# GENERALIZED BOUNDARY TRIPLES FOR ADJOINT PAIRS WITH APPLICATIONS TO NON-SELF-ADJOINT SCHRÖDINGER OPERATORS

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*Happy Birthday, Henk!*
*We dedicate this note to our friend and colleague Henk de Snoo on the occasion of his 80th birthday.*

Abstract. We extend the notion of generalized boundary triples and their Weyl functions from extension theory of symmetric operators to adjoint pairs of operators, and we provide criteria on the boundary parameters to induce closed operators with a nonempty resolvent set. The abstract results are applied to Schrödinger operators with complex $L^p$-potentials on bounded and unbounded Lipschitz domains with compact boundaries.

## 1. Introduction

Abstract boundary value problems for symmetric and self-adjoint operators in Hilbert spaces can be efficiently treated within the framework of boundary triples and their Weyl functions, a technique that is nowadays well developed and applied in various concrete and abstract settings, see, e.g., the monograph [12] for an introduction into this field, some typical applications to differential operators, and further references. Henk's contributions on boundary triples and their generalizations have inspired many mathematicians in modern analysis, differential equations, and spectral theory. In particular, abstract coupling techniques for boundary triples from [30] or the notion of boundary relations and their Weyl families from [31, 32, 33], developed jointly with V.A. Derkach, S. Hassi, and M.M. Malamud, have had a substantial impact on the field.

The aim of this note is to consider a class of non-symmetric abstract boundary value problems in the context of adjoint pairs of operators. Such problems have their roots in the works of M.I. Visik [67], M.S. Birman [20], and G. Grubb [41], and have been further developed in the framework of (ordinary) boundary triples by L.I. Vainerman in [66] and V.E. Lyantse and O.G. Storozh in the monograph [53], see also [10, 24, 25, 26, 43, 44, 55, 56, 57, 60] for more recent developments and, e.g., [2, 3, 4, 5, 6, 7, 9, 36, 37, 42, 49, 50, 51, 54, 58, 62, 63, 64, 65, 68, 69] for other closely related approaches and typical applications. In fact, our main objective is to introduce and to study the notion of generalized boundary triples and their Weyl functions for adjoint pairs of operators, extending the definition by V.A. Derkach and M.M. Malamud from [34] (see also [28, 29]) from the symmetric to the non-symmetric setting. At the same time the present considerations can be viewed as a special case of the treatment in [8], where the notion of quasi boundary triples and their Weyl functions for symmetric operators from [13, 14, 15, 17] was extended to the general framework of adjoint pairs under minimal assumptions on the boundary maps. Although our abstract treatment in Section 2 in this sense is contained in [8], many of the general results from [8] simplify substantially in their assumptions and their formulation.

The abstract notion of generalized boundary triples for adjoint pairs turns out to be useful as it can be applied directly to Schrödinger operators with complex potentials on Lipschitz domains $\Omega$. More precisely, in Section 3 we consider a class of non-selfadjoint relatively bounded perturbations $V \in L^p(\Omega)$ of the Laplacian and provide a generalized boundary triple for the adjoint pair $\{-\Delta + V, -\Delta + \overline{V}\}$, where the boundary maps are the Dirichlet and Neumann trace operators $\tau_D$ and $\tau_N$; here we rely on properties of the trace operators on Lipschitz domains with compact boundaries that can be found in, e.g., [11, 40] and are recalled in the appendix. We apply this generalized boundary triple to obtain sufficient criteria for boundary parameters in $L^2(\partial\Omega)$ to induce closed realizations of $-\Delta + V$ and $-\Delta + \overline{V}$ with nonempty resolvent set in $L^2(\Omega)$. More precisely, if the complex potential $V \in L^p(\Omega)$ satisfies Assumption 3.1, we conclude in Corollary 3.7 that the Robin realization

$$\begin{aligned} A_B &= -\Delta + V, \\ \operatorname{dom} A_B &= \big\{ f \in H^{3/2}(\Omega) : \tau_N f = B\tau_D f,\ (-\Delta + V)f \in L^2(\Omega) \big\}, \end{aligned} \tag{1.1}$$

is a closed operator in $L^2(\Omega)$ with a nonempty resolvent set; here it is assumed that the boundary parameter $B$ is a bounded everywhere defined operator in $L^2(\partial\Omega)$. Furthermore, if $A_0$ denotes the Neumann realization of $-\Delta + V$, and $\gamma, \widetilde{\gamma}$, and $M$ are the $\gamma$-fields and Weyl function, respectively, associated with the generalized boundary triple in Theorem 3.3, then the Krein-type resolvent formula

$$(A_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\widetilde{\gamma}(\bar{\lambda})^* \tag{1.2}$$

is valid for all $\lambda \in \rho(A_0) \cap \rho(A_B)$. One of the key ingredients in the proofs is to ensure a decay of the norm of the Weyl function $M$ along the negative axis, so that the operator $I - BM(\lambda)$ in (1.2) admits a bounded everywhere defined inverse in $L^2(\partial\Omega)$. This technique is inspired by [15], where similar methods were developed for non-self-adjoint extensions of symmetric operators in the context of quasi boundary triples. We point out that the functions in the domain of $A_B$ in (1.1) exhibit $H^{3/2}$-regularity, which is natural for realizations of the Laplacian on (bounded) Lipschitz domains, such as Dirichlet and Neumann realizations; cf. [45, 46] and [11] for more references and details. We also refer the reader to [1, 22, 23, 21, 27, 37, 38, 39, 48, 61] for related recent contributions on self-adjoint and non-self-adjoint Robin type boundary conditions.

**Notations.** For a linear operator $A$ its domain, kernel, and range are denoted by $\operatorname{dom} A$, $\ker A$, and $\operatorname{ran} A$, respectively. If $A$ is an operator in some Hilbert space, then $\sigma_p(A)$ denotes the set of eigenvalues. If, in addition, $A$ is closed, then the symbols $\rho(A)$ and $\sigma(A)$ are used for the resolvent set and the spectrum, respectively. Next, for an open set $\Omega \subset \mathbb{R}^n$ and $s \geq 0$ the sets $H^s(\Omega)$ are the $L^2$-based Sobolev spaces of order $s$, and for a bounded or unbounded Lipschitz domain $\Omega$ with compact boundary and $s \in [-1, 1]$ the $L^2$-based Sobolev spaces on $\partial\Omega$ are $H^s(\partial\Omega)$; cf. [59] for their definitions. Finally, for a Banach space $X$ its dual space is $X^*$.

## 2. Generalized boundary triples for adjoint pairs

Let $\mathcal{H}$ be a separable Hilbert space and assume that $S$ and $\widetilde{S}$ are densely defined closed operators in $\mathcal{H}$ which satisfy the identity

$$(Sf, g) = (f, \widetilde{S}g), \qquad f \in \operatorname{dom} S,\ g \in \operatorname{dom} \widetilde{S}. \tag{2.1}$$

It is clear that (2.1) is equivalent to $\widetilde{S} \subset S^*$ and $S \subset \widetilde{S}^*$. In the following a pair $\{S, \widetilde{S}\}$ with the property (2.1) will be called an *adjoint pair*. Furthermore, we will make use of another pair of operators $\{T, \widetilde{T}\}$ such that

$$\overline{T} = S^* \quad \text{and} \quad \overline{\widetilde{T}} = \widetilde{S}^* \tag{2.2}$$

holds. Note that (2.2) implies $T \subset S^*$ and $\widetilde{T} \subset \widetilde{S}^*$, and that (2.2) is equivalent to $T^* = S$ and $\widetilde{T}^* = \widetilde{S}$. A pair of operators $\{T, \widetilde{T}\}$ with the property (2.2) will be called a core of $\{S^*, \widetilde{S}^*\}$.

2.1. **Generalized boundary triples.** The next definition is a special case of [8, Definition 2.1] and should be viewed as the natural generalization of the concept of generalized boundary triples for symmetric operators from [34], see also [16, 31, 33].

**Definition 2.1.** Let $\{S, \widetilde{S}\}$ be an adjoint pair of operators and let $\{T, \widetilde{T}\}$ be a core of $\{S^*, \widetilde{S}^*\}$. A *generalized boundary triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ for $\{S, \widetilde{S}\}$ consists of a Hilbert space $\mathcal{G}$ and linear mappings

$$\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G} \quad \text{and} \quad \widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$$

such that the following holds:

(i) the abstract *Green identity*

$$(Tf, g)_{\mathcal{H}} - (f, \widetilde{T}g)_{\mathcal{H}} = (\Gamma_1 f, \widetilde{\Gamma}_0 g)_{\mathcal{G}} - (\Gamma_0 f, \widetilde{\Gamma}_1 g)_{\mathcal{G}}$$

is valid for all $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{T}$,

(ii) the mappings $\Gamma_0 : \operatorname{dom} T \to \mathcal{G}$ and $\widetilde{\Gamma}_0 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$ are surjective,

(iii) the operators $A_0 := T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 := \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ satisfy

$$A_0^* = \widetilde{A}_0 \quad \text{and} \quad \widetilde{A}_0^* = A_0. \tag{2.3}$$

Let $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ be a generalized boundary triple for $\{S, \widetilde{S}\}$ and let $\{T, \widetilde{T}\}$ be a core of $\{S^*, \widetilde{S}^*\}$. We briefly recall some immediate properties that follow from the more general treatment in [8] and are known for the symmetric case from [34]. We remark first that (iii) implies

$$\widetilde{S} \subset T \subset S^* \quad \text{and} \quad S \subset \widetilde{T} \subset \widetilde{S}^*.$$

According to [8, Lemma 2.4] we have

$$\operatorname{dom} S = \ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1 \quad \text{and} \quad \operatorname{dom} \widetilde{S} = \ker \Gamma_0 \cap \ker \Gamma_1,$$

and $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top$ and $\operatorname{ran}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top$ are both dense in $\mathcal{G} \times \mathcal{G}$ by [8, Lemma 2.3]. Furthermore, the mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ and $\widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$ are closable with respect to the graph norm of $T$ and $\widetilde{T}$, respectively. It is also important to note that the pair $\{T, \widetilde{T}\}$ is not unique and may even coincide with $\{S^*, \widetilde{S}^*\}$, which is the case if, e.g.,

$$\dim\bigl(\operatorname{dom} S^* / \operatorname{dom} \widetilde{S}\bigr) = \dim\bigl(\operatorname{dom} \widetilde{S}^* / \operatorname{dom} S\bigr) < \infty. \tag{2.4}$$

In the special situation that $\{T, \widetilde{T}\} = \{S^*, \widetilde{S}^*\}$ it follows from [8, Proposition 2.6] that $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top = \mathcal{G} \times \mathcal{G}$ and $\operatorname{ran}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top = \mathcal{G} \times \mathcal{G}$, which implies that the restrictions $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ automatically satisfy (2.3); cf. [57]. Thus, in this situation the notion of generalized boundary triples from Definition 2.1 reduces to the special case treated in [26, 57, 53, 66]. In the following we will be interested in the case

$$\dim\bigl(\operatorname{dom} S^* / \operatorname{dom} \widetilde{S}\bigr) = \dim\bigl(\operatorname{dom} \widetilde{S}^* / \operatorname{dom} S\bigr) = \infty,$$

although our abstract discussion remains valid (in a simplified form) also in the finite dimensional case (2.4).

The next result is a variant of [8, Theorem 2.7] and can be used to verify that a pair $\{T, \widetilde{T}\}$ is a core of the adjoints of suitable operators $S$ and $\widetilde{S}$, respectively.

**Theorem 2.2.** *Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces, let $T$ and $\widetilde{T}$ be operators in $\mathcal{H}$, and assume that there are linear mappings*

$$\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G} \quad \text{and} \quad \widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$$

*such that the following holds:*

(i) *the abstract Green identity*

$$(Tf, g)_{\mathcal{H}} - (f, \widetilde{T}g)_{\mathcal{H}} = (\Gamma_1 f, \widetilde{\Gamma}_0 g)_{\mathcal{G}} - (\Gamma_0 f, \widetilde{\Gamma}_1 g)_{\mathcal{G}}$$

*is valid for all $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{T}$,*

(ii) *the mappings $\Gamma_0 : \operatorname{dom} T \to \mathcal{G}$ and $\widetilde{\Gamma}_0 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$ are surjective,*

(iii) *the operators $A_0 := T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 := \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ satisfy*

$$A_0^* = \widetilde{A}_0 \quad \text{and} \quad \widetilde{A}_0^* = A_0,$$

(iv) $\ker \Gamma_0 \cap \ker \Gamma_1$ *and* $\ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1$ *are dense in $\mathcal{H}$.*

*Then*

$$\begin{aligned} Sf &:= \widetilde{T}f, \quad f \in \operatorname{dom} S = \ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1, \\ \widetilde{S}g &:= Tg, \quad g \in \operatorname{dom} \widetilde{S} = \ker \Gamma_0 \cap \ker \Gamma_1, \end{aligned}$$

*are closed operators in $\mathcal{H}$ and form an adjoint pair $\{S, \widetilde{S}\}$ such that $\{T, \widetilde{T}\}$ is a core of $\{S^*, \widetilde{S}^*\}$, and $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is a generalized boundary triple for $\{S, \widetilde{S}\}$.*

2.2. **$\gamma$-fields and Weyl functions.** In the following let $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ be a generalized boundary triple for $\{S, \widetilde{S}\}$. It follows from (2.3) that the operators $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ are both closed and that the resolvent set of $A_0 = T \upharpoonright \ker \Gamma_0$ is nonempty if and only if the resolvent set $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ is nonempty. More precisely, one has $\lambda \in \rho(A_0)$ if and only if $\bar{\lambda} \in \rho(\widetilde{A}_0)$, and $\mu \in \rho(\widetilde{A}_0)$ if and only if $\bar{\mu} \in \rho(A_0)$.

Now assume that $\rho(A_0) \neq \emptyset$ or, equivalently, $\rho(\widetilde{A}_0) \neq \emptyset$, and recall the direct sum decompositions

$$\begin{aligned} \operatorname{dom} T &= \operatorname{dom} A_0 \dotplus \ker(T - \lambda) = \ker \Gamma_0 \dotplus \ker(T - \lambda), \qquad \lambda \in \rho(A_0), \\ \operatorname{dom} \widetilde{T} &= \operatorname{dom} \widetilde{A}_0 \dotplus \ker(\widetilde{T} - \mu) = \ker \widetilde{\Gamma}_0 \dotplus \ker(\widetilde{T} - \mu), \qquad \mu \in \rho(\widetilde{A}_0). \end{aligned}$$

We introduce the $\gamma$-fields and Weyl functions following the ideas in [8, 13, 31, 34, 55, 56, 57].

**Definition 2.3.** Let $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ be a generalized boundary triple and assume that $\rho(A_0) \neq \emptyset$ or, equivalently, $\rho(\widetilde{A}_0) \neq \emptyset$.

(i) The $\gamma$-fields $\gamma$ and $\widetilde{\gamma}$ are defined by

$$\begin{aligned} \gamma(\lambda) &:= \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1}, \qquad \lambda \in \rho(A_0), \\ \widetilde{\gamma}(\mu) &:= \big(\widetilde{\Gamma}_0 \upharpoonright \ker(\widetilde{T} - \mu)\big)^{-1}, \qquad \mu \in \rho(\widetilde{A}_0). \end{aligned}$$

(ii) The Weyl functions $M$ and $\widetilde{M}$ are defined by

$$\begin{aligned} M(\lambda) &:= \Gamma_1 \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1} = \Gamma_1 \gamma(\lambda), \qquad \lambda \in \rho(A_0), \\ \widetilde{M}(\mu) &:= \widetilde{\Gamma}_1 \big(\widetilde{\Gamma}_0 \upharpoonright \ker(\widetilde{T} - \mu)\big)^{-1} = \widetilde{\Gamma}_1 \widetilde{\gamma}(\mu), \qquad \mu \in \rho(\widetilde{A}_0). \end{aligned}$$

In the next proposition we collect some properties of the $\gamma$-fields and Weyl functions corresponding to a generalized boundary triple, see [8, Proposition 3.3 and Proposition 3.4] for proofs.

**Proposition 2.4.** *Let $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ be a generalized boundary triple and assume that $\rho(A_0)\neq\emptyset$ or, equivalently, $\rho(\widetilde A_0)\neq\emptyset$. Then the following holds for $\lambda\in\rho(A_0)$ and $\mu\in\rho(\widetilde A_0)$:*

(i) *$\gamma(\lambda)$ and $\widetilde\gamma(\mu)$ are everywhere defined bounded operators from $\mathcal{G}$ to $\mathcal{H}$;*

(ii) *the functions $\lambda\mapsto\gamma(\lambda)$ and $\mu\mapsto\widetilde\gamma(\mu)$ are holomorphic on $\rho(A_0)$ and $\rho(\widetilde A_0)$, respectively, and one has*

$$\begin{aligned}\gamma(\lambda)&=\bigl(I+(\lambda-\nu)(A_0-\lambda)^{-1}\bigr)\gamma(\nu), &&\lambda,\nu\in\rho(A_0),\\ \widetilde\gamma(\mu)&=\bigl(I+(\mu-\omega)(\widetilde A_0-\mu)^{-1}\bigr)\widetilde\gamma(\omega), &&\mu,\omega\in\rho(\widetilde A_0);\end{aligned}$$

(iii) *$\gamma(\lambda)^*$ and $\widetilde\gamma(\mu)^*$ are everywhere defined bounded operators from $\mathcal{H}$ to $\mathcal{G}$ and one has*

$$\gamma(\lambda)^*=\widetilde\Gamma_1(\widetilde A_0-\bar\lambda)^{-1}\quad\textit{and}\quad\widetilde\gamma(\mu)^*=\Gamma_1(A_0-\bar\mu)^{-1};$$

(iv) *$M(\lambda)$ and $\widetilde M(\mu)$ are everywhere defined bounded operators in $\mathcal{G}$, and for $f_\lambda\in\ker(T-\lambda)$ and $g_\mu\in\ker(\widetilde T-\mu)$ one has*

$$M(\lambda)\Gamma_0 f_\lambda=\Gamma_1 f_\lambda\quad\textit{and}\quad\widetilde M(\mu)\widetilde\Gamma_0 g_\mu=\widetilde\Gamma_1 g_\mu;$$

(v) *$M(\lambda)=\widetilde M(\bar\lambda)^*$ and $\widetilde M(\mu)=M(\bar\mu)^*$ and one has*

$$\begin{aligned}M(\lambda)-\widetilde M(\mu)^*&=(\lambda-\bar\mu)\widetilde\gamma(\mu)^*\gamma(\lambda),\\ M(\lambda)^*-\widetilde M(\mu)&=(\bar\lambda-\mu)\gamma(\lambda)^*\widetilde\gamma(\mu);\end{aligned}$$

(vi) *the functions $\lambda\mapsto M(\lambda)$ and $\mu\mapsto\widetilde M(\mu)$ are holomorphic on $\rho(A_0)$ and $\rho(\widetilde A_0)$, respectively, and for some fixed $\lambda_0,\mu_0\in\rho(A_0)\cap\rho(\widetilde A_0)$ one has*

$$\begin{aligned}M(\lambda)&=\widetilde M(\lambda_0)^*+\widetilde\gamma(\lambda_0)^*(\lambda-\bar\lambda_0)\bigl(I+(\lambda-\lambda_0)(A_0-\lambda)^{-1}\bigr)\gamma(\lambda_0),\\ \widetilde M(\mu)&=M(\mu_0)^*+\gamma(\mu_0)^*(\mu-\bar\mu_0)\bigl(I+(\mu-\mu_0)(\widetilde A_0-\mu)^{-1}\bigr)\widetilde\gamma(\mu_0).\end{aligned}$$

2.3. **Closed extensions and their resolvents.** Next we introduce two families of operators in $\mathcal{H}$ as restrictions of $T$ and $\widetilde T$ via abstract boundary conditions in $\mathcal{G}$. Let again $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ be a generalized boundary triple for the adjoint pair $\{S,\widetilde S\}$, where $\{T,\widetilde T\}$ is a core of $\{S^*,\widetilde S^*\}$. For linear operators $B$ and $\widetilde B$ in $\mathcal{G}$ we define the restrictions $A_B$ of $T$ and $\widetilde A_{\widetilde B}$ of $\widetilde T$ via abstract boundary conditions in $\mathcal{G}$ by

$$\begin{aligned}A_B f&:=Tf, &&\operatorname{dom}A_B:=\bigl\{f\in\operatorname{dom}T: B\Gamma_1 f=\Gamma_0 f\bigr\},\\ \widetilde A_{\widetilde B} g&:=\widetilde T g, &&\operatorname{dom}\widetilde A_{\widetilde B}:=\bigl\{g\in\operatorname{dom}\widetilde T:\widetilde B\widetilde\Gamma_1 g=\widetilde\Gamma_0 g\bigr\}.\end{aligned}\tag{2.5}$$

It follows from Green's identity that for a densely defined operator $B$ one has

$$A_B\subset(\widetilde A_{B^*})^*\quad\text{and}\quad\widetilde A_{B^*}\subset(A_B)^*.$$

In the next lemma, we formulate an abstract version of the Birman-Schwinger principle to characterize eigenvalues of the extensions $A_B$ and $\widetilde A_{\widetilde B}$ via the Weyl functions; cf. [8, Corollary 4.3] for a proof.

**Lemma 2.5.** *Let $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ be a generalized boundary triple, assume that $\rho(A_0)\neq\emptyset$ or, equivalently, $\rho(\widetilde A_0)\neq\emptyset$, and let $M$ and $\widetilde M$ be the associated Weyl functions. Then the following assertions hold for the operators $A_B$ and $\widetilde A_{\widetilde B}$ in (2.5), and all $\lambda\in\rho(A_0)$ and $\mu\in\rho(\widetilde A_0)$:*

(i) *$\lambda\in\sigma_p(A_B)$ if and only if $\ker(I-BM(\lambda))\neq\{0\}$, and in this case*

$$\ker(A_B-\lambda)=\gamma(\lambda)\ker(I-BM(\lambda)).$$

(ii) $\mu \in \sigma_p(\widetilde{A}_{\widetilde{B}})$ *if and only if* $\ker(I - \widetilde{B}\widetilde{M}(\mu)) \neq \{0\}$, *and in this case*

$$\ker(\widetilde{A}_{\widetilde{B}} - \mu) = \widetilde{\gamma}(\mu) \ker(I - \widetilde{B}\widetilde{M}(\mu)).$$

In the next theorem we impose abstract conditions on $f, g \in \mathcal{H}$, the $\gamma$-fields, Weyl functions, and parameters $B$ and $\widetilde{B}$ such that a Krein-type formula for the inverses of $A_B - \lambda$ and $\widetilde{A}_{\widetilde{B}} - \mu$ applied to $f$ and $g$, respectively, becomes meaningful; cf. [8, Theorem 4.4]. These conditions will be made more explicit in Theorem 2.8.

**Theorem 2.6.** *Let $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ be a generalized boundary triple and assume that $\rho(A_0) \neq \emptyset$ or, equivalently, $\rho(\widetilde{A}_0) \neq \emptyset$. Let $\gamma, \widetilde{\gamma}$ and $M, \widetilde{M}$ be the associated $\gamma$-fields and Weyl functions, respectively. Then the following assertions hold for the operators $A_B$ and $\widetilde{A}_{\widetilde{B}}$ in* (2.5)*, and all $\lambda \in \rho(A_0)$ and $\mu \in \rho(\widetilde{A}_0)$:*

(i) *If $\lambda \notin \sigma_p(A_B)$ and $f \in \mathcal{H}$ is such that $\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{dom} B$ and $B\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{ran}(I - BM(\lambda))$, then $f \in \operatorname{ran}(A_B - \lambda)$ and*

$$(A_B - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\widetilde{\gamma}(\bar{\lambda})^* f.$$

(ii) *If $\mu \notin \sigma_p(\widetilde{A}_{\widetilde{B}})$ and $g \in \mathcal{H}$ is such that $\gamma(\bar{\mu})^* g \in \operatorname{dom} \widetilde{B}$ and $\widetilde{B}\gamma(\bar{\mu})^* g \in \operatorname{ran}(I - \widetilde{B}\widetilde{M}(\mu))$, then $g \in \operatorname{ran}(\widetilde{A}_{\widetilde{B}} - \mu)$ and*

$$(\widetilde{A}_{\widetilde{B}} - \mu)^{-1} g = (\widetilde{A}_0 - \mu)^{-1} g + \widetilde{\gamma}(\mu)\big(I - \widetilde{B}\widetilde{M}(\mu)\big)^{-1} \widetilde{B}\gamma(\bar{\mu})^* g.$$

As a corollary of Theorem 2.6 we formulate the following result; cf. [8, Theorem 4.6].

**Corollary 2.7.** *Let the assumptions be as in Theorem 2.6 and let $B$ be a densely defined operator in $\mathcal{G}$ such that for some $\lambda \in \rho(A_0)$ one has*

$$\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{dom} B \quad \textit{and} \quad B\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{ran}(I - BM(\lambda)) \quad \textit{for all} \quad f \in \mathcal{H},$$

*and for some $\mu \in \rho(\widetilde{A}_0)$ one has*

$$\gamma(\bar{\mu})^* g \in \operatorname{dom} B^* \quad \textit{and} \quad B^*\gamma(\bar{\mu})^* g \in \operatorname{ran}(I - B^*\widetilde{M}(\mu)) \quad \textit{for all} \quad g \in \mathcal{H}.$$

*Then $A_B$ in* (2.5) *is a closed operator with a nonempty resolvent set and one has $A_B = (\widetilde{A}_{B^*})^*$.*

Now we recall a more direct and explicit criterion in terms of the Weyl function and the boundary operator $B$ such that $A_B$ becomes a closed operator with a nonempty resolvent set; cf. [8, Corollary 4.9].

**Theorem 2.8.** *Let $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ be a generalized boundary triple and assume that $\rho(A_0) \neq \emptyset$ or, equivalently, $\rho(\widetilde{A}_0) \neq \emptyset$. Let $\gamma, \widetilde{\gamma}$ and $M, \widetilde{M}$ be the associated $\gamma$-fields and Weyl functions, respectively.*

(i) *Assume that $B$ is a closable operator in $\mathcal{G}$ such that $1 \in \rho(BM(\lambda_0))$ for some $\lambda_0 \in \rho(A_0)$ and $\operatorname{ran}(\Gamma_1 \upharpoonright \ker \Gamma_0) \subset \operatorname{dom} B$. Then $A_B$ in* (2.5) *is a closed operator, $\lambda_0 \in \rho(A_B)$, and the Krein-type resolvent formula*

$$(A_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\widetilde{\gamma}(\bar{\lambda})^*$$

*holds for all $\lambda \in \rho(A_0) \cap \rho(A_B)$.*

(ii) *Assume that $\widetilde{B}$ is a closable operator in $\mathcal{G}$ such that $1 \in \rho(\widetilde{B}\widetilde{M}(\mu_0))$ for some $\mu_0 \in \rho(\widetilde{A}_0)$ and $\operatorname{ran}(\widetilde{\Gamma}_1 \upharpoonright \ker \widetilde{\Gamma}_0) \subset \operatorname{dom} \widetilde{B}$. Then $\widetilde{A}_{\widetilde{B}}$ in* (2.5) *is a closed operator, $\mu_0 \in \rho(\widetilde{A}_{\widetilde{B}})$, and the Krein-type resolvent formula*

$$(\widetilde{A}_{\widetilde{B}} - \mu)^{-1} = (\widetilde{A}_0 - \mu)^{-1} + \widetilde{\gamma}(\mu)\big(I - \widetilde{B}\widetilde{M}(\mu)\big)^{-1} \widetilde{B}\gamma(\bar{\mu})^*$$

*holds for all $\mu \in \rho(\widetilde{A}_0) \cap \rho(\widetilde{A}_{\widetilde{B}})$.*

*Sketch of the proof of* (i). First, the assumption $\operatorname{ran}(\Gamma_1 \upharpoonright \ker \Gamma_0) \subset \operatorname{dom} B$ and Proposition 2.4 (iii) imply that $B\widetilde{\gamma}(\bar{\lambda}_0)^*$ is an everywhere defined operator from $\mathcal{H}$ into $\mathcal{G}$. Moreover, as $B$ is closable and $\widetilde{\gamma}(\bar{\lambda}_0)^*$ is bounded, $B\widetilde{\gamma}(\bar{\lambda}_0)^*$ is closed and hence bounded by the closed graph theorem. Next, using the assumption $1 \in \rho(BM(\lambda_0))$ we conclude $\lambda_0 \notin \sigma_p(A_B)$ from Lemma 2.5 (i) and, moreover, we also have $\operatorname{ran}(I - BM(\lambda_0)) = \mathcal{G}$. Now it follows from Theorem 2.6 (i) that $\operatorname{ran}(A_B - \lambda_0) = \mathcal{H}$ and

$$(A_B - \lambda_0)^{-1} = (A_0 - \lambda_0)^{-1} + \gamma(\lambda_0)\big(I - BM(\lambda_0)\big)^{-1} B\widetilde{\gamma}(\bar{\lambda}_0)^*.$$

Our assumptions ensure that the right hand side is a bounded operator in $\mathcal{H}$ and hence $(A_B - \lambda_0)^{-1}$ is also bounded, which implies that $A_B$ is closed and $\lambda_0 \in \rho(A_B)$. Finally, the same arguments as in [8, Proof of Theorem 4.7] show that the Krein-type resolvent formula

$$(A_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\widetilde{\gamma}(\bar{\lambda})^*$$

holds for all $\lambda \in \rho(A_0) \cap \rho(A_B)$. □

## 3. Schrödinger operators with unbounded complex potentials on Lipschitz domains with compact boundaries

In this section we apply the abstract notion of generalized boundary triples and their Weyl functions for adjoint pairs to Schrödinger operators with complex potentials $V$ on Lipschitz domains. Here we treat the case of bounded Lipschitz domains $\Omega$ and exterior Lipschitz domains $\Omega$ (that is, complements of the closures of bounded Lipschitz domains) at the same time; cf. Assumption 3.1 (i) below and Appendix A. Using the Dirichlet and Neumann trace operators $\tau_D$ and $\tau_N$ on the space $H^{3/2}_\Delta(\Omega)$ (see (3.1) below) we provide a generalized boundary triple for $\{-\Delta + V, -\Delta + \overline{V}\}$ in Theorem 3.3 and conclude sufficient criteria for Robin-type boundary conditions $B\tau_D f = \tau_N f$ in Corollary 3.7 to induce closed realizations with nonempty resolvent set in $L^2(\Omega)$.

The following assumption is crucial in this section.

**Assumption 3.1.** Let $n \in \mathbb{N}$ with $n \geq 2$.

(i) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain or an unbounded Lipschitz domain with compact boundary $\partial\Omega$ (i.e. $\Omega = \mathbb{R}^n \setminus \overline{\Omega_0}$ with $\Omega_0$ being a bounded Lipschitz domain) and denote the unit normal pointing outward of $\Omega$ by $\nu$.

(ii) Let $V : \Omega \to \mathbb{C}$ be a measurable function such that $V \in L^p(\Omega)$ with $p \geq 2n/3$ if $n > 3$ and $p > 2$ if $n = 2, 3$.

The $L^2$-based Sobolev spaces on $\Omega$ will be denoted by $H^s(\Omega)$, $s \geq 0$, and we shall also make use of the Hilbert spaces

$$H^s_\Delta(\Omega) := \big\{f \in H^s(\Omega) : \Delta f \in L^2(\Omega)\big\}, \quad s \geq 0, \tag{3.1}$$

equipped with the norms induced by

$$(f, g)_{H^s_\Delta(\Omega)} := (f, g)_{H^s(\Omega)} + (\Delta f, \Delta g)_{L^2(\Omega)}, \quad f, g \in H^s_\Delta(\Omega). \tag{3.2}$$

It is clear that for $s \geq 2$ the spaces $H^s_\Delta(\Omega)$ coincide with $H^s(\Omega)$.

In the next lemma we show that the Sobolev spaces $H^s(\Omega)$ are contained in the domain of the maximal multiplication operator in $L^2(\Omega)$ induced by some function $W \in L^p(\Omega)$ under suitable assumptions on $s$ and $p$. In particular, if $V = W$ satisfies Assumption 3.1, then $H^{3/2}(\Omega)$ is contained in the domain of the maximal multiplication operator induced by $V$ in $L^2(\Omega)$. The proof is similar as the proof of [18, Proposition 3.8].

**Lemma 3.2.** *Let Assumption 3.1* (i) *be satisfied, let $s > 0$, let $W \in L^p(\Omega)$ such that $p \geq n/s$ if $n > 2s$ and $p > 2$ if $n \leq 2s$, and let $f \in H^s(\Omega)$. Then $Wf, \overline{W}f \in L^2(\Omega)$ and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|Wf\|_{L^2(\Omega)} = \|\overline{W}f\|_{L^2(\Omega)} \leq \varepsilon\|f\|_{H^s(\Omega)} + C_\varepsilon\|f\|_{L^2(\Omega)}, \quad f \in H^s(\Omega). \tag{3.3}$$

*In particular, if $V \in L^p(\Omega)$ is as in Assumption 3.1* (ii)*, then the above statements are true for $s = 3/2$ and $W = V$, moreover,*

$$\begin{aligned} H^{3/2}_\Delta(\Omega) &= \big\{f \in H^{3/2}(\Omega) : -\Delta f + Vf \in L^2(\Omega)\big\} \\ &= \big\{f \in H^{3/2}(\Omega) : -\Delta f + \overline{V}f \in L^2(\Omega)\big\}. \end{aligned} \tag{3.4}$$

*Proof.* Let $W \in L^p(\Omega)$ and let

$$W_m(x) = \begin{cases} W(x), & \text{if } |W(x)| \leq m, \\ 0, & \text{if } |W(x)| > m, \end{cases} \qquad m \in \mathbb{N}.$$

Then we have $W_m \in L^p(\Omega)$, $m \in \mathbb{N}$, and $\|W - W_m\|_{L^p(\Omega)} \to 0$ as $m \to \infty$. For $f \in H^s(\Omega)$ we will make use of the estimate

$$\|f\|_{L^q(\Omega)} \leq C_{q,s}\|f\|_{H^s(\Omega)}, \qquad q \in \begin{cases} [2, 2n/(n-2s)], & \text{if } n > 2s, \\ [2, \infty), & \text{if } n \leq 2s; \end{cases}$$

cf. [19, Theorem 8.12.6.I]. If $n > 2s$ we use the generalized Hölder inequality with $1/p + 1/q = 1/2$ and $2 \leq q \leq 2n/(n-2s)$ (and hence $p \geq n/s$) and obtain

$$\begin{aligned} \|(W - W_m)f\|_{L^2(\Omega)} &\leq \|W - W_m\|_{L^p(\Omega)}\|f\|_{L^q(\Omega)} \\ &\leq C_{q,s}\|W - W_m\|_{L^p(\Omega)}\|f\|_{H^s(\Omega)}; \end{aligned}$$

the same estimate holds also for $n \leq 2s$ with $2 \leq q < \infty$ (and hence $p > 2$). Therefore, in both cases we conclude

$$\begin{aligned} \|Wf\|_{L^2(\Omega)} &\leq \|(W - W_m)f\|_{L^2(\Omega)} + \|W_m f\|_{L^2(\Omega)} \\ &\leq C_{q,s}\|W - W_m\|_{L^p(\Omega)}\|f\|_{H^s(\Omega)} + m\|f\|_{L^2(\Omega)}, \end{aligned}$$

which implies (3.3).

Finally, if $V$ satisfies Assumption 3.1 (i), then the assumptions of this lemma are fulfilled for $s = 3/2$. Since $Vf, \overline{V}f \in L^2(\Omega)$ for any $f \in H^{3/2}(\Omega)$ it is clear from (3.1) with $s = 3/2$ that (3.4) holds. □

We recall that the Dirichlet Laplacian is defined by

$$H_D = -\Delta, \quad \operatorname{dom} H_D = \big\{f \in H^{3/2}_\Delta(\Omega) : \tau_D f = 0\big\}, \tag{3.5}$$

and the Neumann Laplacian is defined by

$$H_N = -\Delta, \quad \operatorname{dom} H_N = \big\{f \in H^{3/2}_\Delta(\Omega) : \tau_N f = 0\big\}, \tag{3.6}$$

where the Dirichlet trace

$$\tau_D : H^{3/2}_\Delta(\Omega) \to H^1(\partial\Omega) \subset L^2(\partial\Omega) \tag{3.7}$$

and the Neumann trace

$$\tau_N : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega) \tag{3.8}$$

are as in (A.2) and (A.8), respectively, with $s = 3/2$. Both operators $H_D$ and $H_N$ in (3.5)–(3.6) are self-adjoint and nonnegative in $L^2(\Omega)$; they coincide with the self-adjoint operators associated to the densely defined closed nonnegative forms

$$\mathfrak{h}_D[f] = \|\nabla f\|^2_{L^2(\Omega)}, \quad \operatorname{dom} \mathfrak{h}_D = H^1_0(\Omega), \tag{3.9}$$

and

$$\mathfrak{h}_N[f] = \|\nabla f\|^2_{L^2(\Omega)}, \quad \operatorname{dom} \mathfrak{h}_N = H^1(\Omega), \tag{3.10}$$

via the first representation theorem [47, Theorem VI.2.1], see, e.g., [11, Theorem 6.9 and Theorem 6.10] for more details and [45, 46] for the $H^{3/2}$-regularity of the operator domains. In the following let $V \in L^p(\Omega)$ be as in Assumption 3.1 and consider the differential expressions

$$-\Delta + V \quad \text{and} \quad -\Delta + \overline{V}.$$

We define the corresponding minimal operator realizations in $L^2(\Omega)$ by

$$\begin{aligned} S &= -\Delta + \overline{V}, \qquad \operatorname{dom} S = H_0^2(\Omega), \\ \widetilde{S} &= -\Delta + V, \qquad \operatorname{dom} \widetilde{S} = H_0^2(\Omega), \end{aligned} \tag{3.11}$$

where $H_0^2(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the $H^2$-norm and we shall also make use of the operators

$$\begin{aligned} T &= -\Delta + V, \qquad \operatorname{dom} T = H_\Delta^{3/2}(\Omega), \\ \widetilde{T} &= -\Delta + \overline{V}, \qquad \operatorname{dom} \widetilde{T} = H_\Delta^{3/2}(\Omega). \end{aligned} \tag{3.12}$$

It is not difficult to check that $\{S, \widetilde{S}\}$ form an adjoint pair and that $\{T, \widetilde{T}\}$ are well-defined; cf. (3.4). In the proof of Theorem 3.3 below it will turn out en passant that $\{T, \widetilde{T}\}$ is a core of $\{S^*, \widetilde{S}^*\}$.

In the following we will make use of Theorem 2.2 to construct a generalized boundary triple for the adjoint pair $\{S, \widetilde{S}\}$ in (3.11). To this end, we again consider the Dirichlet and Neumann trace operators $\tau_D$ and $\tau_N$ from (3.7)–(3.8), and choose the linear mappings

$$\Gamma_0 = \widetilde{\Gamma}_0 = \tau_N \quad \text{and} \quad \Gamma_1 = \widetilde{\Gamma}_1 = \tau_D \tag{3.13}$$

with domain $\operatorname{dom} T = \operatorname{dom} \widetilde{T} = H_\Delta^{3/2}(\Omega)$.

**Theorem 3.3.** *Let Assumption 3.1 be satisfied. Consider the linear mappings*

$$\Gamma_0, \Gamma_1 : \operatorname{dom} T \to L^2(\partial\Omega) \quad \textit{and} \quad \widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \operatorname{dom} \widetilde{T} \to L^2(\partial\Omega)$$

*given by* (3.13)*. Then* $\{L^2(\partial\Omega), (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *is a generalized boundary triple for the adjoint pair* $\{S, \widetilde{S}\}$ *such that*

$$\begin{aligned} A_0 &= T \upharpoonright \ker \Gamma_0 = -\Delta + V, \quad \operatorname{dom} A_0 = \{ f \in H_\Delta^{3/2}(\Omega) : \tau_N f = 0 \}, \\ \widetilde{A}_0 &= \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0 = -\Delta + \overline{V}, \quad \operatorname{dom} \widetilde{A}_0 = \{ g \in H_\Delta^{3/2}(\Omega) : \tau_N g = 0 \}, \end{aligned} \tag{3.14}$$

*coincide with the Neumann realizations of* $-\Delta + V$ *and* $-\Delta + \overline{V}$*, respectively. Moreover,* $A_0$ *and* $\widetilde{A}_0$ *are closed operators in* $L^2(\Omega)$ *and there exists* $\xi_1 < 0$ *such that* $(-\infty, \xi_1) \subset \rho(A_0) \cap \rho(\widetilde{A}_0)$.

*Proof.* We will verify that the operators $T$ and $\widetilde{T}$ in (3.12) and the boundary mappings in (3.13) satisfy the conditions (i)-(iv) in Theorem 2.2. In fact, for $f, g \in \operatorname{dom} T = \operatorname{dom} \widetilde{T} = H_\Delta^{3/2}(\Omega)$ we have $Vf, \overline{V}g \in L^2(\Omega)$ and hence

$$\begin{aligned} (Tf, g)_{L^2(\Omega)} - (f, \widetilde{T}g)_{L^2(\Omega)} &= (-\Delta f + Vf, g)_{L^2(\Omega)} - (f, -\Delta g + \overline{V}g)_{L^2(\Omega)} \\ &= (-\Delta f, g)_{L^2(\Omega)} - (f, -\Delta g)_{L^2(\Omega)} \\ &= (\tau_D f, \tau_N g)_{L^2(\partial\Omega)} - (\tau_N f, \tau_D g)_{L^2(\partial\Omega)} \\ &= (\Gamma_1 f, \widetilde{\Gamma}_0 g)_{L^2(\partial\Omega)} - (\Gamma_0 f, \widetilde{\Gamma}_1 g)_{L^2(\partial\Omega)}, \end{aligned} \tag{3.15}$$

where we have used (A.14). Thus, (i) in Theorem 2.2 holds. It is clear from Theorem A.2 that $\operatorname{ran} \Gamma_0 = \operatorname{ran} \widetilde{\Gamma}_0 = L^2(\partial\Omega)$ and hence also (ii) is satisfied. In order to check (iii) we use the fact that the Neumann Laplacian $H_N$ in (3.6) is self-adjoint and nonnegative in $L^2(\Omega)$. From Lemma 3.2 and (A.10) we conclude that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|Vf\|_{L^2(\Omega)} \leq \varepsilon \|H_N f\|_{L^2(\Omega)} + C_\varepsilon \|f\|_{L^2(\Omega)}, \quad f \in \operatorname{dom} H_N.$$

In other words, $V$ and $\overline{V}$ are both relatively bounded with respect to $H_N$ with bound smaller than 1 (in fact, with bound 0) and hence it follows from [35, § 3, Theorem 8.9] that the restrictions $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ in (3.14) are closed operators with nonempty resolvent sets and there exists $\xi_1 < 0$ such that $(-\infty, \xi_1) \subset \rho(A_0) \cap \rho(\widetilde{A}_0)$. Green's identity (3.15) yields that $A_0 \subset \widetilde{A}_0^*$ and $\widetilde{A}_0 \subset A_0^*$ and hence also

$$A_0 - \lambda \subset \widetilde{A}_0^* - \lambda \quad \text{and} \quad \widetilde{A}_0 - \lambda \subset A_0^* - \lambda, \quad \lambda \in (-\infty, \xi_1).$$

As $A_0 - \lambda$ and $\widetilde{A}_0 - \lambda$ are bijective we conclude $A_0 = \widetilde{A}_0^*$ and $\widetilde{A}_0 = A_0^*$, that is, (iii) in Theorem 2.2 holds. Finally, condition (iv) is satisfied as $C_0^\infty(\Omega) \subset \ker \Gamma_0 \cap \ker \Gamma_1$ and $C_0^\infty(\Omega) \subset \ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1$.

Now it follows from Theorem 2.2 that the operators

$$T \upharpoonright \ker \Gamma_0 \cap \ker \Gamma_1 \quad \text{and} \quad \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1 \tag{3.16}$$

form an adjoint pair and $\{L^2(\partial\Omega), (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is a generalized boundary triple. One finds with Lemma A.3 that the domains of the operators in (3.16) are $H_0^2(\Omega)$ and hence these restrictions coincide with the minimal operators in (3.11). Therefore, $\{T, \widetilde{T}\}$ is a core of $\{S^*, \widetilde{S}^*\}$ and $\{L^2(\partial\Omega), (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is a generalized boundary triple for $\{S, \widetilde{S}\}$. □

In the next proposition we identify the operators $A_0$ and $\widetilde{A}_0$ in (3.14) as representing operators of the closed sectorial forms (3.17)–(3.18) below, and thus $A_0$ and $\widetilde{A}_0$ are automatically both $m$-sectorial. In addition, we provide, as a variant of [47, Lemma VI.3.1], a useful representation of their resolvents in terms of (the square root) of the resolvent of the Neumann Laplacian $H_N$ in (3.6).

**Proposition 3.4.** *Let Assumption 3.1 be satisfied. Let the operators $A_0, \widetilde{A}_0$, and $H_N$ be as in* (3.14) *and* (3.6)*, respectively. Then $A_0$ and $\widetilde{A}_0$ are both $m$-sectorial operators associated with the closed sectorial forms*

$$\mathfrak{a}_0[f] := \|\nabla f\|^2_{L^2(\Omega)} + \int_\Omega V|f|^2 \,\mathrm{d}x, \quad \operatorname{dom} \mathfrak{a}_0 = H^1(\Omega), \tag{3.17}$$

*and*

$$\widetilde{\mathfrak{a}}_0[f] := \|\nabla f\|^2_{L^2(\Omega)} + \int_\Omega \overline{V}|f|^2 \,\mathrm{d}x, \quad \operatorname{dom} \widetilde{\mathfrak{a}}_0 = H^1(\Omega), \tag{3.18}$$

*respectively. Moreover, there exists $\xi_2 < 0$ such that for all $\lambda \in (-\infty, \xi_2)$ one has $\lambda \in \rho(A_0) \cap \rho(\widetilde{A}_0)$ and there is a bounded operator $C_1(\lambda)$ with $\|C_1(\lambda)\| \leq 1/2$ such that*

$$(A_0 - \lambda)^{-1} = (H_N - \lambda)^{-1/2}(I + C_1(\lambda))^{-1}(H_N - \lambda)^{-1/2} \tag{3.19}$$

*and*

$$(\widetilde{A}_0 - \lambda)^{-1} = (H_N - \lambda)^{-1/2}(I + C_1(\lambda)^*)^{-1}(H_N - \lambda)^{-1/2}. \tag{3.20}$$

*Proof.* Recall the definition of the form $\mathfrak{h}_N$ in (3.10) and that the Neumann Laplacian in (3.6) is the self-adjoint operator associated to $\mathfrak{h}_N$ via the first representation theorem. Next, we consider the form

$$\mathfrak{v}[f] := \int_\Omega V|f|^2 \,\mathrm{d}x, \quad \operatorname{dom} \mathfrak{v} := \operatorname{dom} \mathfrak{h}_N,$$

and employ Lemma 3.2 for $W = |V|^{1/2} \in L^{2p}(\Omega)$ and $s = 1$. It follows that for any $\delta > 0$, there exists $\widetilde{C}_\delta > 0$ such that

$$|\mathfrak{v}[f]| \le \| |V|^{\frac{1}{2}} f\|^2_{L^2(\Omega)} \le \delta \mathfrak{h}_N[f] + \widetilde{C}_\delta \|f\|^2_{L^2(\Omega)}, \quad f \in \operatorname{dom} \mathfrak{h}_N. \tag{3.21}$$

This shows that the form $\mathfrak{v}$ is a relatively bounded perturbation of $\mathfrak{h}_N$ with bound 0. By [47, Theorem VI.3.4], the form $\mathfrak{a}_0 = \mathfrak{h}_N + \mathfrak{v}$ is closed and sectorial, and hence it defines an $m$-sectorial operator $\widehat{A}_0$. Via the first Green's identity (A.15), one verifies that $A_0 \subset \widehat{A}_0$. Since there exists $\lambda_0 < 0$ such that $\lambda_0 \in \rho(A_0) \cap \rho(\widehat{A}_0)$, it follows that $A_0 = \widehat{A}_0$. This justifies the first claim.

To show (3.19), we slightly adjust the proof of [47, Theorem VI.3.2]. To this end, recall first that by the second representation theorem, see [47, Theorem VI.2.23 and Problem VI.2.25], we have for all $\lambda < 0$ that

$$\operatorname{dom} \mathfrak{h}_N = \operatorname{dom} H_N^{1/2} = \operatorname{dom} (H_N - \lambda)^{1/2}$$

and

$$(\mathfrak{h}_N - \lambda)[f, g] = ((H_N - \lambda)^{1/2} f, (H_N - \lambda)^{1/2} g)_{L^2(\Omega)}, \quad f, g \in \operatorname{dom} \mathfrak{h}_N. \tag{3.22}$$

Next, it follows from (3.21) with $\delta = 1/4$ that for all $\lambda < \lambda_1 := -4\widetilde{C}_{1/4}$,

$$|\mathfrak{v}[f]| \le \frac{1}{4}(\mathfrak{h}_N - \lambda)[f] + \left(\frac{1}{4}\lambda + \widetilde{C}_{1/4}\right) \|f\|^2_{L^2(\Omega)} \le \frac{1}{4}(\mathfrak{h}_N - \lambda)[f], \quad f \in \operatorname{dom} \mathfrak{h}_N.$$

Hence [47, Lemma VI.3.1] yields that for all $\lambda < \lambda_1$ there exists a bounded operator $C_1(\lambda)$ with $\|C_1(\lambda)\| \le 1/2$ such that

$$\mathfrak{v}[f, g] = \big(C_1(\lambda)(H_N - \lambda)^{1/2} f, (H_N - \lambda)^{1/2} g\big)_{L^2(\Omega)}, \quad f, g \in \operatorname{dom} \mathfrak{h}_N. \tag{3.23}$$

Combing (3.22) and (3.23), we obtain for all $\lambda < \lambda_1$ and all $f, g \in \operatorname{dom} \mathfrak{h}_N$ that

$$\begin{aligned}(\mathfrak{a}_0 - \lambda)[f, g] &= (\mathfrak{h}_N - \lambda)[f, g] + \mathfrak{v}[f, g] \\ &= \big((I + C_1(\lambda))(H_N - \lambda)^{1/2} f, (H_N - \lambda)^{1/2} g\big)_{L^2(\Omega)}.\end{aligned}$$

Let further $f \in \operatorname{dom} A_0 \subset \operatorname{dom} \mathfrak{h}_N$. Then for all $g \in \operatorname{dom} \mathfrak{h}_N$,

$$((A_0 - \lambda) f, g)_{L^2(\Omega)} = \big((I + C_1(\lambda))(H_N - \lambda)^{1/2} f, (H_N - \lambda)^{1/2} g\big)_{L^2(\Omega)},$$

and hence, since $(H_N - \lambda)^{1/2}$ is self-adjoint,

$$(A_0 - \lambda) f = (H_N - \lambda)^{1/2} (I + C_1(\lambda))(H_N - \lambda)^{1/2} f, \quad f \in \operatorname{dom} A_0.$$

Therefore, we have shown that

$$A_0 - \lambda \subset (H_N - \lambda)^{1/2} (I + C_1(\lambda))(H_N - \lambda)^{1/2}. \tag{3.24}$$

Recall that $\|C_1(\lambda)\| \le 1/2$ for all $\lambda < \lambda_1$, thus $I + C_1(\lambda)$, and therefore also the operator on the right hand side of (3.24), is boundedly invertible for such $\lambda$. On the other hand, since $A_0 - \lambda$ is boundedly invertible for all $\lambda < \lambda_0$ (see above), we arrive at

$$A_0 - \lambda = (H_N - \lambda)^{1/2} (I + C_1(\lambda))(H_N - \lambda)^{1/2} \tag{3.25}$$

for all $\lambda < \xi_2 := \min\{\lambda_0, \lambda_1\}$. Finally, (3.19) follows by inverting (3.25), and the assertions for $\widetilde{\mathfrak{a}}_0$ and $\widetilde{A}_0$ follow in the same way by replacing $V$ with $\overline{V}$ and taking adjoints. □

In the rest of this section we use the triple $\{L^2(\partial\Omega), (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ to show that restrictions of $T$ that satisfy Robin-type boundary conditions $\tau_N f = B \tau_D f$ for a bounded operator $B$ in $L^2(\partial\Omega)$ are closed and have nonempty resolvent set, and we provide an explicit Krein-type resolvent formula for these operators; cf. Corollary 3.7. For that purpose we use decay properties of the associated Weyl function; for this we need the following preparatory lemma.

**Lemma 3.5.** *Let Assumption 3.1 be satisfied. Then there exists $\xi_3 \leq -1$ such that for all $\lambda \in (-\infty, \xi_3)$ and all $\delta > 0$ the densely defined operator*

$$\Gamma_1(H_N - \lambda)^{-1/4-\delta} = \widetilde{\Gamma}_1(H_N - \lambda)^{-1/4-\delta}$$

*admits an everywhere defined bounded extension*

$$C_2(\lambda) : L^2(\Omega) \to L^2(\partial\Omega),$$

*which is uniformly bounded in $\lambda \in (-\infty, \xi_3)$.*

*Proof.* Throughout the proof, we assume that $\lambda \leq \xi_1 - 1$, where $\xi_1 < 0$ is as in Theorem 3.3. First, we claim for any $a \geq 0$ that

$$(H_N - \lambda)^{-a} : L^2(\Omega) \to H^{\min\{1,2a\}}(\Omega) \tag{3.26}$$

is well-defined and uniformly bounded in $\lambda \leq \xi_1 - 1$. For $a = 0$ this is clearly true. Next, we consider the case $a = 1/2$. Let $f \in L^2(\Omega)$ and let $\mathfrak{h}_N$ be the form in (3.10). Then one gets with the help of the second representation theorem [47, Theorem VI.2.23] applied for the nonnegative operator $H_N - \lambda$, $\lambda \leq \xi_1 - 1$ that

$$\begin{aligned}\|(H_N - \lambda)^{-1/2} f\|^2_{H^1(\Omega)} &\leq (\mathfrak{h}_N - \lambda)[(H_N - \lambda)^{-1/2} f] + \|(H_N - \lambda)^{-1/2} f\|^2_{L^2(\Omega)} \\ &\leq 2\|f\|^2_{L^2(\Omega)},\end{aligned}$$

which yields the claim in (3.26) for $a = 1/2$. Thus, the statement for $a \in (0, 1/2)$ follows from an interpolation argument, see, e.g., [52]. Eventually, the claim in (3.26) is also true for $a > 1/2$, as then $(H_N - \lambda)^{-a+1/2}$ is uniformly bounded in $L^2(\Omega)$ in $\lambda \leq \xi_1 - 1$, and thus

$$\begin{aligned}&\|(H_N - \lambda)^{-a}\|_{L^2(\Omega)\to H^1(\Omega)} \\ &\qquad \leq \|(H_N - \lambda)^{-1/2}\|_{L^2(\Omega)\to H^1(\Omega)} \|(H_N - \lambda)^{-a+1/2}\|_{L^2(\Omega)\to L^2(\Omega)} \leq C.\end{aligned}$$

To proceed, denote by $\tau_D$ the Dirichlet trace defined on $H^{\min\{1/2+2\delta,1\}}$; cf. (A.1). Clearly, $\tau_D$ is an extension of $\Gamma_1 = \widetilde{\Gamma}_1$, and taking (3.26) for $a = 1/4+\delta$ into account, we find that

$$C_2(\lambda) := \tau_D(H_N - \lambda)^{-1/4-\delta} : L^2(\Omega) \to L^2(\partial\Omega)$$

is well-defined and uniformly bounded with respect to $\lambda \in (-\infty, \xi_3)$, if $\xi_3$ is chosen smaller than $\xi_1 - 1$. □

In the next proposition we collect some properties of the Weyl functions corresponding to the generalized boundary triple $\{L^2(\partial\Omega), (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$. In particular, in item (iv) we prove decay estimates for $M$ and $\widetilde{M}$.

**Proposition 3.6.** *Let Assumption 3.1 be satisfied. Let $M$ and $\widetilde{M}$ be the Weyl functions corresponding to the generalized boundary triple $\{L^2(\partial\Omega), (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ in Theorem 3.3. Then the following holds for all $\lambda \in \rho(A_0)$ and $\mu \in \rho(\widetilde{A}_0)$:*

(i) *$M(\lambda)\tau_N f_\lambda = \tau_D f_\lambda$ for $f_\lambda \in H^{3/2}_\Delta(\Omega)$ such that $(-\Delta + V)f_\lambda = \lambda f_\lambda$;*
(ii) *$\widetilde{M}(\mu)\tau_N g_\mu = \tau_D g_\mu$ for $g_\mu \in H^{3/2}_\Delta(\Omega)$ such that $(-\Delta + \overline{V})g_\mu = \mu g_\mu$;*
(iii) *$\operatorname{ran} M(\lambda) \subset H^1(\partial\Omega)$ and $\operatorname{ran} \widetilde{M}(\mu) \subset H^1(\partial\Omega)$, and, in particular, the operators $M(\lambda)$ and $\widetilde{M}(\mu)$ are compact in $L^2(\partial\Omega)$;*
(iv) *For all $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that*

$$\|M(\lambda)\| \leq C|\lambda|^{-1/2+\varepsilon} \quad \text{and} \quad \|\widetilde{M}(\mu)\| \leq C|\mu|^{-1/2+\varepsilon}, \quad \lambda, \mu \to -\infty.$$

*Proof.* Items (i) and (ii) are immediate consequences from Proposition 2.4 (iv). It is also clear from the mapping properties of the Dirichlet trace $\tau_D$ in Theorem A.1 that $\operatorname{ran} M(\lambda) \subset H^1(\partial\Omega)$ and $\operatorname{ran} \widetilde{M}(\mu) \subset H^1(\partial\Omega)$. Furthermore, it is easy to check that $M(\lambda)$ and $\widetilde{M}(\mu)$ are closed as operators from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$ and hence bounded. Since $\partial\Omega$ is compact the embedding $H^1(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ is a compact operator and this implies (iii). In order to verify the claim on $M$ in (iv) (the assertion on $\widetilde{M}$ can be shown in a similar way) we employ the resolvent formula (3.20) and Lemma 3.5. In detail, with $\varepsilon > 0$ small and $\lambda < 0$ sufficiently negative,

$$\begin{aligned}
M(\lambda) &= \Gamma_1\gamma(\lambda)^{**} \\
&= \Gamma_1\big(\widetilde{\Gamma}_1(\widetilde{A}_0-\lambda)^{-1}\big)^* \\
&= \Gamma_1\big(\widetilde{\Gamma}_1(H_N-\lambda)^{-1/2}(I+C_1(\lambda)^*)^{-1}(H_N-\lambda)^{-1/2}\big)^* \\
&= \Gamma_1\big(\widetilde{\Gamma}_1(H_N-\lambda)^{-1/4-\varepsilon/2}(H_N-\lambda)^{-1/4+\varepsilon/2}(I+C_1(\lambda)^*)^{-1} \\
&\qquad\qquad \times (H_N-\lambda)^{-1/4+\varepsilon/2}(H_N-\lambda)^{-1/4-\varepsilon/2}\big)^* \\
&= \Gamma_1(H_N-\lambda)^{-1/4-\varepsilon/2}(H_N-\lambda)^{-1/4+\varepsilon/2}(I+C_1(\lambda))^{-1} \\
&\qquad\qquad \times (H_N-\lambda)^{-1/4+\varepsilon/2}(\widetilde{\Gamma}_1(H_N-\lambda)^{-1/4-\varepsilon/2})^*.
\end{aligned}$$

By Lemma 3.5 the operator $\Gamma_1(H_N-\lambda)^{-1/4-\varepsilon/2} = \widetilde{\Gamma}_1(H_N-\lambda)^{-1/4-\varepsilon/2}$ admits a uniformly bounded extension $C_2(\lambda)$, thus for all $\lambda < 0$ sufficiently negative (recall that $\|C_1(\lambda)\| \le 1/2$ from Proposition 3.4)

$$\|M(\lambda)\| \le \|C_2(\lambda)\||\lambda|^{-1/4+\varepsilon/2}2|\lambda|^{-1/4+\varepsilon/2}\|C_2(\lambda)^*\| = 2\|C_2(\lambda)\|^2|\lambda|^{-1/2+\varepsilon},$$

as claimed. □

Finally, we formulate a corollary of Theorem 2.8 in the context of Schrödinger operators with complex potentials satisfying Assumption 3.1. For simplicity we assume that the parameter $B$ in the boundary condition is a bounded everywhere defined operator in $L^2(\partial\Omega)$, so that the condition $1 \in \rho(BM(\lambda_0))$ in Theorem 2.8 is satisfied for all $\lambda_0 < 0$ sufficiently negative by Proposition 3.6 (iv).

**Corollary 3.7.** *Let Assumption 3.1 be satisfied and let $B$ be a bounded everywhere defined operator in $L^2(\partial\Omega)$. Then*

$$A_B = -\Delta + V, \quad \operatorname{dom} A_B = \big\{f \in H^{3/2}_\Delta(\Omega) : \tau_N f = B\tau_D f\big\},$$

*is a closed operator with a nonempty resolvent set, there exists $\xi_4 < 0$ such that $(-\infty,\xi_4) \subset \rho(A_0)\cap\rho(A_B)$, and the Krein-type resolvent formula*

$$(A_B-\lambda)^{-1} = (A_0-\lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1}B\widetilde{\gamma}(\bar{\lambda})^* \tag{3.27}$$

*is valid for all $\lambda \in \rho(A_0)\cap\rho(A_B)$, where $\gamma$ and $\widetilde{\gamma}$ are the $\gamma$-fields associated to the generalized boundary triple $\{L^2(\partial\Omega),(\Gamma_0,\Gamma_1),(\widetilde{\Gamma}_0,\widetilde{\Gamma}_1)\}$ and $M$ is the Weyl function.*

For completeness we note that the operator $\widetilde{\gamma}(\bar{\lambda})^* = \Gamma_1(A_0-\lambda)^{-1}$ is closed and hence bounded as an operator from $L^2(\Omega)$ to $H^1(\partial\Omega)$. Since it is assumed that the boundary of the Lipschitz domain $\Omega$ is compact it follows that the embedding $H^1(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact, and hence $\widetilde{\gamma}(\bar{\lambda})^*$ is a compact operator from $L^2(\Omega)$ to $L^2(\partial\Omega)$. Therefore, under the assumptions in Corollary 3.7 the perturbation term in the resolvent formula (3.27) is compact in $L^2(\Omega)$ and it follows that $A_B$ is a compact perturbation of $A_0$ in resolvent sense.

**Remark 3.8.** An alternative approach to define Robin-type operators is via quadratic forms. Notice first that by the perturbation arguments in the proof of Proposition 3.4 and the boundedness of $B$ and of $\tau_D : H^1(\Omega) \to L^2(\partial\Omega)$, it follows that the form

$$\mathfrak{a}_B[f] := \|\nabla f\|^2_{L^2(\Omega)} + \int_\Omega V|f|^2 - (B\tau_D f,\tau_D f)_{L^2(\partial\Omega)}, \quad \operatorname{dom}\mathfrak{a}_B := H^1(\Omega), \tag{3.28}$$

is closed and sectorial. Thus $\mathfrak{a}_B$ defines an $m$-sectorial operator $\widehat{A}_B$ via the first representation theorem [47, Theorem VI.2.1], in detail

$$\begin{aligned}\widehat{A}_B f &= (-\Delta + V)f, \\ \operatorname{dom} \widehat{A}_B &= \{f \in H^1(\Omega) : (-\Delta + V)f \in L^2(\Omega) \text{ and} \\ &\qquad \mathfrak{a}_B[f,\phi] = ((-\Delta+V)f,\phi)_{L^2(\Omega)} \text{ for all } \phi \in H^1(\Omega)\}.\end{aligned} \tag{3.29}$$

Using the definition of the Neumann trace, it is straightforward to verify that $A_B \subset \widehat{A}_B$ and since $\rho(A_B) \cap \rho(\widehat{A}_B) \neq \emptyset$, we arrive at $A_B = \widehat{A}_B$. One advantage of the generalized boundary triple approach lies in the explicit description of the operator domain, including the $H^{3/2}$-regularity and the boundary condition, as well as the immediate availability of the resolvent formula (3.27).

## Appendix A. Dirichlet and Neumann trace maps on Lipschitz domains with compact boundary

In this appendix we briefly collect some properties of the Dirichlet and Neumann trace maps on Lipschitz domains; for bounded Lipschitz domains the results are known from [11, 40]. Recall first that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, it is well known that for $s \in (1/2, 3/2)$ the Dirichlet trace map $f \mapsto f|_{\partial\Omega}$ for $f \in C^\infty(\overline{\Omega})$ admits a unique continuous extension

$$\tau_D : H^s(\Omega) \to H^{s-1/2}(\partial\Omega), \quad f \mapsto \tau_D f, \quad s \in (1/2, 3/2), \tag{A.1}$$

and this extension has a continuous right inverse; cf. [59, Theorem 3.38]. Similarly, if $\Omega$ is an unbounded Lipschitz domain with compact boundary and $\chi : \Omega \to [0,1]$ is a smooth function equal to 1 near $\partial\Omega$ and equal to 0 sufficiently far away from $\partial\Omega$, then one considers $\tau_D f := \tau_D(\chi f)$, so that (A.1) extends naturally also to such domains.

Now it will be explained that the Dirichlet trace operator can be extended to the endpoints $s = 1/2$ and $s = 3/2$ if one assumes some additional slight regularity in the $H^s$ spaces, that is, one considers the spaces $H^{1/2}_\Delta(\Omega)$ and $H^{3/2}_\Delta(\Omega)$ from (3.1). For bounded Lipschitz domains the next theorem is a variant of [11, Theorem 3.6 and Corollary 3.7], see also [40, Lemma 3.1].

**Theorem A.1.** *Let Assumption 3.1* (i) *be satisfied. Then for all $s \in [1/2, 3/2]$ the Dirichlet trace map* (A.1) *gives rise to a bounded, surjective operator*

$$\tau_D : H^s_\Delta(\Omega) \to H^{s-1/2}(\partial\Omega) \tag{A.2}$$

*(where $H^s_\Delta(\Omega)$ is equipped with the norm induced by* (3.2)*), with bounded right-inverse. In addition, for each $s \in [1/2, 3/2]$, we have*

$$\ker \tau_D \subset H^{3/2}(\Omega) \tag{A.3}$$

*and there exists $C > 0$ such that if $f \in H^{1/2}_\Delta(\Omega)$ and $\tau_D f = 0$, then $f \in H^{3/2}(\Omega)$ and*

$$\|f\|_{H^{3/2}(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|\Delta f\|_{L^2(\Omega)}\big). \tag{A.4}$$

*Proof.* If $\Omega$ is a bounded Lipschitz domain, then the assertions are contained in [11, Corollary 3.7]. In the case that $\Omega$ is unbounded with compact boundary the assertions follow with standard localization methods. We briefly sketch the main arguments: First of all choose a $C^\infty(\Omega)$-function $\chi : \Omega \to [0,1]$ such that $\chi(x) = 1$ for all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < 1$ and $\chi(x) = 0$ for all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > 2$. For $f \in H^s_\Delta(\Omega)$ we obtain from $\Delta f \in L^2(\Omega)$ and standard elliptic regularity that

$f \in H^2_{\mathrm{loc}}(\Omega)$ and hence, as $\nabla\chi$ is compactly supported in $\Omega$, $\nabla\chi \cdot \nabla f \in H^1(\Omega)$. Therefore,

$$\Delta(\chi f) = (\Delta\chi)f + 2\nabla\chi \cdot \nabla f + \chi(\Delta f) \in L^2(\Omega), \tag{A.5}$$

and as $\chi f \in H^s(\Omega)$, see [59, Theorem 3.20], it follows that $\chi f \in H^s_\Delta(\Omega)$, and thus also $(1-\chi)f \in H^s_\Delta(\Omega)$. In particular, as $(1-\chi)f$ vanishes near $\partial\Omega$ one considers the Dirichlet trace

$$\tau_D f := \tau_D(\chi f), \qquad f = \chi f + (1-\chi)f \in H^s_\Delta(\Omega).$$

Then the assertions of the theorem follow when taking into account that $\chi f$ vanishes sufficiently far away from $\partial\Omega$, so that the properties for the Dirichlet trace on a bounded Lipschitz domain can be used. For the estimate (A.4) one uses that

$$\|\chi g\|_{L^2(\Omega)} + \|\Delta(\chi g)\|_{L^2(\Omega)} \le C\big(\|g\|_{L^2(\Omega)} + \|\Delta g\|_{L^2(\Omega)}\big), \quad g \in L^2_\Delta(\Omega) := H^0_\Delta(\Omega). \tag{A.6}$$

Indeed, as in (A.5) one finds for $g \in L^2_\Delta(\Omega)$ that $\chi g \in L^2_\Delta(\Omega)$. Moreover, as $L^2_\Delta(\Omega)$ is continuously embedded in $L^2(\Omega)$ and the multiplication by $\chi$ gives rise to a bounded operator in $L^2(\Omega)$, one can verify that the multiplication by $\chi$ is a closed operator in $L^2_\Delta(\Omega)$. Therefore, by the closed graph theorem, one gets that (A.6) is true. Therefore, using (A.4) on bounded Lipschitz domains for $\chi f$ and (A.6) we conclude

$$\begin{aligned} \|\chi f\|_{H^{3/2}(\Omega)} &\le C\big(\|\chi f\|_{L^2(\Omega)} + \|\Delta(\chi f)\|_{L^2(\Omega)}\big) \\ &\le C\big(\|f\|_{L^2(\Omega)} + \|\Delta f\|_{L^2(\Omega)}\big). \end{aligned} \tag{A.7}$$

To get a similar estimate for $(1-\chi)f$, note first that, as $1-\chi$ is zero in a neighborhood of $\partial\Omega$, one has for the zero extension $\widetilde{g}$ of $(1-\chi)f$ by standard elliptic regularity arguments that $\widetilde{g} \in H^2(\mathbb{R}^n)$. As the graph norm associated with $-\Delta$ in $\mathbb{R}^n$ is equivalent with the norm in $H^2(\mathbb{R}^n)$, the estimate

$$\begin{aligned} \|(1-\chi)f\|_{H^{3/2}(\Omega)} \le \|\widetilde{g}\|_{H^2(\mathbb{R}^n)} &\le C\big(\|\Delta\widetilde{g}\|_{L^2(\mathbb{R}^n)} + \|\widetilde{g}\|_{L^2(\mathbb{R}^n)}\big) \\ &= C\big(\|\Delta((1-\chi)f)\|_{L^2(\Omega)} + \|(1-\chi)f\|_{L^2(\Omega)}\big) \end{aligned}$$

holds. Next, one finds as in (A.6) that the multiplication by $1-\chi$ gives rise to a bounded operator in $L^2_\Delta(\Omega)$. By combining this with the last displayed formula one verifies $\|(1-\chi)f\|_{H^{3/2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|\Delta f\|_{L^2(\Omega)})$, which finally implies with (A.7) the estimate in (A.4) for unbounded domains. □

Now we turn to the Neumann trace operator $f \mapsto \nu \cdot \nabla f|_{\partial\Omega}$ for $f \in C^\infty(\Omega) \cap L^2(\Omega)$. For the case of a bounded Lipschitz domain $\Omega$ the next result is contained in [11, Theorem 5.4 and Corollary 5.7] (see also [40, Lemma 3.2]) and for the case that $\Omega$ is unbounded with compact boundary the same localization arguments as in the proof of Theorem A.1 can be applied to verify the statement; we leave the details to the reader. We recall that $(H^t(\partial\Omega))^* = H^{-t}(\partial\Omega)$ for $t \in [-1,1]$.

**Theorem A.2.** *Let Assumption 3.1* (i) *be satisfied. Then for all $s \in [1/2, 3/2]$ the Neumann trace map induces a bounded, surjective operator*

$$\tau_N : H^s_\Delta(\Omega) \to H^{s-3/2}(\partial\Omega) \tag{A.8}$$

*(where $H^s_\Delta(\Omega)$ is equipped with the norm induced by* (3.2)*), with bounded right-inverse. In addition, for each $s \in [1/2, 3/2]$, we have*

$$\ker \tau_N \subset H^{3/2}(\Omega) \tag{A.9}$$

*and there exists $C > 0$ such that if $f \in H^{1/2}_\Delta(\Omega)$ and $\tau_N f = 0$, then $f \in H^{3/2}(\Omega)$ and*

$$\|f\|_{H^{3/2}(\Omega)} \le C\big(\|f\|_{L^2(\Omega)} + \|\Delta f\|_{L^2(\Omega)}\big). \tag{A.10}$$

*Furthermore, if* $f \in H^{3/2}_\Delta(\Omega)$*, then* $\tau_N f = \nu \cdot \tau_D(\nabla f)$.

Next, we recall a version of the second Green identity for the trace operators in Theorem A.1 and Theorem A.2. Observe first that for $s \in [1/2, 3/2]$ and $f \in H^s_\Delta(\Omega)$ we have

$$\tau_D f \in H^{s-1/2}(\partial\Omega) \quad \text{and} \quad \tau_N f \in H^{s-3/2}(\partial\Omega) = \big(H^{3/2-s}(\partial\Omega)\big)^*, \tag{A.11}$$

and in the same way for $g \in H^{2-s}_\Delta(\Omega)$ we have

$$\tau_D g \in H^{3/2-s}(\partial\Omega) \quad \text{and} \quad \tau_N g \in H^{1/2-s}(\partial\Omega) = \big(H^{s-1/2}(\partial\Omega)\big)^*. \tag{A.12}$$

Then for $s \in [1/2, 3/2]$ and $f \in H^s_\Delta(\Omega)$, $g \in H^{2-s}_\Delta(\Omega)$ one has

$$\begin{aligned}(-\Delta f, g)_{L^2(\Omega)} - (f, -\Delta g)_{L^2(\Omega)} &= \langle \tau_D f, \tau_N g\rangle_{H^{s-1/2}(\partial\Omega)\times(H^{s-1/2}(\partial\Omega))^*} \\ &\quad - \langle \tau_N f, \tau_D g\rangle_{(H^{3/2-s}(\partial\Omega))^*\times H^{3/2-s}(\partial\Omega)};\end{aligned} \tag{A.13}$$

cf. [11, Corollary 5.7] for bounded $\Omega$; the case of unbounded Lipschitz domains with compact boundary can again be handled with a localization argument as in the proof of Theorem A.1. In particular, for $f, g \in H^{3/2}_\Delta(\Omega)$ the traces in (A.11) and (A.12) are contained in $L^2(\partial\Omega)$ and (A.13) takes the form

$$(-\Delta f, g)_{L^2(\Omega)} - (f, -\Delta g)_{L^2(\Omega)} = (\tau_D f, \tau_N g)_{L^2(\partial\Omega)} - (\tau_N f, \tau_D g)_{L^2(\partial\Omega)}. \tag{A.14}$$

Furthermore, for $f \in H^1_\Delta(\Omega)$ and $g \in H^1(\Omega)$ the first Green identity

$$(-\Delta f, g)_{L^2(\Omega)} = (\nabla f, \nabla g)_{L^2(\Omega;\mathbb{R}^n)} - \langle \tau_N f, \tau_D g\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}$$

holds; cf. [59, Theorem 4.4 (i)]. If, in addition, $f \in H^{3/2}_\Delta(\Omega)$, then $\tau_N f$ is contained in $L^2(\partial\Omega)$ and one has

$$(-\Delta f, g)_{L^2(\Omega)} = (\nabla f, \nabla g)_{L^2(\Omega;\mathbb{R}^n)} - (\tau_N f, \tau_D g)_{L^2(\partial\Omega)}. \tag{A.15}$$

In the next lemma we collect an additional regularity property for the functions $f \in H^s_\Delta(\Omega)$ that satisfy $\tau_D f = \tau_N f = 0$. The assertion follows from (A.3), (A.9), and [11, Theorem 6.12 and Remark 5.8] for bounded Lipschitz domains and extends with the help of localization arguments as in the proof of Theorem A.1 also to unbounded Lipschitz domains with compact boundary.

**Lemma A.3.** *Let Assumption 3.1* (i) *be satisfied and let* $s \in [1/2, 3/2]$. *Then the Dirichlet and Neumann trace operators*

$$\tau_D : H^s_\Delta(\Omega) \to H^{s-1/2}(\partial\Omega) \quad \textit{and} \quad \tau_N : H^s_\Delta(\Omega) \to H^{s-3/2}(\partial\Omega)$$

*satisfy* $\ker \tau_D \cap \ker \tau_N = H^2_0(\Omega)$.

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