On the law of the iterated logarithm for the discrepancy of sequences $\langle n_{\mathbf{k}} x \rangle$ with multidimensional indices

Christoph Aistleitner *

Abstract

By a classical result of Weyl (1916), for any increasing sequence (n_k) of positive integers, $(n_k x)$ is uniformly distributed mod 1 for almost all x. The precise asymptotics of the discrepancy of this sequence is known only in a few cases, e.g. for $n_k = k$ (Khinchin (1924)) and for lacunary (n_k) (Philipp (1975)). In this paper we extend Philipp's result to lacunary sequences with multidimensional indices.

1 Introduction

Let $(n_k)_{k \geq 1}$ be an increasing sequence of positive integers and for $x \in (0,1)$ we set

$$\eta_k = \eta_k(x) := \langle n_k x \rangle, \tag{1}$$

where $\langle \cdot \rangle$ denotes fractional part. The discrepancy of the first N elements of the sequence (η_k) is defined as

$$D_N = D_N(x) := \sup_{0 \leq t \leq 1} \left| \frac{1}{N} \operatorname{card}\left(k \leq N : \eta_k(x) \leq t\right) - t \right|. \tag{2}$$

By a classical result of H. Weyl [11], $D_N(x) \to 0$ for almost all $x \in (0,1)$, i.e. $(n_k x)$ is uniformly distributed mod 1 for all $x \in (0,1)$ except for a set of Lebesgue measure zero. Estimating the speed of the convergence of $D_N(x)$ to 0 is a difficult problem requiring sophisticated analytic and number theoretic tools and the precise order of magnitude of $D_N(x)$ is known only for a few special sequences (n_k). In the case $n_k = k$ Khinchin [5] proved that

$$D_n(x) = O((\log N)^{1+\varepsilon}) \quad \text{a.e.} \qquad (\varepsilon > 0),$$

and this becomes false for $\varepsilon = 0$. On the other hand, Philipp [8] proved that if $(n_k)_{k \geq 1}$ is a lacunary sequence of integers, i.e. a sequence of integers satisfying

$$n_{k+1}/n_k \geq q > 1 \qquad k = 1, 2 \ldots, \tag{3}$$

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*Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: aistleitner@finanz.math.tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N13.

then $D_N(x)$ satisfies the law of the iterated logarithm (LIL), i.e.

$$\frac{1}{\sqrt{32}} \leq \limsup_{N\to\infty} \frac{ND_N(x)}{\sqrt{N\log\log N}} \leq C_q \qquad \text{a.e.,} \tag{4}$$

where C_q is a positive number depending on q. Except the value of the limsup in (4), this behavior is the same as that of the discrepancy of independent random variables, where the limsup is $1/2$ (see e.g. [9], p. 504). If (n_k) grows much faster than exponential, the limsup equals $1/2$ (this follows, e.g., from the results of Gaposhkin [4] or from the approximation theorems in Berkes [1]). However, assuming only the Hadamard gap condition (3), the limsup is generally different from $1/2$, see Fukuyama [3]. It is an open problem if the limsup is a constant almost everywhere.

The purpose of this paper is to extend the theorem of Philipp for sequences (n_k) with multidimensional indices. Most results in the theory of uniform distribution and discrepancy extend for sequences with values in $\mathbb{R}^d$, although usually there is a price in accuracy to pay for the high dimensional result. In contrast, there are very few results on the discrepancy of sequences with multidimensional indices, even though the corresponding problem, namely the uniform asymptotic behavior of random fields, has been extensively studied in probability theory (see e.g. Khoshnevisan [6]). In view of this fact, it seems to be of considerable interest to study the multiparameter version Philipp's theorem, one of the sharp and delicate results in metric discrepancy theory.

Let $\mathbb{N}^d$ denote the set of d-dimensional vectors with positive integer components and let $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ be a sequence of integers with d-dimensional indices. Letting $\mathbf{k} = (k_1, \dots, k_d)$ and $\mathbf{k}' = (k'_1, \dots, k'_d)$, we say that $\mathbf{k} \leq \mathbf{k}'$ if $k_i \leq k'_i$, $1 \leq i \leq d$ and $\mathbf{k} < \mathbf{k}'$ if $\mathbf{k} \leq \mathbf{k}'$ and $\mathbf{k} \neq \mathbf{k}'$. We say that $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is nondecreasing if $n_{\mathbf{k}'} \leq n_{\mathbf{k}}$ provided $\mathbf{k}' \leq \mathbf{k}$. Let $\mathbf{1}$ denote the d-dimensional vector $(1, \dots, 1)$ and for $\mathbf{N} = (N_1, \dots, N_d)$ we set $|\mathbf{N}| = \prod_{i=1}^d N_i$. The discrepancy $D_{\mathbf{N}}(x)$ of the finite sequence $(n_{\mathbf{k}})_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{N}}$ is defined, similarly to the one-parameter case, as

$$D_{\mathbf{N}}(x) = \sup_{0\leq a<b\leq 1} \left| \frac{\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} \mathbb{1}_{[a,b)}(\langle n_{\mathbf{k}} x\rangle)}{|\mathbf{N}|} - (b-a) \right|,$$

where $\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} = \sum_{\mathbf{k}:\ \mathbf{1}\leq\mathbf{k}\leq\mathbf{N}}$. Our main result is

Theorem 1 *Let $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ be a nondecreasing sequence of positive integers for which*

$$\#\{\mathbf{k}\in\mathbb{N}^d:\ 2^r \leq n_{\mathbf{k}} < 2^{r+1}\} \leq Q, \qquad r = 1, 2, \dots \tag{5}$$

with a constant Q. Then

$$\limsup_{|\mathbf{N}|\to\infty} \frac{|\mathbf{N}| D_{\mathbf{N}}(x)}{\sqrt{|\mathbf{N}|\log\log|\mathbf{N}|}} \leq C_{Q,d} \qquad \text{a.e.,} \tag{6}$$

where $C_{Q,d}$ is a positive number depending on Q and d.

Note that the one-dimensional Hadamard gap condition (3) has been replaced by condition (5) which has a different character. In one dimension, (5) is satisfied if and only if (n_k) is the union of finitely many sequences each of which satisfies the Hadamard gap condition (3). Note that any such sequence will satisfy the upper bound in Philipp's result (4). Thus Theorem 1 really is a generalization of Philipp's result, and condition (5) can be seen as a generalization of the concept of lacunary sequences to the case of sequences with multidimensional indices. We emphasize that the lower bound in (4) may not necessarily hold for sequences that satisfy only (5) instead of (3).

It would be tempting to define the Hadamard gap condition for a sequence $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ with multidimensional indices by requiring that

$$n_{\mathbf{k}'}/n_{\mathbf{k}} \geq q > 1 \quad \text{for } \mathbf{k}' > \mathbf{k}. \tag{7}$$

However, with definition (7), Theorem 1 fails. Let e.g. $d = 2$ and $n_{\mathbf{k}} = 2^{k_1+k_2}$ for $\mathbf{k} = (k_1, k_2)$. In Section 4 we will show that for almost all $x \in (0,1)$, the inequality $|\mathbf{N}|D_{\mathbf{N}}(x) \geq \text{const}\,|\mathbf{N}|^{3/4}$ holds for infinitely many $\mathbf{N}$, and thus (6) is not valid.

To prove Theorem 1 we use techniques developed by Takahashi [10], Philipp [8] and Erdős and Gál [2].

2 Exponential bounds

In the following let a d-dimensional vector $\mathbf{N} = (N_1, \ldots, N_d)$ of positive integers be given, let $f(x)$ denote an even function satisfying

$$f(x+1) = f(x), \quad \text{Var } f \leq 2, \quad \|f\|_\infty \leq 1, \quad \int_0^1 f(x)\ dx = 0 \tag{8}$$

and let

$$f(x) \sim \sum_{j=1}^\infty c_j \cos 2\pi j x$$

be its Fourier series. Additionally we assume

$$2^{-h-2} \leq \int_0^1 f(x)^2\ dx \leq 2^{-h-1}, \tag{9}$$

where h is a positive integer with $h \leq (\log_2 |\mathbf{N}|)/2$; this condition will play a crucial role in the chaining argument in Section 3. Let

$$g(x) = \sum_{j=1}^{|\mathbf{N}|^3} c_j \cos 2\pi j x.$$

Then $\|g\|_\infty \leq \|f\|_\infty + \text{Var}\, f \leq 3$. By (8) and Zygmund [12, p. 48]

$$|c_j| \leq \frac{\text{Var } f}{2j} \leq \frac{1}{j}, \qquad j \geq 1,$$

and thus

$$\sum_{j=1}^\infty c_j^2 \leq 2$$

and for any $J \geq 1$

$$\sum_{j=J+1}^\infty c_j^2 \leq \int_J^\infty \frac{1}{t^2}\ dt = \frac{1}{J}. \tag{10}$$

Lemma 1

$$\mathbb{P}\left\{\max_{\mathbf{M}\leq\mathbf{N}} \left|\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \Big(f(n_{\mathbf{k}}x) - g(n_{\mathbf{k}}x)\Big)\right| > h^{-2}\sqrt{|\mathbf{N}| \log\log |\mathbf{N}|}\right\} \quad \leq \quad \frac{(\log_2 |\mathbf{N}|)^4}{|\mathbf{N}|}.$$

Proof: For $\mathbf{1} \leq \mathbf{M} \leq \mathbf{N}$ we have

$$\begin{aligned}
\left\| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \Big(f(n_{\mathbf{k}} x) - g(n_{\mathbf{k}} x) \Big) \right\|_2 &\leq |\mathbf{M}| \; \|f-g\|_2 \\
&\leq |\mathbf{M}| \sqrt{\sum_{j=|\mathbf{N}|^3+1}^{\infty} c_j^2} \\
&\leq \frac{|\mathbf{M}|}{|\mathbf{N}|^{3/2}} \leq \frac{1}{|\mathbf{N}|^{1/2}}.
\end{aligned}$$

Thus by the Markov inequality and $h \leq (\log_2 |\mathbf{N}|)/2$

$$\begin{aligned}
& \mathbb{P}\left\{ \max_{\mathbf{M} \leq \mathbf{N}} \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \Big(f(n_{\mathbf{k}} x) - g(n_{\mathbf{k}} x) \Big) \right| > h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \right\} \\
\leq & \sum_{\mathbf{M} \leq \mathbf{N}} \mathbb{P}\left\{ \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \Big(f(n_{\mathbf{k}} x) - g(n_{\mathbf{k}} x) \Big) \right| > h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \right\} \\
\leq & |\mathbf{N}| \frac{(\log_2 |\mathbf{N}|)^4}{|\mathbf{N}|^2} \leq \frac{(\log_2 |\mathbf{N}|)^4}{|\mathbf{N}|}.
\end{aligned}$$

The following lemma, which extends [10] and [8, Proposition], is the key technical step in the proof of Theorem 1.

Lemma 2 *Let $\widetilde{K}$ be a finite set of d-dimensional vectors of positive integers. Then*

$$\mathbb{P}\left\{ \left| \sum_{\mathbf{k} \in \widetilde{K}} g(n_{\mathbf{k}} x) \right| > C_1 h^{-2} \sqrt{|\widetilde{K}| \log\log |\widetilde{K}|} \right\} \leq 2 e^{-2(d+1) h \log\log |\widetilde{K}|}, \tag{11}$$

provided $|\mathbf{N}|^{1/3} \leq |\widetilde{K}| < |\mathbf{N}|$ and $|\mathbf{N}|$ is sufficiently large.

Here and in the following $C_1, C_2, \dots$ denote suitable positive numbers that may depend on d and Q but not on $\mathbf{k}, \widetilde{K}, \mathbf{N}$ or anything else. $|\widetilde{K}|$ denotes the number of elements of $\widetilde{K}$.

Proof: We write $\{n_{\mathbf{k}} : \mathbf{k} \in \widetilde{K}\}$ as a nondecreasing sequence with 1-dimensional indices $(n_k)_{1 \leq k \leq |\widetilde{K}|}$. It suffices to prove

$$\mathbb{P}\left\{ \left| \sum_{k=1}^{|\widetilde{K}|} g(n_k x) \right| > C_1 h^{-2} \sqrt{|\widetilde{K}| \log\log |\widetilde{K}|} \right\} \leq 2 e^{-2(d+1) h \log\log |\widetilde{K}|} \tag{12}$$

for sufficiently large $|\mathbf{N}|$. We put

$$U_m(x) = \sum_{k=M_m+1}^{M_{m+1}} g(n_k x),$$

where M_m is the smallest integer greater or equal $m|\widetilde{K}|^{1/3}$, $m = 0, \dots, \lfloor |\widetilde{K}|^{2/3} \rfloor$, and $M_{\lfloor |\widetilde{K}|^{2/3} \rfloor + 1} = |\widetilde{K}|$. We put $m^+ = \lfloor |\widetilde{K}|^{2/3} \rfloor / 2$,

$$I_1(\lambda) = \int_0^1 \exp\left(2\lambda \sum_{m=0}^{\lfloor m^+ \rfloor} U_{2m}(x) \right) dx$$

and

$$I_2(\lambda) = \int_0^1 \exp\left(2\lambda \sum_{m=1}^{\lceil m^+ \rceil} U_{2m-1}(x)\right)\ dx.$$

For $|z| \le 1$ we have

$$e^z \le 1 + z + z^2,$$

and since $2\lambda |U_{2m}(x)| \le 2\lambda \|g\|_\infty (M_{2m+1} - M_{2m}) \le 6\lambda(|\widetilde{K}|^{1/3} + 1) \le 1$ for

$$\lambda \le \frac{1}{6(|\widetilde{K}|^{1/3} + 1)}, \tag{13}$$

we obtain

$$I_1(\lambda) \le \int_0^1 \prod_{m=0}^{\lfloor m^+ \rfloor} \left(1 + 2\lambda U_{2m} + 4\lambda^2 U_{2m}^2\right)\ dx,$$

provided that (13) holds. For any m

$$U_m^2(x) \le 2 \sum_{k=M_m+1}^{M_{m+1}} \sum_{k'=k}^{M_{m+1}} g(n_k x) g(n_{k'} x) = W_m(x) + V_m(x),$$

where W_m is a sum of trigonometric functions whose frequencies lie between n_{M_m+1} and $2|\mathbf{N}|^3 n_{M_{m+1}}$, and where V_m is a sum of trigonometric functions with frequencies at most $n_{M_m+1} - 1$.

$$\begin{aligned}
|V_m(x)| &\le 2 \sum_{k=M_m+1}^{M_{m+1}} \sum_{k'=k}^{M_{m+1}} \sum_{\substack{1 \le j, j' \le |\mathbf{N}|^3, \\ |n_k j - n_{k'} j'| < n_{M_m+1}}} |c_j c_{j'}| \\
&\le 2 \sum_{k=M_m+1}^{M_{m+1}} \sum_{k'=k}^{M_{m+1}} \sum_{\substack{1 \le j, j' \le |\mathbf{N}|^3, \\ |j - \frac{n_{k'}}{n_k} j'| < 1}} |c_j c_{j'}| \\
&\le 4 \sum_{k=M_m+1}^{M_{m+1}} \sum_{k'=k}^{M_{m+1}} \left(\sum_{j'=1}^{\infty} c_{j'}^2\right)^{1/2} \left(\sum_{j > n_{k'}/n_k - 1} c_j^2\right)^{1/2}.
\end{aligned}$$

For fixed k there are at most $2Q$ integers $k' \ge k$ for which $\frac{n_k}{n_{k'}} \ge 1$ (for these $n_{k'} = n_k$), at most $2Q$ for which $1 > \frac{n_k}{n_{k'}} \ge \frac{1}{2}$, at most $2Q$ for which $\frac{1}{2} > \frac{n_k}{n_{k'}} \ge \frac{1}{4}$ and so on. Thus

$$|V_m(x)| \le 4\sqrt{2} \|f\|_2 (M_{m+1} - M_m) 2Q \left(2\sqrt{2} + \sum_{i=1}^{\infty} \sqrt{\frac{1}{2^i - 1}}\right) \ \le\ 64Q \|f\|_2 (M_{m+1} - M_m).$$

Therefore

$$I_1(\lambda) \le \int_0^1 \prod_{m=0}^{\lfloor m^+ \rfloor} \left(1 + 2\lambda U_{2m}(x) + 4\lambda^2 W_{2m}(x) + 256\lambda^2 Q \|f\|_2 (M_{2m+1} - M_{2m})\right).$$

If $d_{2m} \cos 2\pi u_{2m} x$ is any term of the trigonometric polynomial $2\lambda U_{2m}(x) + 4\lambda^2 W_{2m}$, then

$$u_{2m} - \sum_{k=0}^{m-1} u_{2k} \ge n_{M_{2m}} - 2|\mathbf{N}|^3 \sum_{k=0}^{m-1} n_{M_{2k}} \ge n_{M_{2m}} \left(1 - 2|\mathbf{N}|^3 \sum_{k=0}^{m-1} \left(2^{-\lfloor |\widetilde{K}|^{1/3}/Q \rfloor}\right)^{m-k}\right) > 0$$

for sufficiently large $|\mathbf{N}|$, since by assumption $|\widetilde{K}| > |\mathbf{N}|^{1/3}$. Hence

$$I_1(\lambda) \le \prod_{m=0}^{\lfloor m^+ \rfloor} \left(1 + 256\lambda^2 Q \|f\|_2 (M_{2m+1} - M_{2m})\right) \le \exp\left(\sum_{m=0}^{\lfloor m^+ \rfloor} 256\lambda^2 Q \|f\|_2 (M_{2m+1} - M_{2m})\right).$$

In the same way we can prove a similar inequality for $I_2(\lambda)$, and thus by the Cauchy-Schwarz-inequality

$$\int_0^1 \exp\left(\lambda \sum_{k=1}^{|\widetilde{K}|} g(n_k x)\right) dx \le \sqrt{I_1(\lambda) I_2(\lambda)} \le \exp\left(128\lambda^2 Q \|f\|_2 |\widetilde{K}|\right),$$

valid for sufficiently large $|\mathbf{N}|$ and any λ satisfying (13). We choose

$$\lambda = h^3 \sqrt{\frac{\log\log |\widetilde{K}|}{|\widetilde{K}|}},$$

and observe that this λ satisfies (13) for sufficiently large $|\mathbf{N}|$. Thus we get by Markov's inequality

$$\begin{aligned} & \mathbb{P}\left(\sum_{k=1}^{|\widetilde{K}|} g(n_k x) > C_1 h^{-2} \sqrt{|\widetilde{K}| \log\log |\widetilde{K}|}\right) \\ \le \quad & \exp\left(128 Q h^6 \|f\|_2 \log\log |\widetilde{K}| - C_1 h \log\log |\widetilde{K}|\right) \\ \le \quad & \exp\left(-2(d+1) h \log\log |\widetilde{K}|\right) \end{aligned}$$

for a sufficiently large C_1 that satisfies $128 Q h^6 2^{(-h-1)/2} - C_1 h \le -2(d+1)h$ for $h \ge 1$ and sufficiently large $|\mathbf{N}|$. A similar result for $-g(x)$ instead of $g(x)$ yields (12), which proves Lemma 2.

Until now we considered only even functions f. Since any function f satisfying (8) can be written as the sum of an even and an odd function both of which satisfy (8) and our previous estimates remain valid for odd functions f, we get as a consequence of Lemma 1 and Lemma 2

Corollary 1 *Let $f(x)$ be a function which satisfies (8) and which can be divided into an even and an odd part both of which satisfy (9). Write $g(x)$ for the $|\mathbf{N}|^3$-th partial sum of the Fourier series of f. Then we have*

$$\mathbb{P}\left\{\max_{\mathbf{M} \le \mathbf{N}} \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \left(f(n_{\mathbf{k}} x) - g(n_{\mathbf{k}} x)\right) \right| > 2h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|}\right\} \le \frac{2(\log_2 |\mathbf{N}|)^4}{|\mathbf{N}|}$$

Let $\widetilde{K}$ be a finite set of d-dimensional vectors of positive integers. Then

$$\mathbb{P}\left\{\left| \sum_{\mathbf{k} \in \widetilde{K}} g(n_{\mathbf{k}} x) \right| > 2C_1 h^{-2} \sqrt{|\widetilde{K}| \log\log |\widetilde{K}|}\right\} \le 4e^{-2(d+1)h \log\log |\widetilde{K}|},$$

provided $|\mathbf{N}|^{1/3} \le |\widetilde{K}| < |\mathbf{N}|$ and $|\mathbf{N}|$ is sufficiently large.

3 Proof of Theorem 1

Using the inequalities in Section 2 it is not difficult to prove that

$$\left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} \left(\mathbb{1}_{[a,b)}(\langle n_{\mathbf{k}} x \rangle) - (b-a) \right) \right| = \mathcal{O}\left(\sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \right) \quad a.e.$$

for any *fixed* $0 \le a < b \le 1$. To prove the uniformity in a, b we will use a multiparameter chaining argument which extends the arguments in Erdős and Gál [2] and Philipp [8], but the multiparameter setting presents considerable difficulties.

Let $\mathbf{N} = (N_1, \dots, N_d)$ with $|\mathbf{N}| = 2^n$ be given. We put $H = (\log_2 |\mathbf{N}|)/2 = n/2$. Every $a \in [0,1)$ can be written in dyadic expansion

$$a = \sum_{h=1}^{\infty} 2^{-h} a_h, \qquad a_h \in \{0,1\},$$

and obviously

$$\sum_{h=1}^{H} 2^{-h} a_h \le a \le \sum_{h=1}^{H} 2^{-h} a_h + 2^{-H}.$$

We define functions

$$\varrho_h^{(j)}(x) = \mathbb{1}_{[(j-1)2^{-h}, j2^{-h})}(x), \qquad 1 \le j \le 2^h, \ 1 \le h \le H,$$

where $\mathbb{1}_{[a,b)}$ denotes the indicator of the interval $[a,b)$, extended with period 1, and

$$\varphi_h^{(j)}(x) = \varrho_h^{(j)}(x) - \int_0^1 \varrho_h^{(j)}(x) \ dx, \qquad 1 \le j \le 2^h, \ 1 \le h \le H.$$

Then for any a there exist coefficients $\varepsilon_h = \varepsilon_h(a) \in \{0,1\}$ and indices $j_h = j_h(a), 1 \le h \le H$, plus an additional index $\bar{j_H} = \bar{j_H}(a)$ such that

$$\sum_{h=1}^{H} \varepsilon_h \varrho_h^{(j_h)}(x) \le \mathbb{1}_{[0,a)}(x) \le \sum_{h=1}^{H} \varepsilon_h \varrho_h^{(j_h)}(x) + \varrho_H^{(\bar{j_H})}(x). \tag{14}$$

The functions $\varphi_h^{(j)}(x)$ satisfy the conditions of Corollary 1. We write $\hat{\varphi}_h^{(j)}$ for the $|\mathbf{N}|^3$-th partial sum of the Fourier series of $\varphi_h^{(j)}$ (corresponding to the function g in Section 2) and $\bar{\varphi}_h^{(j)}$ for the remainder terms (corresponding to $f-g$).

We define sets

$$\hat{K}_i = \bigcup_{L=0}^{\log_2 N_i} \bigcup_{l=0}^{\frac{N_i}{2^L}-1} \left\{ \left\{ x \in \mathbb{N} : l2^L + 1 \le x \le (l+1)2^L \right\} \right\}, \qquad i = 1, \dots, d,$$

(that means for example if $N_i = 4$ then $\hat{K}_i = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{3,4\},\{1,2,3,4\}\}$) and put

$$\hat{K} = \hat{K}(\mathbf{N}) = \left\{ \widetilde{K}_1 \times \dots \times \widetilde{K}_d : \ \widetilde{K}_i \in \hat{K}_i, i = 1, \dots, d \right\}.$$

Now let any $\mathbf{M} = (M_1, \dots, M_d) \le \mathbf{N}$ be given. We write each M_i in dyadic expansion $M_i = \sum_{l=0}^{\infty} M_{i,l} 2^l$, put

$$\hat{M}_i = \bigcup_{L=0}^{\log_2 N_i} \left\{ \left\{ x \in \mathbb{N} : \sum_{l=L}^{\log_2 N_i} M_{i,l} 2^l + 1 \le x \le \sum_{l=L-1}^{\log_2 N_i} M_{i,l} 2^l \right\} \right\},$$

write the set $\{x \in \mathbb{N}^d : x \leq \mathbf{M}\}$ as an union of disjoint sets

$$\widetilde{K}(\mathbf{M}) \in \hat{M} = \left\{ \widetilde{M_1} \times \cdots \times \widetilde{M_d} : \ \widetilde{M_i} \in \hat{M}_i, i = 1, \ldots, d \right\}$$

and write $\bar{K}(\mathbf{M})$ for the class of sets $\widetilde{K}(\mathbf{M})$. (For example if $d = 2$ and $\mathbf{M} = (7,5)$, then $\hat{M}_1 = \{\{1,2,3,4\},\{5,6\},\{7\}\}, \hat{M}_2 = \{\{1,2,3,4\},\{5\}\}$ and

$$\begin{aligned}\bar{K}(\mathbf{M}) = \{&\{1,2,3,4\} \times \{1,2,3,4\}\}, \{\{1,2,3,4\} \times \{5\}\}, \{\{5,6\} \times \{1,2,3,4\}\}, \{\{5,6\} \times \{5\}\}, \\ &\{\{7\} \times \{1,2,3,4\}\}, \{\{7\} \times \{5\}\}.\end{aligned}$$

We emphasize that the elements $\widetilde{K}$ of $\bar{K}(\mathbf{M})$ are contained in $\hat{K}$ as well.) Thus

$$\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} f(n_{\mathbf{k}}x) = \sum_{\widetilde{K} \in \bar{K}(\mathbf{M})} \sum_{\mathbf{k} \in \widetilde{K}} f(n_{\mathbf{k}}x).$$

The number of sets $\widetilde{K} = \widetilde{K}(\mathbf{M})$ with $|\widetilde{K}| = 2^l$ is at most $d!l^{d-1}$ (by the construction of the sets $\widetilde{K}$ it is clear that $|\widetilde{K}|$ always is an integer power of 2). Thus

$$\left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} f(n_{\mathbf{k}}x) \right| \leq \left(\sum_{\widetilde{K}(\mathbf{M}):\ |\widetilde{K}|>|\mathbf{N}|^{1/3}} \left| \sum_{\mathbf{k} \in \widetilde{K}} f(n_{\mathbf{k}}x) \right| \right) + 2d! \left(\log_2 \left(|\mathbf{N}|^{1/3} \right) \right)^{d-1} |\mathbf{N}|^{1/3} \|f\|_\infty.$$

In $\widetilde{K}(\mathbf{M})$ there are at most $d!l^{d-1}$ sets with $|\widetilde{K}| = 2^{n-l}$. We define

$$G_{\mathbf{N}} = \bigcup_{\widetilde{K} \in \hat{K}(\mathbf{N}), |\widetilde{K}|>|\mathbf{N}|^{1/3}} \bigcup_{h=1}^{H} \bigcup_{j=1}^{2^h} \left\{ \left| \sum_{\mathbf{k} \in \widetilde{K}} \hat{\varphi}_h^{(j)}(n_{\mathbf{k}}x) \right| > \frac{C_2}{h^2} \left(\log_2 \frac{|\mathbf{N}|}{|\widetilde{K}|} \right)^{-d-1} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \right\}$$

$$H_{\mathbf{N}} = \bigcup_{h=1}^{H} \bigcup_{j=1}^{2^h} \left\{ \max_{\mathbf{M} \leq \mathbf{N}} \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \bar{\varphi}_h^{(j)} \right| > 2h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \right\}$$

where C_2 will be chosen later. Here and in the sequel, $\log_2$ is meant as $\max(1, \log_2 x)$. For $x \in (0,1)$ in $G_{\mathbf{N}}^c \cap H_{\mathbf{N}}^c$ (A^c denotes the complement of A) we have

$$\begin{aligned}
&\left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \varphi_h^{(j)}(n_{\mathbf{k}}x) \right| \\
= &\left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \left(\hat{\varphi}_h^{(j)}(n_{\mathbf{k}}x) + \bar{\varphi}_h^{(j)}(n_{\mathbf{k}}x) \right) \right| \\
\leq &\left(\sum_{\widetilde{K} \in \bar{K}(\mathbf{M}), |\widetilde{K}|>|\mathbf{N}|^{1/3}} \left| \sum_{\mathbf{k} \in \widetilde{K}} \hat{\varphi}_h^{(j)}(n_{\mathbf{k}}x) \right| \right) \\
&+2d! \left(\log_2 \left(|\mathbf{N}|^{1/3} \right) \right)^{d-1} |\mathbf{N}|^{1/3} + \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \bar{\varphi}_h^{(j)}(n_{\mathbf{k}}x) \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\sum_{l=0}^{2n/3} \sum_{\widetilde{K}:\ \frac{|\mathbf{N}|}{|\widetilde{K}|}=2^l} C_2 h^{-2} l^{-d-1} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \right) \\
&\quad +2d! \left(\log_2 \left(|\mathbf{N}|^{1/3} \right) \right)^{d-1} |\mathbf{N}|^{1/3} + 2h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \\
&\le \left(\sum_{l=0}^{2n/3} C_2 h^{-2} l^{-d-1} d! l^{d-1} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \right) \\
&\quad +2d! \left(\log_2 \left(|\mathbf{N}|^{1/3} \right) \right)^{d-1} |\mathbf{N}|^{1/3} + 2h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \\
&\le C_3 h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} \quad \text{for all} \quad \mathbf{M} \le \mathbf{N},\ h = 1, \dots, H,\ j = 1, \dots, 2^h.
\end{aligned}
$$

Hence by (14) we have for such x

$$
\begin{aligned}
& \sup_{a \in [0,1)} \max_{\mathbf{M}: |\mathbf{N}|/2 < |\mathbf{M}| \le |\mathbf{N}|} \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \mathbb{1}_{[0,a)}(n_{\mathbf{k}} x) - |\mathbf{M}| a \right| \\
\le\ & \sup_{a \in [0,1)} \max_{\mathbf{M}: |\mathbf{N}|/2 < |\mathbf{M}| \le |\mathbf{N}|} \sum_{h=1}^{H} \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \varphi_h^{(j_h)}(n_{\mathbf{k}} x) \right| + \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \varphi_H^{(\bar{j}_H)}(n_{\mathbf{k}} x) \right| + 2\sqrt{|\mathbf{N}|} \\
\le\ & \sum_{h=1}^{H} C_3 h^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} + C_3 H^{-2} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|} + 2\sqrt{|\mathbf{N}|} \\
\le\ & C_4 \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|}
\end{aligned}
$$

and thus

$$
\left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \mathbb{1}_{[0,a)}(n_{\mathbf{k}} x) - |\mathbf{M}| a \right| \le 2C_4 \sqrt{|\mathbf{M}| \log\log |\mathbf{M}|}
$$

for all $\mathbf{M} \le \mathbf{N}$ with $|\mathbf{N}|/2 < |\mathbf{M}| \le |\mathbf{N}|$ and all $a \in [0,1)$. We write

$$
G_n = \bigcup_{\mathbf{N}:\ |\mathbf{N}|=2^n} G_{\mathbf{N}}, \qquad H_n = \bigcup_{\mathbf{N}:\ |\mathbf{N}|=2^n} H_{\mathbf{N}}.
$$

Then for all $x \in [0,1)$ in $G_n^c \cap H_n^c$

$$
\left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{M}} \mathbb{1}_{[0,a)}(\langle n_{\mathbf{k}} x \rangle) - |\mathbf{M}| a \right| \le 2C_4 \sqrt{|\mathbf{M}| \log\log |\mathbf{M}|} \tag{15}
$$

for all $\mathbf{M}$ with $2^{n-1} < |\mathbf{M}| \le 2^n$ and all $a \in [0,1)$. If we can show

$$
\sum_{n=1}^{\infty} \mathbb{P}(G_n) < \infty, \qquad \sum_{n=1}^{\infty} \mathbb{P}(H_n) < \infty \tag{16}
$$

then, by the Borel-Cantelli lemma, for almost all $x \in [0,1)$ there exists an $n_0 = n_0(x)$ such that $x \notin (G_n \cup H_n)$ for all $n > n_0$, and thus by (15)

$$
\limsup_{\mathbf{N} \ge \mathbf{1}} \frac{|\mathbf{N}| D_{\mathbf{N}}(x)}{\sqrt{|\mathbf{N}| \log\log |\mathbf{N}|}} \le 4C_4 \quad a.e.,
$$

which proves (6). It remains to show (16). There are at most $d! n^{d-1}$ different vectors $\mathbf{N}$

with $|\mathbf{N}| = 2^n$. By Corollary 1

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{P}(H_n) &\leq d! \sum_{n=1}^{\infty} n^{d-1} \max_{\mathbf{N}:\ |\mathbf{N}|=2^n} \mathbb{P}(H_{\mathbf{N}}) \\ &\leq d! \sum_{n=1}^{\infty} n^{d-1} \sum_{h=1}^{H} \sum_{j=1}^{2^h} \frac{2n^4}{2^n} \\ &\leq 2d! \sum_{n=1}^{\infty} n^{d-1} \frac{n}{2}\ 2^{n/2}\ \frac{n^4}{2^n}\ <\ \infty.\end{aligned}$$

For any $\mathbf{N}$ with $|\mathbf{N}| = 2^n$ and $\widetilde{K} \in \hat{K}(\mathbf{N})$ with $|\widetilde{K}| = 2^{n-l}, l \geq n/3$, and any $1 \leq h \leq H$ and $1 \leq j \leq 2^h$ by Corollary 1

$$\begin{aligned}&\mathbb{P}\left\{\left|\sum_{\mathbf{k}\in\widetilde{K}} \hat{\varphi}_h^{(j)}(n_{\mathbf{k}}x)\right| > C_2 h^{-2} \left(\log_2 \frac{|\mathbf{N}|}{|\widetilde{K}|}\right)^{-d-1} \sqrt{|\mathbf{N}| \log\log |\mathbf{N}|}\right\} \\ \leq\ &\mathbb{P}\left\{\left|\sum_{\mathbf{k}\in\widetilde{K}} \hat{\varphi}_h^{(j)}(n_{\mathbf{k}}x)\right| > C_2 h^{-2} (\log_2 2^l)^{-d-1}\ \sqrt{2^l}\sqrt{|\widetilde{K}| \log\log |\widetilde{K}|}\right\} \\ \leq\ &\mathbb{P}\left\{\left|\sum_{\mathbf{k}\in\widetilde{K}} \hat{\varphi}_h^{(j)}(n_{\mathbf{k}}x)\right| > 2C_1 h^{-2} \sqrt{|\widetilde{K}| \log\log |\widetilde{K}|}\right\} \\ \leq\ &4e^{-2(d+1)h \log\log |\widetilde{K}|},\end{aligned}$$

if n is sufficiently large and if C_2 is chosen such that $C_2(\log_2 2^l)^{-d-1}\sqrt{2^l} > 2C_1$ for $l = 0, 1, 2, \ldots$, and so

$$\begin{aligned}\mathbb{P}(G_{\mathbf{N}}) &\leq \sum_{\widetilde{K}\in\hat{K}(\mathbf{N}):|\widetilde{K}|>|\mathbf{N}|^{1/3}} \sum_{h=1}^{H} \sum_{j=1}^{2^h} 4e^{-2(d+1)h \log\log |\widetilde{K}|} \\ &\leq 4 \sum_{l=0}^{2n/3} d! l^{d-1} \sum_{h=1}^{H} 2^h \left(\frac{1}{\log 2^{n-l}}\right)^{2(d+1)h} \\ &\leq 4\ d! n^d \sum_{h=1}^{H} \left(\frac{2}{\log 2^{n/3}}\right)^{2(d+1)h} \\ &\leq 4\ d! n^d \left(\frac{9}{n}\right)^{2(d+1)}\end{aligned}$$

for sufficiently large n. Thus

$$\begin{aligned}\mathbb{P}(G_n) &\leq d! n^{d-1} \max_{\mathbf{N}:\ |\mathbf{N}|=2^n} \mathbb{P}(G_{\mathbf{N}}) \\ &\leq d!\ 4\ d!\ 9^{2(d+1)} n^{d-1} n^d n^{-2(d+1)}\ <\ C_5 n^{-3}\end{aligned}$$

for sufficiently large n, which implies that

$$\sum_{n=1}^{\infty} \mathbb{P}(G_n) < \infty.$$

This proves the theorem.

In conclusion we prove the remark made in the introduction, namely that Theorem 1 fails for the sequence $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^2}$ defined by $n_{\mathbf{k}} = 2^{k_1+k_2}$ for $\mathbf{k} = (k_1, k_2)$. To see this, let $\mathbf{N} = (n, n)$ and $f(x) = \mathbb{1}_{[0,1/2)}(x) - 1/2$, extended with period 1. Then

$$\sum_{\mathbf{1}\le\mathbf{k}\le\mathbf{N}} f(n_{\mathbf{k}}x) = \sum_{j=2}^{2n} c_j^{(n)} f\left(2^j x\right) \tag{17}$$

where $c_j^{(n)} = j-1$ for $2 \le j \le n+1$ and $c_j^{(n)} = 2n-j+1$ for $n+2 \le j \le 2n$. Observe that

$$f\left(2^j x\right) = \frac{1}{2}\, r_{j+1}(x), \quad j \ge 1,$$

where r_j denotes the j-th Rademacher function, and thus the sequence $(r_j(x))_{j\ge1}$ is a sequence of i.i.d. random variables. Hence using the central limit theorem with Berry-Esseen remainder term (see e.g. Petrov [7, p. 149]) we get that

$$\left|\mu\left(x\in(0,1) : B_n^{-1/2}\sum_{j=2}^{2n} c_j^{(n)} f(2^j x) < t\right) - \Phi(t)\right| \le C\, L_n \le C' n^{-1/2}, \tag{18}$$

where μ is the Lebesgue measure, Φ is the standard normal distribution function, C, C' are absolute constants,

$$B_n = \int_0^1 \left(\sum_{j=2}^{2n} c_j^{(n)} f(2^j x)\right)^2 dx = \sum_{j=2}^{2n}\left(c_j^{(n)}\right)^2 \int_0^1 f(x)^2 dx = \frac{2n^3+n}{12}.$$

and

$$\begin{aligned} L_n &= B_n^{-3/2}\sum_{j=2}^{2n}\int_0^1 \left|c_j^{(n)} f(2^j x)\right|^3 dx \\ &= B_n^{-3/2}\sum_{j=2}^{2n}\left(c_j^{(n)}\right)^3 \int_0^1 |f(x)|^3\, dx = \left(\frac{12}{2n^3+n}\right)^{3/2} \frac{n^4+n^2}{16}. \end{aligned}$$

Given $\varepsilon > 0$ choose $a > 0$ so small that $\Phi(a) - \Phi(-a) \le \varepsilon$, then by (18) we get

$$\mu\left(x\in(0,1) : \left|\sum_{j=2}^{2n} f\left(2^j x\right)\right| \ge a B_n^{1/2}\right) \ge 1 - 2\varepsilon$$

for sufficiently large n. Since $B_n \ge n^3/6 \ge |\mathbf{N}|^{3/2}/6$, the last relation implies

$$\mu\left(x\in(0,1) : \left|\sum_{\mathbf{1}\le\mathbf{k}\le\mathbf{N}} f(n_{\mathbf{k}}x)\right| \ge \frac{a|\mathbf{N}|^{3/4}}{\sqrt{6}}\right) \ge 1 - 2\varepsilon \tag{19}$$

for sufficiently large n. Letting F_n denote the set in the brackets in (19), it follows that

$$\mu(\cap_{n=1}^\infty \cup_{k=n}^\infty F_k) \ge 1 - 2\varepsilon,$$

i.e. the set of $x \in (0,1)$ such that

$$\left|\sum_{\mathbf{1}\le\mathbf{k}\le\mathbf{N}} f(n_{\mathbf{k}}x)\right| \ge \text{const}\,|\mathbf{N}|^{3/4} \qquad \text{for infinitely many } \mathbf{N} \tag{20}$$

has measure $\geq 1-2\varepsilon$. Since ε was arbitrary, we get

$$\limsup_{|\mathbf{N}|\to\infty} \frac{|\mathbf{N}|D_{\mathbf{N}}(x)}{\sqrt{|\mathbf{N}|\log\log|\mathbf{N}|}} \geq \limsup_{|\mathbf{N}|\to\infty} \frac{\left|\sum_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{N}} f(n_{\mathbf{k}}x)\right|}{\sqrt{|\mathbf{N}|\log\log|\mathbf{N}|}} = +\infty \qquad \text{a.e.},$$

i.e. the conclusion of Theorem 1 fails for the sequence $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^2}$.

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