Convergence of $\sum c_k f(kx)$ and the Lip α class

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Abstract

By Carleson's theorem a trigonometric series $\sum_{k=1}^{\infty} c_k \cos 2\pi kx$ or $\sum_{k=1}^{\infty} c_k \sin 2\pi kx$ is a.e. convergent if

$$\sum_{k=1}^{\infty} c_k^2 < \infty. \tag{1}$$

Gaposhkin generalized this result to series of the form

$$\sum_{k=1}^{\infty} c_k f(kx) \tag{2}$$

for functions f satisfying $f(x+1) = f(x)$, $\int_0^1 f(x) = 0$ and belonging to the Lip α class for some $\alpha > 1/2$. In the case $\alpha \leq 1/2$ condition (1) is in general no longer sufficient to guarantee the a.e. convergence of (2).
For $0 < \alpha < 1/2$ Gaposhkin showed that (2) is a.e. convergent if

$$\sum_{k=1}^{\infty} c_k^2 k^{1-2\alpha} (\log k)^{\beta} < \infty \qquad \text{for some} \qquad \beta > 1 + 2\alpha. \tag{3}$$

In this paper we show that condition (3) can be significantly weakened for $\alpha \in [1/4, 1/2)$. In fact, we show that in this case the factor $k^{1-2\alpha}(\log k)^{\beta}$ can be replaced by a factor which is asymptotically smaller than any positive power of k.

1 Introduction and statement of results

In 1966 Carleson showed that trigonometric series of the form

$$\sum_{k=1}^{\infty} c_k \cos 2\pi kx \qquad \text{or} \qquad \sum_{k=1}^{\infty} c_k \sin 2\pi kx$$

are a.e. convergent if

$$\sum_{k=1}^{\infty} c_k^2 < \infty \tag{4}$$

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(see [7] for Carleson's paper; cf. also the monographs of Mozzochi [15] and Arias de Reyna [4]). Gaposhkin [11] showed that (4) also implies the a.e. convergence of series of the form

$$\sum_{k=1}^{\infty} c_k f(kx), \tag{5}$$

where, here and throughout the paper, f is a measurable function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\, dx = 0,$$

and belonging to the Lip α class for some $\alpha > 1/2$. If the condition that $f \in \mathrm{Lip}_\alpha$ for some $\alpha > 1/2$ is dropped the convergence of the sum (4) will in general no longer be sufficient to guarantee the a.e. convergence of (5). One possibility to meet this fact is to consider series of the form

$$\sum_{k=1}^{\infty} c_k f(n_k x) \tag{6}$$

instead of (5), where $(n_k)_{k \geq 1}$ is a fast growing sequence of positive integers. A typical growth condition in this case is Hadamard's condition, requesting that

$$\frac{n_{k+1}}{n_k} > q, \quad k \geq 1, \qquad \text{for some} \qquad q > 1.$$

Under this condition (4) is still sufficient to have a.e. convergence of (6) for f belonging to the Lip_α class for any $\alpha > 0$ (Kac [13]; for recent results in the field cf. e.g. Fukuyama [9], Aistleitner and Berkes [2], Aistleitner [1]).

An other possibility to get a.e. convergence results for series of the form (5) is to impose a stronger condition than (4) on the sequence $(c_k)_{k \geq 1}$, typically depending on the modulus of continuity of f. For example, for $f \in \mathrm{Lip}_{1/2}$ Gaposhkin [10] proved the a.e. convergence of (5) under the condition

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^\beta < \infty, \qquad \beta > 3$$

(later, Berkes and Weber [6] showed that it is sufficient to assume $\beta > 2$).
On the other hand, Berkes [5] showed that there exist a function $f \in \mathrm{Lip}_{1/2}$ and a sequence $(c_k)_{k \geq 1}$ satisfying $\sum_{k=1}^{\infty} c_k^2 < \infty$ such that (5) is a.e. divergent.
In the case $0 < \alpha < 1/2$, Gaposhkin [10] showed that (5) is a.e. convergent, provided

$$\sum_{k=1}^{\infty} c_k^2 k^{1-2\alpha} (\log k)^\beta < \infty \qquad \text{for some} \qquad \beta > 1 + 2\alpha,$$

and Berkes proved that there exists a function $f \in \mathrm{Lip}_\alpha$ and a sequence $(c_k)_{k \geq 1}$ such that (5) is a.e. divergent, although

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^\gamma < \infty \qquad \text{for all} \qquad \gamma < 1 - 2\alpha.$$

Recapitulating these results, we see that there is a large gap between the necessary and sufficient conditions on $(c_k)_{k\geq 1}$ to guarantee the a.e. convergence of (5), where $f \in \mathrm{Lip}_\alpha$, $0 < \alpha \leq 1/2$. In the case $\alpha = 1/2$ the necessary and sufficient condition has to be somewhere between

$$\sum_{k=1}^{\infty} c_k^2 < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} c_k^2 (\log k)^\beta < \infty, \quad \beta > 2,$$

and in the case $0 < \alpha < 1/2$ between

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^\gamma < \infty, \quad \gamma \in [0, 1-2\alpha)$$

and

$$\sum_{k=1}^{\infty} c_k^2 k^{1-2\alpha} (\log k)^\beta < \infty, \quad \beta > 1+2\alpha.$$

Concerning this problem Berkes and Weber [6] wrote:

> It is possible that in the case $0 < \alpha < 1/2$ the condition $\sum_{k=1}^{\infty} c_k^2 (\log k)^\gamma < \infty$ for a suitable $\gamma > 0$ suffices for the a.e. convergence of $\sum_{k=1}^{\infty} c_k f(n_k x)$, but this remains open.

The purpose of this paper is to give a strong improvement of Gaposhkin's result for the case $\alpha \in [1/4, 1/2)$. We show that in this case for convergence in (2) the factor $k^{1-2\alpha}$ in (3) can be replaced by k^ε for any $\varepsilon > 0$, or even by

$$\exp\left(\frac{2 \log k}{\log \log k}\right) \tag{7}$$

(Remark: here, and in the sequel, we write $\exp(x)$ for e^x. Also, to simplify notation, we understand $\log x$ as $\max(1, \log x)$). Observe that the function in (7) is asymptotically smaller than any positive power of k.

More precisely, we will prove the following theorem:

Theorem 1 *Let $f \in \mathrm{Lip}_\alpha$ for some $\alpha \in [1/4, 1/2)$. Then*

$$\sum_{k=1}^{\infty} c_k f(kx)$$

converges a.e. provided

$$\sum_{k=1}^{\infty} c_k^2 \ \exp\left(\frac{2 \log k}{\log \log k}\right) < \infty.$$

We note that, despite our improvement, the exact best possible condition for $(c_k)_{k\geq 1}$ to imply a.e. convergence of (2) remains unknown. In particular, the function

$$\exp\left(\frac{2 \log k}{\log \log k}\right)$$

in our theorem grows faster than $(\log n)^\beta$ for any $\beta \in \mathbb{R}$. Therefore, the question whether

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^\beta < \infty$$

is sufficient to have the a.e. convergence of (2) remains open (cf. the aforementioned remark of Berkes and Weber).

2 Auxiliary results

Lemma 1 *Let $f \in \mathrm{Lip}_\alpha$ for some $\alpha \in [1/4, 1/2)$, and write*

$$f \sim \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos 2\pi j x + b_j \sin 2\pi j x.$$

Then for any $m \geq 1$

$$\sum_{j=2^m+1}^{2^{m+1}} \left(a_j^2 + b_j^2\right) \leq c\ 2^{-2m\alpha}$$

for some constant c.

This is formula (3.3) in Zygmund [16, p. 241]

Lemma 2 *Let J and N be positive integers and let $(n_k)_{1 \leq k \leq N}$ be a sequences of distinct non-zero integers. Then the number of solutions to*

$$j_1 n_{k_1} = j_2 n_{k_2}$$

with

$$1 \leq j_1, j_2 \leq J, \qquad 1 \leq k_1, k_2 \leq N$$

is bounded by

$$cJN \exp\left(\frac{5 \log N}{2 \log\log N}\right),$$

where c is a constant.

This is a special case of Harman and Dyer [8, Theorem 2] (also contained in Harmans monograph [12] as Theorem 3.9). We have already used this result in an earlier paper in a related context (cf. Aistleitner, Mayer and Ziegler [3]).

3 Preparations

Throughout the rest of the paper we will assume that f and $(c_k)_{k\geq 1}$ are fixed, and that $f \in \mathrm{Lip}_\alpha$ for some $\alpha \in [1/4, 1/2)$. W.l.o.g. we will assume that f is an even function, i.e. that the Fourier series of f is a pure cosine-series (the proof in the general case is exactly the same), and that $|f| \leq 1$ and $|c_k| \leq 1,\ k \geq 1$.
We write

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x \tag{8}$$

for the Fourier series of f and define

$$\|f(x)\| = \left(\int_0^1 f(x)^2 dx \right)^{1/2}.$$

Furthermore, we define

$$\tilde{f}(x) = \sum_{j=1}^{\infty} |a_j| \cos 2\pi j x.$$

Let $\mathcal{K}$ be a set of positive integers. Then by the orthogonality of the trigonometric system

$$\left\| \sum_{k \in \mathcal{K}} c_k f(kx) \right\| \leq \left\| \sum_{k \in \mathcal{K}} |c_k| \tilde{f}(kx) \right\|. \tag{9}$$

We will write c for appropriate positive constants, not always the same.

First we will prove the following

Lemma 3 *Let $(n_k)_{k \geq 1}$ be a strictly increasing sequence of positive integers, and let $M < N$ be positive integers. Let*

$$\mathcal{K} = \{k : \ n_k \in [M+1, N]\} \qquad \textit{and} \qquad K = \#\mathcal{K}.$$

Then

$$\left\| \sum_{k \in \mathcal{K}} \tilde{f}(n_k x) \right\| \leq c\sqrt{K} \ \exp\left(\frac{5 \log K}{4 \log\log K} \right).$$

Using this we can show

Lemma 4 *Let $M < N$ be positive integers. Let $L > 1$. Then*

$$\left\| \sum_{\substack{M < k \leq N \\ |c_k| \geq L^{-1}}} c_k f(kx) \right\| \leq c\sqrt{\log L} \ \exp\left(\frac{5 \log N}{4 \log\log N} \right) \left(\sum_{M < k \leq N} c_k^2 \right)^{1/2}.$$

This yields

Lemma 5 *Let $N_2 > N_1$ be positive integers. Then*

$$\left\| \max_{N_1 < M \leq N_2} \left| \sum_{N_1 < k \leq M} c_k f(kx) \right| \right\| \leq N_2^{-2} + c \ \exp\left(\frac{3 \log N_2}{2 \log\log N_2} \right) \left(\sum_{N_1 < k \leq N_2} c_k^2 \right)^{1/2}.$$

To deduce Theorem 1 from Lemma 5 we will finally need the following

Lemma 6 *Assume that for every given $\varepsilon > 0$ there exists an M_0 such that*

$$\left\| \max_{M > M_0} \left| \sum_{k=M_0+1}^{M} c_k f(kx) \right| \right\| \leq \varepsilon.$$

Then

$$\sum_{k=1}^{\infty} c_k f(kx)$$

is a.e. convergent.

Proof of Lemma 3: For $s = 0, 1, 2, \dots$ we define

$$r_s(x) = \sum_{j=2^s+1}^{2^{s+1}} |a_j| \cos 2\pi j x$$

(the a_j's are defined in (8)). Then by Minkowski's inequality

$$\left\| \sum_{k \in \mathcal{K}} \tilde{f}(n_k x) \right\| \leq \sum_{s \geq 0} \left\| \sum_{k \in \mathcal{K}} r_s(n_k x) \right\| \tag{10}$$

By Lemma 1 we have

$$\sum_{j=2^s+1}^{2^{s+1}} a_j^2 \leq c 2^{-2s\alpha}, \tag{11}$$

where c is a constant. Let $\eta \in [0, 1]$ (we will choose the exact value of η later). By Minkowski's inequality and the orthogonality of the trigonometric system

$$\begin{aligned}
& \left\| \sum_{k \in \mathcal{K}} r_s(n_k x) \right\| & (12) \\
\leq & \left\| \sum_{k \in \mathcal{K}} \sum_{\substack{2^{-2s\alpha - 2s\alpha(1-\eta)} < a_j^2, \\ 2^s < j \leq 2^{s+1}}} |a_j| \cos 2\pi j n_k x \right\| + \left\| \sum_{k \in \mathcal{K}} \sum_{\substack{2^{-2s\alpha - 2s\alpha(1-\eta)} \geq a_j^2, \\ 2^s < j \leq 2^{s+1}}} |a_j| \cos 2\pi j n_k x \right\| & \\
\leq & \left\| \sum_{\substack{2^{-2s\alpha - 2s\alpha(1-\eta)} < a_j^2, \\ 2^s < j \leq 2^{s+1}}} |a_j| \sum_{k \in \mathcal{K}} \cos 2\pi j n_k x \right\| + \left\| \sum_{k \in \mathcal{K}} \sum_{j=2^s+1}^{2^{s+1}} 2^{-s\alpha - s\alpha(1-\eta)} \cos 2\pi j n_k x \right\| & \\
\leq & \sum_{j:\ 2^{-2s\alpha - 2s\alpha(1-\eta)} < a_j^2} |a_j| \left\| \sum_{k \in \mathcal{K}} \cos 2\pi j n_k x \right\| & (13) \\
& + 2^{-2s\alpha + s\alpha\eta} \left\| \sum_{k \in \mathcal{K}} \sum_{j=2^s+1}^{2^{s+1}} \cos 2\pi j n_k x \right\|. & (14)
\end{aligned}$$

By the orthogonality of the trigonometric system and (11) the term (13) is bounded by

$$\begin{aligned}
\sqrt{K} \sum_{\substack{2^{-2s\alpha - 2s\alpha(1-\eta)} < a_j^2, \\ 2^s < j \leq 2^{s+1}}} |a_j| & \leq \sqrt{K} 2^{s\alpha + s\alpha(1-\eta)} \sum_{j=2^s+1}^{2^{s+1}} a_j^2 \\
& \leq c \sqrt{K} 2^{-s\alpha\eta}.
\end{aligned}$$

By Lemma 2 the term (14) is bounded by

$$\begin{aligned}
& 2^{-2s\alpha+s\alpha\eta}\left(\sum_{k_1,k_2\in\mathcal{K}}\sum_{j_1,j_2=2^s+1}^{2^{s+1}} \mathbb{1}(j_1 n_{k_1}=j_2 n_{k_2})\right)^{1/2} \\
\leq\ & 2^{-2s\alpha+s\alpha\eta} c 2^{s/2}\sqrt{K}\ \exp\left(\frac{5\log K}{4\log\log K}\right) \\
\leq\ & c\sqrt{K}2^{-s(2\alpha-1/2)+s\alpha\eta}\ \exp\left(\frac{5\log K}{4\log\log K}\right)
\end{aligned}$$

Therefore (12) is bounded by

$$c\sqrt{K}2^{-s\alpha\eta}+c\sqrt{K}2^{-s(2\alpha-1/2)+s\alpha\eta}\ \exp\left(\frac{5\log K}{4\log\log K}\right).$$

We choose

$$\eta = 1-\frac{1}{4\alpha},$$

and see that (12) is bounded by

$$c\sqrt{K}2^{-s\alpha+s/4}\ \exp\left(\frac{5\log K}{4\log\log K}\right).$$

By (10) this implies

$$\begin{aligned}
\left\|\sum_{k\in\mathcal{K}}\tilde{f}(n_k x)\right\| &\leq \sum_{s\geq 0} c\sqrt{K}2^{-s(\alpha-1/4)}\ \exp\left(\frac{5\log K}{4\log\log K}\right) \\
&\leq c\sqrt{K}\ \exp\left(\frac{5\log K}{4\log\log K}\right),
\end{aligned}$$

which proves Lemma 3. $\square$

Proof of Lemma 4 :
Let $M<N$ and L be given. By (9) we have

$$\begin{aligned}
\left\|\sum_{\substack{M<k\leq N\\|c_k|\geq L^{-1}}} c_k f(kx)\right\| &\leq \left\|\sum_{\substack{M<k\leq N\\|c_k|\geq L^{-1}}} |c_k|\tilde{f}(kx)\right\| \\
&\leq \sum_{s:\ 1\leq 2^s\leq L}\left\|\sum_{\substack{M<k\leq N\\2^{-s-1}\leq|c_k|<2^{-s}}} |c_k| f(kx)\right\| \\
&\leq \sum_{s:\ 1\leq 2^s\leq L} 2^{-s}\left\|\sum_{\substack{M<k\leq N\\2^{-s-1}\leq|c_k|<2^{-s}}} f(kx)\right\|.
\end{aligned}$$

Using Lemma 3 we get

$$\left\| \sum_{\substack{M<k\leq N \\ 2^{-s-1}\leq |c_k|<2^{-s}}} \tilde{f}(kx) \right\| \leq c\sqrt{K(s)}\ \exp\left(\frac{5\log K(s)}{4\log\log K(s)}\right),$$

where

$$K(s) = \#\left\{M<k\leq N:\ 2^{-s-1}\leq c_k < 2^{-s}\right\}.$$

Trivially $K(s)\leq N-M$.

Using the Cauchy-Schwarz inequality we finally get,

$$\begin{aligned}
\left\| \sum_{\substack{M<k\leq N \\ |c_k|\geq L^{-1}}} c_k f(kx) \right\| &\leq c \sum_{s:\ 1\leq 2^s\leq L} 2^{-s}\sqrt{K(s)}\ \exp\left(\frac{5\log K(s)}{4\log\log K(s)}\right) \\
&\leq c\sqrt{\log L}\ \exp\left(\frac{5\log N}{4\log\log N}\right)\left(\sum_{s:\ 1\leq 2^s\leq L} 2^{-2s}K(s)\right)^{1/2} \\
&\leq c\sqrt{\log L}\ \exp\left(\frac{5\log N}{4\log\log N}\right)\left(\sum_{M<k\leq N} c_k^2\right)^{1/2}.
\end{aligned}$$

This proves Lemma 4. $\square$

Proof of Lemma 5: Let $N_2>N_1$ be given. We have

$$\begin{aligned}
&\left\| \max_{N_1\leq M\leq N_2}\left|\sum_{N_1<k\leq M} c_k f(kx)\right|\right\| \\
\leq\ &\left\| \max_{N_1\leq M\leq N_2}\left|\sum_{\substack{N_1<k\leq M \\ |c_k|\leq N_2^{-3}}} c_k f(kx)\right|\right\| + \left\| \max_{N_1\leq M\leq N_2}\left|\sum_{\substack{N_1<k\leq M \\ |c_k|\geq N_2^{-3}}} c_k f(kx)\right|\right\|. \qquad (15)
\end{aligned}$$

The first term in (15) is bounded by

$$\begin{aligned}
\left\| \max_{N_1\leq M\leq N_2}\left|\sum_{\substack{N_1<k\leq M \\ |c_k|\leq N_2^{-3}}} c_k f(kx)\right|\right\| &\leq \sum_{\substack{N_1<k\leq N_2 \\ |c_k|\leq N_2^{-3}}} |c_k| \\
&\leq N_2^{-2}. \qquad (16)
\end{aligned}$$

To estimate the value of the second term in (15) we use Lemma 4, where we chose $L=N_2^3$.

Then we have

$$\left\| \sum_{\substack{N_1<k\leq N_2 \\ |c_k|\geq N_2^{-3}}} c_k f(kx) \right\| \leq c(\log N_2)^{1/2} \ \exp\left(\frac{5\log N_2}{4\log\log N_2}\right) \left(\sum_{N_1<k\leq N_2} c_k^2\right)^{1/2}. \tag{17}$$

Imitating the proof of the Rademacher-Menshov inequality (see [14, p. 123]), we can easily show that

$$\begin{aligned} \left\| \max_{N_1\leq M\leq N_2} \left| \sum_{\substack{N_1<k\leq M \\ |c_k|\geq N_2^{-3}}} c_k f(kx) \right| \right\| &\leq c(\log N_2)^{3/2} \ \exp\left(\frac{5\log N_2}{4\log\log N_2}\right) \left(\sum_{N_1<k\leq N_2} c_k^2\right)^{1/2} \\ &\leq c \ \exp\left(\frac{3\log N_2}{2\log\log N_2}\right) \left(\sum_{N_1<k\leq N_2} c_k^2\right)^{1/2}. \end{aligned}$$

Combining this with (16) we have

$$\left\| \max_{N_1\leq M\leq N_2} \left| \sum_{N_1<k\leq M} c_k f(kx) \right| \right\| \leq N_2^{-2} + c \ \exp\left(\frac{3\log N_2}{2\log\log N_2}\right) \left(\sum_{N_1<k\leq N_2} c_k^2\right)^{1/2}.$$

This proves Lemma 5. $\square$

Proof of Lemma 6:

Assume that for every given $\varepsilon > 0$ there exists an M_0 such that

$$\left\| \sup_{N>M_0} \left| \sum_{k=M_0+1}^{N} c_k f(kx) \right| \right\| \leq \varepsilon. \tag{18}$$

By Minkowski's inequality this implies

$$\left\| \sup_{N_2>N_1>M_0} \left| \sum_{k=N_1}^{N_2} c_k f(kx) \right| \right\| \leq 2\varepsilon.$$

Therefore

$$\left\| \inf_{M} \sup_{N_2>N_1>M} \left| \sum_{k=N_1}^{N_2} c_k f(kx) \right| \right\| = 0$$

and

$$\inf_{M} \sup_{N_2>N_1>M} \left| \sum_{k=N_1}^{N_2} c_k f(kx) \right| = 0 \qquad \text{a.e.},$$

which implies the a.e. convergence of

$$\sum_{k=1}^{\infty} c_k f(kx).$$

4 Proof of Theorem 1

Assume that $(c_k)_{k\geq 1}$ satisfies

$$\sum_{k=1}^{\infty} c_k^2 \ \exp\left(\frac{2\log k}{\log\log k}\right) < \infty. \tag{19}$$

As a consequence of (19) we have for $r \geq 1$

$$\sum_{k=2^r+1}^{2^{r+1}} c_k^2 \leq c \ \exp\left(\frac{-2\log(2^r)}{\log\log(2^r)}\right).$$

By the monotone convergence theorem and Minkowski's inequality we have, for any $m \geq 1$,

$$\begin{aligned}
\left\| \sup_{M>2^m} \left| \sum_{k=2^m+1}^{M} c_k f(kx) \right| \right\| &= \lim_{w\to\infty} \left\| \sup_{2^m<M\leq 2^{m+w}} \left| \sum_{k=2^m+1}^{M} c_k f(kx) \right| \right\| \\
&\leq \lim_{w\to\infty} \sum_{m\leq r\leq m+w-1} \left\| \max_{2^r<M\leq 2^{r+1}} \left| \sum_{k=2^r+1}^{M} c_k f(kx) \right| \right\|.
\end{aligned}$$

Together with Lemma 5 this implies

$$\begin{aligned}
& \left\| \sup_{M>2^m} \left| \sum_{k=2^m+1}^{M} c_k f(kx) \right| \right\| \\
\leq \ & \sum_{r\geq m} \left\| \max_{2^r<M\leq 2^{r+1}} \left| \sum_{k=2^r+1}^{M} c_k f(kx) \right| \right\| \\
\leq \ & \sum_{r\geq m} \left((2^r)^{-2} + c \ \exp\left(\frac{3\log(2^r)}{2\log\log(2^r)}\right) \left(\sum_{2^r<k\leq 2^{r+1}} c_k^2 \right)^{1/2} \right) \\
\leq \ & 2^{-2m} + \sum_{r\geq m} c \ \exp\left(\frac{-\log(2^r)}{2\log\log(2^r)}\right)
\end{aligned} \tag{20}$$

For any given $\varepsilon > 0$, we can choose m so large that (20) is smaller than ε. Therefore, by Lemma 6 the series

$$\sum_{k=1}^{\infty} c_k f(kx)$$

is a.e. convergent.

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