_{k \geq 1}$ be a sequence of positive integers satisfying

$$\frac{n_{k+1}}{n_k} \geq q > 1, \qquad k \geq 1. \tag{2}$$

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Then by a result of Erdős and Gál [10]

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \cos 2\pi n_k x\right|}{\sqrt{2N\log\log N}} = \frac{1}{\sqrt{2}} \qquad \text{a.e.} \tag{3}$$

Observe that

$$\|\cos 2\pi\cdot\| := \left(\int_0^1 (\cos 2\pi x)^2\ dx\right)^{1/2} = \frac{1}{\sqrt{2}},$$

and hence the LIL in (3) is in perfect accordance with the LIL for i.i.d. random variables (where the standard deviation appears on the right-hand side). However, this analogy is not perfect if $\cos 2\pi x$ is replaced by a general 1-periodic function $f(x)$. In fact, the precise LIL of the form (3) may even fail for trigonometric polynomials: by an example of Erdős and Fortet (see [17]) we have for $p(x) = \cos 2\pi x + \cos 4\pi x$ and $n_k = 2^k + 1,\ k \geq 1$

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N p(n_k x)}{\sqrt{2N\log\log N}} = \sqrt{2}|\cos\pi x| \qquad \text{a.e.,} \tag{4}$$

and by an observation of Fortet [11] (see also Kac [16] and Maruyama [19]) for a large class of 1-periodic functions f and for $n_k = 2^k,\ k \geq 1$,

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N\log\log N}} = \sigma \qquad \text{a.e.,} \tag{5}$$

where

$$\sigma^2 = \|f\|^2 + 2\sum_{j=1}^{\infty}\int_0^1 f(x) f\left(2^j x\right)\ dx$$

(observe that in this case the number σ will in general be different from $\|f\|$). There is a significant difference between (4) and (5): while the lim sup in (5) still equals a constant a.e., this is not true in (4). This difference can be explained by considering the Diophantine structure of the sequences $(2^k)_{k\geq1}$ and $(2^k+1)_{k\geq1}$: for the sequence $n_k = 2^k,\ k \geq 1$, only Diophantine equations of the (homogeneous) form

$$2^\nu n_{k_1} - n_{k_2} = 0$$

have many solutions (k_1, k_2) (for arbitrary, fixed $\nu \geq 1$), while the Diophantine equations for which many solutions exist in the case $n_k = 2^k + 1,\ k \geq 1$, are of the (inhomogeneous) form

$$2^\nu n_{k_1} - n_{k_2} = 2^\nu - 1.$$

Generally it can be said that the probabilistic behavior of systems $(f(n_k x))_{k\geq1}$ is particularly similar to the behavior of i.i.d. random systems if the number of solutions (k_1, k_2) of linear Diophantine equations of the form

$$a n_{k_1} \pm b n_{k_2} = c \tag{6}$$

is "small" (see [5, 7, 15]), while "irregular" probabilistic behavior as in (4) may occur if the number of solutions of such Diophantine equations is "large" (see [3, 8, 14]). This also carries over to the LIL for the discrepancy of $(n_k x)_{k\geq1}$: in [4] we showed that if the number of solutions (k_1, k_2), $k_1, k_2 \leq N$, of equations of the form (6) is bounded by $\mathcal{O}\left(N(\log N)^{-1-\delta}\right)$ for some $\delta > 0$, then

$$\limsup_{N\to\infty} \frac{ND_N(n_kx)}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad \text{a.e.} \tag{7}$$

(note that the constant $1/2$ on the right-hand side of (7) is the same as in the Chung-Smirnov LIL for i.i.d. random variables, see [21, p.504]). On the other hand Fukuyama [12] proved that

$$\limsup_{N\to\infty} \frac{ND_N(2^kx)}{\sqrt{2N\log\log N}} = \frac{\sqrt{42}}{9} \qquad \text{a.e.,} \tag{8}$$

and there even exist lacunary sequences $(n_k)_{k\geq 1}$ for which a non-constant function $\psi(x)$ appears on the right-hand side of (7) instead of the number $1/2$ (see [1, 2, 14]).

The purpose of this paper is to give a unifying explanation of these phenomena. More precisely, we will provide exact formulas for the LIL for $f(n_kx)$ and $D_N(n_kx)$ in the case when the relative number of solutions of Diophantine equations of the form (6) converges to appropriate coefficients at a certain speed, i.e. if there exist numbers $\gamma_{j_1,j_2,\nu}$ such that

$$\frac{\#\left\{(k_1,k_2),\ (j_1,k_1)\neq(j_2,k_2),\ 1\leq k_1,k_2\leq N:\ j_1n_{k_1}-j_2n_{k_2}=\nu\right\}}{N} \to \gamma_{j_1,j_2,\nu} \tag{9}$$

as $N\to\infty$, sufficiently fast. Our result covers all the aforementioned examples, and gives a complete solution of the problem in the case of "stationary" Diophantine behavior (i.e. in the case when the quotients on the left-hand side of (9) converge sufficiently fast; if these quotients do not converge at all the situation can be extraordinarily complicated, and as far as we know there exist no results at all for this case).

For $j_1, j_2, N \geq 1$ and $\nu \in \mathbb{Z}$ set

$$S(j_1,j_2,\nu,N) := \#\left\{(k_1,k_2),\ (j_1,k_1)\neq(j_2,k_2),\ 1\leq k_1,k_2\leq N:\ j_1n_{k_1}-j_2n_{k_2}=\nu\right\}. \tag{10}$$

We say that $(n_k)_{k\geq 1}$ satisfies condition $\mathbf{D}_d$ if there exist real numbers $\gamma_{j_1,j_2,\nu}$ such that for $1 \leq j_1, j_2 \leq d$

$$\left|\frac{S(j_1,j_2,\nu,N)}{N} - \gamma_{j_1,j_2,\nu}\right| = \mathcal{O}\left(\frac{1}{(\log N)^{1+\delta}}\right) \tag{11}$$

for some $\delta > 0$, uniformly for $\nu \in \mathbb{Z}$. We say that $(n_k)_{k\geq 1}$ satisfies condition $\mathbf{D}$ if it satisfies $\mathbf{D}_d$ for every $d \geq 1$.

Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying (2) and condition $\mathbf{D}_d$. Let $p(x)$ be a trigonometric polynomial of the form

$$p(x) = \sum_{j=1}^{d} a_j \cos 2\pi jx + b_j \sin 2\pi jx. \tag{12}$$

Set

$$\sigma_p^2(x) \;=\; \|p\|^2 + \sum_{\nu=-\infty}^{\infty}\sum_{j_1,j_2=1}^{d} \frac{\gamma_{j_1,j_2,\nu}}{2}\Big((a_{j_1}a_{j_2}+b_{j_1}b_{j_2})\cos 2\pi\nu x \tag{13}$$

$$+ (b_{j_1} a_{j_2} - a_{j_1} b_{j_2}) \sin 2\pi\nu x \Big).$$

Let $f(x)$ be a function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\ dx = 0, \qquad \mathrm{Var}_{[0,1]} f < \infty, \tag{14}$$

and write

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx + b_j \sin 2\pi jx$$

for the Fourier series of f. For $(n_k)_{k\geq 1}$ satisfying (2) and condition $\mathbf{D}$ set

$$\begin{aligned} \sigma_f^2(x) &= \|f\|^2 + \sum_{\nu=-\infty}^{\infty} \sum_{j_1,j_2=1}^{\infty} \frac{\gamma_{j_1,j_2,\nu}}{2} \Big((a_{j_1} a_{j_2} + b_{j_1} b_{j_2}) \cos 2\pi\nu x \\ &\quad + (b_{j_1} a_{j_2} - a_{j_1} b_{j_2}) \sin 2\pi\nu x \Big). \end{aligned} \tag{15}$$

We will prove at the beginning of Section 2 that the limits in (13) and (15) are well-defined, provided the sequence $(n_k)_{k\geq 1}$ satisfies (2) and condition $\mathbf{D}_d$ and $\mathbf{D}$, respectively. We emphasize that the functions $\sigma_p(x)$ and $\sigma_f(x)$ depend on the numbers $\gamma_{j_1,j_2,\nu}$ and hence on the sequence $(n_k)_{k\geq 1}$.

Theorem 1 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying (2) and condition $\mathbf{D}_d$. Then*

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p(n_k x)\right|}{\sqrt{2N \log\log N}} = \sigma_p(x) \quad \text{a.e.} \tag{16}$$

As a consequence of Theorem 1 we obtain the following result for general functions f:

Theorem 2 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying (2) and condition $\mathbf{D}$, and let $f(x)$ be a function satisfying (14). Then*

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} = \sigma_f(x) \quad \text{a.e.}$$

The next theorem gives a similar result for the discrepancies D_N^* and D_N. For $0 \leq a \leq b \leq 1$ set

$$\mathbf{I}_{[a,b)}(x) = \mathbb{1}_{[a,b)}(\langle x \rangle) - (b-a),$$

where $\langle \cdot \rangle$ denotes the fractional part. For a finite sequence $(x_1, \dots, x_N)$ of real numbers the star-discrepancy D_N^* and the (extremal) discrepancy D_N of $(x_1, \dots, x_N)$ are defined as

$$D_N^*(x_1, \dots, x_N) := \sup_{0\leq a\leq 1} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[0,a)}(x_k)}{N} \right|$$

and

$$D_N(x_1, \dots, x_N) := \sup_{0\leq a\leq b\leq 1} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[a,b)}(x_k)}{N} \right|.$$

If $(x_k)_{k\geq 1}$ is an infinite sequence, we write $D_N(x_k)$ for $D_N(x_1,\ldots,x_N)$. For some fixed sequence $(n_k)_{k\geq 1}$ satisfying (2) and condition **D** we will write $\sigma_{\mathbf{I}_{[a,b)}}(x)$ for the function $\sigma_f(x)$ with $f=\mathbf{I}_{[a,b)}$, corresponding to (15). For general basic information on discrepancy theory (and the theory of uniform distribution modulo one) we refer the reader to [9] and [18].

Theorem 3 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying (2) and condition* **D**. *Then*

$$\limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N\log\log N}} = \sup_{0\leq a\leq 1} \sigma_{\mathbf{I}_{[0,a)}}(x) \qquad \text{a.e.} \tag{17}$$

and

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} = \sup_{0\leq a\leq b\leq 1} \sigma_{\mathbf{I}_{[a,b)}}(x) \qquad \text{a.e.} \tag{18}$$

As an application we show that our results are in accordance with the example of Erdős and Fortet (4):
Let $n_k=2^k-1$ and $p(x)=\cos 2\pi x+\cos 4\pi x$. Calculating the values of $\gamma_{j_1,j_2,\nu},\ 1\leq j_1,j_2\leq 2,\ \nu\in\mathbb{Z}$, for this sequence we get

$$\gamma_{j_1,j_2,\nu} = \begin{cases} 1 & \text{if} \quad j_1=1, j_2=2, \nu=1 \quad \text{or} \quad j_1=2, j_2=1, \nu=-1 \\ 0 & \text{otherwise} \end{cases}$$

Thus we have $\sigma_p(x)^2=1+\cos 2\pi x$, and hence (16) yields

$$\begin{aligned} \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p(n_k x)\right|}{\sqrt{2N\log\log N}} &= \sqrt{1+\cos 2\pi x} \quad \text{a.e.} \\ &= \sqrt{2}|\cos \pi x| \quad \text{a.e.,} \end{aligned}$$

which is the same as (4).

Remark 1: As in [4] we have to assume a bound of the form $\mathcal{O}\left((\log N)^{-1-\delta}\right)$ in our Diophantine condition. It is unclear how far this is from optimality. It is possible that the optimal condition is $o(1)$ (as in the case of the CLT, see [5]), but we have doubts that this actually is the case.

Remark 2: Obviously the coefficients $\gamma_{j_1,j_2,\nu}$ in (11) are symmetric in the sense that

$$\gamma_{j_1,j_2,\nu}=\gamma_{j_2,j_1,-\nu} \qquad \text{for any} \qquad j_1,j_2,\nu\in\mathbb{Z}.$$

Thus (13) and (15) can be rewritten in the form

$$\begin{aligned} \sigma_p^2(x) &= \|p\|^2 + \sum_{j_1,j_2=1}^d \gamma_{j_1,j_2,0} \\ &\quad + \sum_{\nu=1}^\infty \sum_{j_1,j_2=1}^d \gamma_{j_1,j_2,\nu}\left((a_{j_1}a_{j_2}+b_{j_1}b_{j_2})\cos 2\pi\nu x + (b_{j_1}a_{j_2}-a_{j_1}b_{j_2})\sin 2\pi\nu x\right) \end{aligned}$$

and

$$\sigma_f^2(x) = \|f\|^2 + \sum_{j_1,j_2=1}^\infty \gamma_{j_1,j_2,0}$$

$$+\sum_{\nu=1}^{\infty}\sum_{j_1,j_2=1}^{\infty}\gamma_{j_1,j_2,\nu}\left((a_{j_1}a_{j_2}+b_{j_1}b_{j_2})\cos 2\pi\nu x+(b_{j_1}a_{j_2}-a_{j_1}b_{j_2})\sin 2\pi\nu x\right),$$

respectively.

Remark 3: As mentioned before, we do not know of any results for a lacunary sequence $(n_k)_{k\geq 1}$ for which the quotients

$$\frac{\#\left\{(k_1,k_2),\ k_1\neq k_2,\ 1\leq k_1,k_2\leq N:\ j_1n_{k_1}-j_2n_{k_2}=\nu\right\}}{N}$$

are not convergent. By the properties of lacunary sequences these quotients are bounded (as $N\to\infty$), but they can converge to different numbers $\gamma_{j_1,j_2,\nu}$ along different subsequences of $\mathbb{N}$. In this situation it can happen that there exist several limiting functions $\sigma_f^{(m)}(x),\ 1\leq m\leq M$ along different subsequences, and that

$$\limsup_{N\to\infty}\frac{\sum_{k=1}^{N}f(n_kx)}{\sqrt{2N\log\log N}}=\max_{1\leq m\leq M}\sigma_f^{(m)}(x)\quad \text{a.e.},$$

but the situation can be even more complicated. It is hardly imaginable that a complete solution of the problem is possible in this general case.

Remark 4: Theorem 3 is a general LIL for the discrepancy of lacunary sequences, which includes several known results. However it can be extremely difficult to calculate the explicit value of the functions on the right-hand side of (17) and (18). For example, it is by no means easy to deduce Fukuyama's result (8) from (18), i.e. to show that for $n_k=2^k,\ k\geq 1$ we get $\sup_{0\leq a\leq 1}\sigma_{\mathbf{I}_{[0,a)}}(x)=\sup_{0\leq a\leq b\leq 1}\sigma_{\mathbf{I}_{[a,b)}}(x)=\frac{\sqrt{42}}{9}$ a.e.

2 Preliminaries

In this section we will show that the functions $\sigma_p(x)$ and $\sigma_f(x)$ in (13) and (15) are well-defined and bounded. This follows directly from the following

Lemma 1 *Assume that $(n_k)_{k\geq 1}$ satisfies (2) and condition $\mathbf{D}$, and $f(x)$ satisfies (14). Then*

$$\sum_{j_1,j_2=1}^{\infty}\sum_{\nu=-\infty}^{\infty}\gamma_{j_1,j_2,\nu}\left(|a_{j_1}a_{j_2}|+|b_{j_1}b_{j_2}|+|b_{j_1}a_{j_2}|+|a_{j_1}b_{j_2}|\right)<\infty.$$

Proof: By assumption we have $\mathrm{Var}_{[0,1]}f<K$ for some number K, which by [22, Vol. I, p. 48] implies

$$|a_j|\leq Kj^{-1},\quad |b_j|\leq Kj^{-1},\qquad j\geq 1. \tag{19}$$

We will show that for fixed $j_1\geq 1$ and $r\geq 0$

$$\sum_{j_1q^r\leq j_2<j_1q^{r+1}}\sum_{\nu\in\mathbb{Z}}\gamma_{j_1,j_2,\nu}\leq 1. \tag{20}$$

Together with (19) this would imply

$$\sum_{1\leq j_1\leq j_2\leq\infty}\sum_{\nu=-\infty}^{\infty}\gamma_{j_1,j_2,\nu}\left(|a_{j_1}a_{j_2}|+|b_{j_1}b_{j_2}|+|b_{j_1}a_{j_2}|+|a_{j_1}b_{j_2}|\right)$$

$$
\begin{aligned}
&\leq \ 4K \sum_{j_1=1}^{\infty} \sum_{r=0}^{\infty} \sum_{j_1 q^r \leq j_2 < j_1 q^{r+1}} \sum_{\nu=-\infty}^{\infty} \frac{\gamma_{j_1,j_2,\nu}}{j_1 j_2} \\
&\leq \ 4K \sum_{j=1}^{\infty} \sum_{r=0}^{\infty} \frac{1}{j^2 q^r} \\
&\leq \ \frac{8Kq}{q-1},
\end{aligned}
$$

which together with Remark 2 proves the lemma. Thus it remains to show (20).

Now assume that there exist some $j_1 \geq 1$ and $r \geq 0$ such that

$$
\sum_{j_1 q^r \leq j_2 < j_1 q^{r+1}} \sum_{\nu \in \mathbb{Z}} \gamma_{j_1,j_2,\nu} > 1. \tag{21}
$$

We will show that this leads to a contradiction. If (21) holds, then there has to exist a *finite* set of triplets $\left(j_1^{(i)}, j_2^{(i)}, \nu^{(i)}\right)$, $j_1^{(i)} q^r \leq j_2^{(i)} < j_1^{(i)} q^{r+1}$, $\nu^{(i)} \in \mathbb{Z}$, such that

$$
\sum_i \gamma_{j_1^{(i)}, j_2^{(i)}, \nu^{(i)}} > 1
$$

Let

$$
A = \bigcup_i \nu^{(i)}.
$$

Then A is finite. Since by (2) for $k_1 \neq k_2$ we have

$$
\frac{n_{k_1}}{n_{k_2}} \not\in [1/q, q],
$$

for sufficiently large k_1 is is not possible that there exist numbers j_1, j_2, j_3, satisfying $j_1^{(i)} q^r \leq j_2, j_3 < j_1^{(i)} q^{r+1}$ and two different indices k_2, k_3 such that for $\nu_1, \nu_2 \in A$

$$
j_1 n_{k_1} - j_2 n_{k_2} = \nu_1, \qquad j_1 n_{k_1} - j_3 n_{k_3} = \nu_2.
$$

But this clearly implies

$$
\sum_i \gamma_{j_1^{(i)}, j_2^{(i)}, \nu^{(i)}} \leq 1,
$$

which is in contradiction with (21). This proves the lemma. □

3 Proof of Theorem 1

The proof of Theorem 1 is somewhat similar to the proof of the main lemma (Lemma 2.4) of [4]. However, the situation is more difficult in the present case, and several adjustments and refinements are necessary.

Let $\varepsilon > 0$ be given. For simplicity of writing we consider only the case when $p(x)$ is an even function, i.e. when p is of the form

$$
p(x) = \sum_{j=1}^{d} a_j \cos 2\pi j x. \tag{22}
$$

The general case can be treated in exactly the same way; in fact, the only major difference is that in the general case $p(x) = \sum_{j=1}^{d} a_j \cos 2\pi jx + b_j \sin 2\pi jx$ the terms with small frequencies in equation (29) are of the form $(a_{j_1}a_{j_2} + b_{j_1}b_{j_2}) \cos 2\pi(j_1 n_{k_1} - j_2 n_{k_2})x + (b_{j_1}a_{j_2} - a_{j_1}b_{j_2}) \sin 2\pi(j_1 n_{k_1} - j_2 n_{k_2})x$, which is in perfect accordance with the definition of σ in (13).

For (22) by (13) we have

$$\sigma_p = \|p\|^2 + \sum_{\nu=-\infty}^{\infty} \sum_{j_1,j_2=1}^{d} \gamma_{j_1,j_2,\nu} \, (a_{j_1}a_{j_2}) \cos 2\pi\nu x.$$

We will assume that $\|p\| > 0$, since otherwise the theorem is trivial. We will also assume w.l.o.g. that $\|p\|_\infty \leq 1$ and $|a_j| \leq 1,\ 1 \leq j \leq d$. Throughout the rest of the paper C will denote positive constants, not always the same, depending (at most) on p, d and q, but not on i, k, N, etc.

We divide the set of positive integers into consecutive blocks

$$\Delta_1', \Delta_1, \Delta_2', \Delta_2, \ldots, \Delta_i', \Delta_i, \ldots$$

of lengths $\lceil 4 \log_q i \rceil$ and i, respectively. More precisely, for any $i \geq 1$ set

$$\Delta_i' = \left\{ k:\ 1 + \sum_{l<i} \left(\lceil 4 \log_q l \rceil + l \right) \leq k \leq \lceil 4 \log_q i \rceil + \sum_{l<i} \left(\lceil 4 \log_q l \rceil + l \right) \right\}$$

and

$$\Delta_i = \left\{ k:\ 1 + \lceil 4 \log_q i \rceil + \sum_{l<i} \left(\lceil 4 \log_q l \rceil + l \right) \leq k \leq i + \lceil 4 \log_q i \rceil + \sum_{l<i} \left(\lceil 4 \log_q l \rceil + l \right) \right\}.$$

Furthermore, set

$$\Delta = \bigcup_{i \geq 1} \Delta_i, \qquad \Delta' = \bigcup_{i \geq 1} \Delta_i'.$$

Then obviously $\Delta \cup \Delta' = \mathbb{N}$. Letting i^- and i^+ denote the smallest resp. largest integer in Δ_i, we have

$$\frac{n_{(i-1)^+}}{n_{i^-}} \leq q^{-4 \log_q i} = i^{-4}, \quad i \geq 2.$$

For every $k \in \Delta$, there exists a uniquely defined index i such that $k \in \Delta_i$. For every $k \in \Delta$, let $i = i(k)$ denote this index. Put $m(k) = \lceil \log_2 n_k + 2 \log_2 i \rceil$, and approximate $p(n_k x)$ by a discrete function $\varphi_k(x)$ such that the following properties are satisfied:

(P1) $\varphi_k(x)$ is $\mathcal{G}_i$-measurable
(P2) $\|\varphi_k(x) - p(n_k x)\|_\infty \leq C i^{-2}$
(P3) $\mathbb{E}(\varphi_k(x) | \mathcal{G}_{i-1}) = 0$

Here $\mathcal{G}_i$ denotes the σ-field generated by the intervals $[v 2^{-m(i^+)}, (v+1) 2^{-m(i^+)}),\ 0 \leq v < 2^{m(i^+)}$. The existence of such functions $\varphi_k(x)$ is explained in detail in the proof of [4, Lemma 2.4].

For $i \geq 1$, $k \in \Delta_i$ we define

$$\eta_k = \varepsilon i^{-1/2} \operatorname{sgn}\left(\cos 4\pi 2^{m(i^+)} x\right), \qquad \psi_k(x) = \varphi_k(x) + \eta_k(x), \tag{23}$$

and let $\mathcal{F}_i$ denote the σ-field generated by the intervals $[v2^{-m(i^+)-1}, (v+1)2^{-m(i^+)-1})$, $0 \leq v < 2^{m(i^+)+1}$. For notational convenience we also set $\eta_k \equiv 0$ for $k \in \Delta'$. Then (P1),(P2) and (P3) imply

(P1*) $\psi_k(x)$ is $\mathcal{F}_i$-measurable
(P2*) $\|\psi_k(x) - p(n_k x)\|_\infty \leq \varepsilon + Ci^{-2}$
(P3*) $\mathbb{E}(\psi_k(x)|\mathcal{F}_{i-1}) = 0$.

We set

$$Y_i = \sum_{k\in\Delta_i} \psi_k(x), \quad T_i = \sum_{k\in\Delta_i} p(n_k x), \quad T_i' = \sum_{k\in\Delta_i'} p(n_k x), \quad V_M = \sum_{i=1}^{M} \mathbb{E}(Y_i^2|\mathcal{F}_{i-1}).$$

Then $(Y_i, \mathcal{F}_i, i \geq 1)$ is a martingale difference sequence. The reason for using the functions ψ_k instead of φ_k (which was not necessary in [4]) is to guarantee that V_M is "not too small". In fact, it is easily seen that (23) implies

$$V_M \geq \sum_{k=1}^{M} \left(\frac{\varepsilon i^{1/2}}{2}\right)^2 \geq \frac{\varepsilon^2}{4} \frac{M(M-1)}{2}, \qquad M \geq 1. \tag{24}$$

By [7, Lemma 2.2], Minkowski's inequality and (P2*),

$$\mathbb{E}Y_M^4 \leq C|\Delta_M|^2 \leq CM^2,$$

where $|\Delta_M|$ denotes the number of elements of Δ_M. Thus by (24) and the trivial estimate

$$V_M \leq \sum_{i=1}^{M} |Y_i|^2 \leq C \sum_{i=1}^{M} |\Delta_i|^2 \leq CM^3$$

wo obtain

$$\sum_{M=1}^{\infty} \frac{(\log V_M)^{10}}{V_M^2} \mathbb{E}Y_M^4 \leq \sum_{M=1}^{\infty} C \frac{(\log M)^{10}}{M^2} < +\infty.$$

Hence by [1, Lemma 11]

$$\limsup_{M\to\infty} \frac{\left|\sum_{i=1}^{M} Y_i\right|}{\sqrt{2V_M \log\log V_M}} = 1 \quad \text{a.e.,}$$

which can be rewritten as

$$\limsup_{M\to\infty} \frac{\left|\sum_{1\leq k\leq M^+, k\in\Delta} (\varphi_k + \eta_k)\right|}{\sqrt{2V_M \log\log V_M}} = 1 \quad \text{a.e.}$$

We add the sum of the "short blocks" T'_i, for which by [20, Theorem 1] and Koksma's inequality (see [18, p. 143]),

$$\left|\sum_{i=1}^{M} T'_i\right| = \mathcal{O}\left(\sqrt{M(\log M)\log\log(M\log M)}\right) \quad \text{a.e.},$$

change from φ_k to $p(n_k x)$, which is possible by (P2), and get

$$\limsup_{M\to\infty} \frac{\left|\sum_{k=1}^{M^+} (p(n_k x) + \eta_k)\right|}{\sqrt{2V_M \log\log V_M}} = 1 \quad \text{a.e.}$$

Since

$$\sum_{k\in\Delta_i\cup\Delta'_i} |(p(n_k x) + \eta_k)| \le C\left(|\Delta_i| + |\Delta'_i|\right) \le Ci$$

it follows by (24) that

$$\limsup_{M\to\infty} \frac{\left|\max_{(M-1)^+ < N \le M^+} \ \sum_{k=1}^{N} (p(n_k x) + \eta_k)\right|}{\sqrt{2V_M \log\log V_M}} = 1 \quad \text{a.e.}$$

For $N \ge 1$ we define $M(N)$ as the index m, for which N is contained in $\Delta_m \cup \Delta'_m$. Then

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} (p(n_k x) + \eta_k)\right|}{\sqrt{2V_{M(N)} \log\log V_{M(N)}}} = 1 \quad \text{a.e.} \tag{25}$$

Finally, we want to replace $V_{M(N)}$ by $N\sigma_p(x)$. We choose a positive number A such that

$$\sum_{j_1,j_2=1}^{d} \sum_{|\nu|\ge A} \gamma_{j_1,j_2,\nu} \le \varepsilon, \tag{26}$$

which is always possible by Lemma 1. Set

$$\sigma_{p,A}^2(x) = \|p\|^2 + \sum_{j_1,j_2=1}^{d} \sum_{|\nu|\le A} \frac{\gamma_{j_1,j_2,\nu}}{2}\ a_{j_1} a_{j_2} \cos 2\pi\nu x.$$

Then by (26)

$$|\sigma_{p,A}(x)^2 - \sigma_p(x)^2| = \sum_{j_1,j_2=1}^{d} \sum_{|\nu|> A} \frac{\gamma_{j_1,j_2,\nu}}{2}\ a_{j_1} a_{j_2} \cos 2\pi\nu x \le \varepsilon. \tag{27}$$

We have

$$\begin{aligned} & T_i(x)^2 - \|p\|_2^2 |\Delta_i| \\ = \quad & \left(\sum_{k\in\Delta_i} p(n_k x)\ dx\right)^2 - \|p\|_2^2 |\Delta_i| \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{k\in\Delta_i}\sum_{j=1}^{d} a_j \cos 2\pi j n_k x\right)^2 - \left(\frac{1}{2}\sum_{i=1}^{d} a_j^2\right)|\Delta_i| \\
&= \sum_{\substack{1\le j_1,j_2\le d,\ k_1,k_2\in\Delta_i,\ (j_1,k_1)\neq(j_2,k_2),\\ 0\le|j_1n_{k_1}-j_2n_{k_2}|\le A}} \frac{1}{2}\, a_{j_1}a_{j_2}\cos 2\pi(j_1n_{k_1}-j_2n_{k_2})x \\
&\quad + \sum_{\substack{1\le j_1,j_2\le d,\ k_1,k_2\in\Delta_i,\\ A<|j_1n_{k_1}-j_2n_{k_2}|\le n_{(i-1)^+}}} \frac{1}{2}\, a_{j_1}a_{j_2}\cos 2\pi(j_1n_{k_1}-j_2n_{k_2})x \\
&\quad + \sum_{\substack{1\le j_1,j_2\le d,\ k_1,k_2\in\Delta_i,\\ n_{(i-1)^+}<|j_1n_{k_1}-j_2n_{k_2}|<n_{i^-}}} \frac{1}{2}\, a_{j_1}a_{j_2}\cos 2\pi(j_1n_{k_1}-j_2n_{k_2})x \\
&\quad + \sum_{\pm}\sum_{\substack{1\le j_1,j_2\le d,\ k_1,k_2\in\Delta_i,\\ n_{i^-}\le|j_1n_{k_1}\pm j_2n_{k_2}|}} \frac{1}{2}\, a_{j_1}a_{j_2}\cos 2\pi(j_1n_{k_1}\pm j_2n_{k_2})x \qquad (28)\\
&=: A_i(x)+U_i(x)+W_i(x)+R_i(x), \qquad (29)
\end{aligned}
$$

where the sum $\sum_\pm$ in (28) should be understood as a sum over both possible choices of the signs "+" and "−" in the second sum in (28) (note that for the sign "+" we *always* have $n_{i^-}\le j_1n_{k_1}+j_2n_{k_2}$, and thus R_i contains *all* frequencies of the form $j_1n_{k_1}+j_2n_{k_2}$).

Like in the proof of [4, Lemma 2.4] we can show

$$\left|\sum_{i=1}^{M}\mathbb{E}(R_i|\mathcal{F}_{i-1})\right| \le CM \qquad (30)$$

and

$$\left\|\sum_{i=1}^{M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\right\| \le CM^{3/2}. \qquad (31)$$

By the Diophantine condition $\mathbf{D}_d$ we have, for $1\le j_1,j_2\le d$,

$$|S(j_1,j_2,\nu,N)-\gamma_{j_1,j_2,\nu}N| \le C(\log N)^{-1-\delta}N, \qquad (32)$$

where S is defined in (10). We note that U_i is a sum of trigonometric functions with frequencies at most $n_{(i-1)^+}$, i.e.

$$U_i = \sum_{\nu=0}^{n_{(i-1)^+}} c_\nu \cos 2\pi\nu x,$$

where $\sum_\nu |c_\nu|\le C|\Delta_i|$. Hence the fluctuation of U_i on any atom of $\mathcal{F}_{i-1}$ is at most

$$\sum_{\nu=0}^{n_{(i-1)^+}} |c_\nu| 2\pi\nu 2^{-m((i-1)^+)-1} \le Ci\frac{n_{(i-1)^+}}{i^2 n_{(i-1)^+}} \le Ci^{-1},$$

and consequently

$$|\mathbb{E}(U_i|\mathcal{F}_{i-1})-U_i| \le Ci^{-1},$$

which gives

$$\left|\sum_{i=1}^{M} U_i(x) - \sum_{i=1}^{M} \mathbb{E}(U_i|\mathcal{F}_{i-1})\right| \leq C\log M. \tag{33}$$

By (32) we can decompose

$$\sum_{i=1}^{M} U_i(x) = \underbrace{\sum_{\nu=A+1}^{n_{(M-1)^+}} d_\nu \cos 2\pi\nu x}_{=:U_M^{(1)}} + \underbrace{\sum_{\nu=A+1}^{n_{(M-1)^+}} e_\nu \cos 2\pi\nu x}_{=:U_M^{(2)}},$$

where

$$|d_\nu| \leq \sum_{1\leq j_1,j_2\leq d} \sum_{|\nu|\geq A} \gamma_{j_1,j_2,\nu} M^2$$

and

$$|e_\nu| \leq C(\log M)^{-1-\delta} M^2, \qquad \sum_\nu |e_\nu| \leq CM^2. \tag{34}$$

Then by (26)

$$\left|U_M^{(1)}\right| \leq \varepsilon M^2,$$

and by (34)

$$\left\|U_M^{(2)}\right\| \leq \left(\sum_\nu |e_\nu|^2\right)^{1/2} \leq C(\log M)^{-1/2-\delta/2} M^2. \tag{35}$$

In the same way as (33) we can also show

$$\left|\sum_{i=1}^{M} A_i(x) - \sum_{i=1}^{M} \mathbb{E}(A_i|\mathcal{F}_{i-1})\right| \leq C\log M. \tag{36}$$

It is easy to see that for $1 \leq j_1, j_2 \leq d$ and for all ν, $|\nu| \leq A$,

$$\left|\sum_{i=1}^{M} \frac{\#\left\{k_1,k_2 \in \Delta_i,\ (j_1,k_1) \neq (j_2,k_2):\ j_1 n_{k_1} - j_2 n_{k_2} = \nu\right\}}{M^+} - \gamma_{j_1,j_2,\nu}\right| = \mathcal{O}\left(\frac{1}{(\log M)^{1+\delta}}\right),$$

i.e. that the contribution of the indices in Δ' is negligible. Using this observation we obtain

$$\left\|M^+\sigma_{p,A}^2 - \sum_{i=1}^{M} A_i(x)\right\| \leq C(\log M)^{-1/2-\delta/2}. \tag{37}$$

We choose an $\alpha > 0$ such that

$$\left(1+\frac{\delta}{2}\right)^{-1} < \alpha < \left(1+\frac{\delta}{4}\right)^{-1} \tag{38}$$

and define numbers

$$M_l = \lfloor 2^{(l^\alpha)} \rfloor, \quad l \geq 0,$$

and sets

$$S_l = \bigcup_{M_l \le M \le M_{l+1}} \left\{ x \in (0,1): \ |V_M - M^+ \sigma_p^2| > 2C^* \varepsilon M^2 \right\}, \quad l \ge 0,$$

where C^* (which denotes a positive constant) will be chosen later. We also define

$$S_l^* = \left\{ x \in (0,1): \ |V_{M_l} - M_l^+ \sigma_p^2| > C^* \varepsilon M_l^2 \right\}, \quad l \ge 0.$$

Since $\alpha < 1$ and since V_M and $M^+ \sigma_p^2$ grow at most polynomially in M, for all sufficiently large l

$$S_l \subset S_l^*. \tag{39}$$

By Hölder's inequality and (P2),

$$\begin{aligned}
& \left| V_M - \mathbb{E}\left(\sum_{k=1}^M T_i^2 | \mathcal{F}_{i-1} \right) \right| \\
\le \ & 2\mathbb{E}\left(\sum_{i=1}^M T_i \left(\sum_{k \in \Delta_i} \varphi_k - p(n_k x) + \eta_k \right) | \mathcal{F}_{i-1} \right) \\
& + \underbrace{\mathbb{E}\left(\sum_{i=1}^M \left(\sum_{k \in \Delta_i} \varphi_k - p(n_k x) + \eta_k \right)^2 | \mathcal{F}_{i-1} \right)}_{\le C\varepsilon^2 M^2} \\
\le \ & 2 \underbrace{\left(\mathbb{E}\left(\sum_{i=1}^M T_i^2 | \mathcal{F}_{i-1} \right) \right)^{1/2}}_{\le CM} \underbrace{\left(\mathbb{E}\left(\sum_{i=1}^M \left(\sum_{k \in \Delta_i} \varphi_k - p(n_k x) + \eta_k \right)^2 | \mathcal{F}_{i-1} \right) \right)^{1/2}}_{\le C\varepsilon M} + C\varepsilon^2 M^2 \\
\le \ & C\varepsilon M^2.
\end{aligned}$$

Using the decomposition

$$\sum_{i=1}^M T_i^2 = \|p\|_2^2 \sum_{i=1}^M |\Delta_i| + \sum_{i=1}^M A_i + \underbrace{U_M^{(1)}}_{\le \varepsilon M^2} + U_M^{(2)} + \sum_{i=1}^M W_i + \sum_{i=1}^M R_i$$

we have, using (27), (30), (31), (33), (36),

$$\begin{aligned}
& \left| V_M - M^+ \sigma_p^2 \right| \\
\le \ & \underbrace{\left(\|p\|_2^2 \left(M^+ - \sum_{i=1}^M |\Delta_i| \right) \right)}_{\le C\varepsilon M \log M} + \underbrace{\left(M^+ |\sigma_{p,A}^2 - \sigma_p^2| \right)}_{\le C\varepsilon M^2} + \left| M^+ \sigma_{p,A}^2 - \sum_{i=1}^M A_i \right| \\
& + \underbrace{\left| \sum_{i=1}^M A_i - \sum_{i=1}^M \mathbb{E}\left(A_i | \mathcal{F}_{i-1} \right) \right|}_{\le C \log M} + \underbrace{\left| \sum_{i=1}^M U_i - \sum_{i=1}^M \mathbb{E}\left(U_i | \mathcal{F}_{i-1} \right) \right|}_{\le C \log M} + \varepsilon M^2
\end{aligned} \tag{40}$$

$$+\left|U_M^{(2)}\right| + \underbrace{\left|\mathbb{E}\left(\sum_{i=1}^M W_i|\mathcal{F}_{i-1}\right)\right|}_{\leq CM^{3/2}} + \underbrace{\left|\mathbb{E}\left(\sum_{i=1}^M R_i|\mathcal{F}_{i-1}\right)\right|}_{\leq CM}$$

$$\leq \left|M^+\sigma_{p,A}^2 - \sum_{i=1}^M A_i\right| + \left|U_M^{(2)}\right| + C\varepsilon M^2. \tag{41}$$

We choose a constant C^* for which $C^* > C + 2$, where C is the constant in (41). Then, since by Chebyshev's inequality and (35), (37),

$$\mathbb{P}\left(\left|M^+\sigma_{p,A}^2 - \sum_{i=1}^M A_i\right| > \varepsilon M^2\right) \leq C\varepsilon^{-2}(\log M)^{-1-\delta}$$

and

$$\mathbb{P}\left(\left|U_M^{(2)}\right| > \varepsilon M^2\right) \leq C\varepsilon^{-2}(\log M)^{-1-\delta},$$

we obtain

$$\begin{aligned} & \mathbb{P}\left(\left|V_M - M^+\sigma_p^2\right| > C^*\varepsilon M^2\right) \\ \leq\ & \mathbb{P}\left(\left|M^+\sigma_{p,A}^2 - \sum_{i=1}^M A_i\right| > \varepsilon M^2\right) + \mathbb{P}\left(\left|U_M^{(2)}\right| > \varepsilon M^2\right) \\ \leq\ & C\varepsilon^{-2}(\log M)^{-1-\delta}. \end{aligned}$$

Thus

$$\mathbb{P}(S_l^*) \leq Cl^{-\alpha(1+\delta)}$$

and by (38)

$$\sum_{l=1}^\infty \mathbb{P}(S_l) < +\infty.$$

Thus the Borel-Cantelli Lemma implies that the set of those $x \in (0,1)$, which are contained in infinitely many sets $S_l^*, l \geq 1$, has Lebesgue measure 0, and by (39) the set of those x which are contained in infinitely many sets $S_l,\ l \geq 1$, also has measure zero. This implies

$$|V_M - M^+\sigma_p^2| \leq 2C^*\varepsilon M^2$$

for sufficiently large M for a.e. x. Together with (25) this implies

$$\sigma_p - \sqrt{2C^*\varepsilon} \leq \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N (p(n_k x) + \eta_k)\right|}{\sqrt{2N\log\log N}} \leq \sigma_p + \sqrt{2C^*\varepsilon} \quad \text{a.e.}$$

Since the functions η_{k_1} and η_{k_2} are independent for $k_1 \in \Delta_{i_1}, k_2 \in \Delta_{i_2}, i_1 \neq i_2$ (similar to the Rademacher function system), by Kolmogorov's law of the iterated logarithm

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \eta_k\right|}{\sqrt{2N\log\log N}} = \varepsilon \qquad \text{a.e.},$$

which implies

$$\sigma_p - \sqrt{2C^*\varepsilon} - \varepsilon \leq \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N (p(n_k x))\right|}{\sqrt{2N\log\log N}} \leq \sigma_p + \sqrt{2C^*\varepsilon} + \varepsilon \quad \text{a.e.}$$

Since ε was arbitrary, this proves Theorem 1.

4 Proof of Theorems 2 and 3

Again we assume for simplicity of writing that f is an even function. Let $f(x) = p(x) + r(x)$, where

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx, \quad p(x) = \sum_{j=1}^{d} a_j \cos 2\pi jx, \quad r(x) = \sum_{j=d+1}^{\infty} a_j \cos 2\pi jx$$

for some d. We clearly have

$$\begin{aligned}
\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} &\le \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p(n_k x)\right| + \left|\sum_{k=1}^N r(n_k x)\right|}{\sqrt{2N \log\log N}} \\
&\le \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p(n_k x)\right|}{\sqrt{2N \log\log N}} + \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N r(n_k x)\right|}{\sqrt{2N \log\log N}}. \qquad (42)
\end{aligned}$$

Similarly, we also have

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} \ge \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p(n_k x)\right|}{\sqrt{2N \log\log N}} - \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N r(n_k x)\right|}{\sqrt{2N \log\log N}}. \qquad (43)$$

By (19) and [4, Lemma 3.1],

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N r(n_k x)\right|}{\sqrt{2N \log\log N}} \le C d^{-1/4} \qquad \text{a.e.,}$$

for some constant C, and by Theorem 1

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p(n_k x)\right|}{\sqrt{2N \log\log N}} = \sigma_p \qquad \text{a.e.}$$

Thus, by (42) and (43),

$$\sigma_p - C d^{-1/4} \le \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N f(n_k x)\right|}{\sqrt{2N \log\log N}} \le \sigma_p + C d^{-1/4} \qquad \text{a.e.}$$

By Lemma 1 we have

$$\sigma_p \to \sigma_f \qquad \text{as} \qquad d \to \infty,$$

which proves Theorem 2. Theorem 3 follows directly from Theorem 2 and [13, Theorem 1].

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