BOUNDARY VALUE PROBLEMS FOR ADJOINT PAIRS OF OPERATORS

JUSSI BEHRNDT

Abstract. The notion of quasi boundary triples and their Weyl functions from extension theory of symmetric operators is extended to the general framework of adjoint pairs of operators under minimal conditions on the boundary maps. With the help of the corresponding abstract Titchmarsh-Weyl M-functions sufficient conditions for the unique solvability of the related boundary value problems are obtained and the solutions are expressed via Krein-type resolvent formulae. The abstract theory developed in this manuscript can be applied to a large class of elliptic differential operators.

1. Introduction

Boundary value problems for elliptic partial differential operators are often treated within the abstract framework of adjoint pairs of operators. This approach has its roots in the works of M.I. Visik [74], M.S. Birman [24] and G. Grubb [49], and has been further developed in the context of boundary triples and their Weyl functions for adjoint pairs of abstract operators in, e.g., [25, 26, 27, 59]. The notion of (ordinary) boundary triples for adjoint pairs of operators goes back to L.I. Vainerman [73] and the monograph [58] by V.E. Lyantse and O.G. Storozh; the corresponding Weyl functions and Krein-type resolvent formulae were provided later by M.M. Malamud and V.I. Mogilevskii in [60, 61, 62, 65]; see also [14, 52, 53]. For the special case of symmetric operators and the spectral analysis of their self-adjoint extensions the boundary triple technique is nowadays very well established [15, 28, 29, 32, 36, 48, 51, 54, 72] and has been applied and extended in various directions. Among many generalizations of the notion of ordinary boundary triples for symmetric operators are the so-called quasi boundary triples, generalized boundary triples, and boundary relations for symmetric operators and relations from [16, 33, 37], see [17, 20, 21, 22, 23, 30, 31, 34, 35] for subsequent developments and in this context we also refer to [1, 2, 6, 7, 8, 9, 10, 11, 13, 18, 45, 46, 47, 50, 55, 56, 57, 63, 66, 67, 68, 69, 70, 71, 75, 76] for other closely related approaches and typical applications. Extension theory problems for adjoint pairs of operators are also connected to a class of (abstract) positive first order symmetric systems, so-called Friedrichs systems [42, 43, 44], and in this context we also mention the more recent operator theoretic treatment in [4, 5, 38, 39, 40] inspired by [3, 41].

The main objective of this paper is to extend the notion of quasi boundary triples and their Weyl functions for symmetric operators from [16, 17, 20, 22] to the general framework of adjoint pairs of operators, and to develop the abstract theory around this concept; in particular, the aim is to provide sufficient conditions for boundary parameters and boundary mappings to induce closed extensions with nonempty resolvent sets, and to describe their resolvents via Krein-type resolvent formulae. Here we shall work under minimal assumptions on the boundary operators in the triple, that is, we require an abstract version of Green's second identity (G), a weaker density condition (D) on the range of the boundary mappings than usual, and a certain maximality condition (M). In our results we shall always state explicitely which assumptions (G), (D), or (M) are needed for the actual statement.

Let us briefly explain and motivate our approach and the main difference to the concept of ordinary boundary triples for adjoint pairs of operators. For this, consider two densely defined closed operators S and $\widetilde{S}$ in a Hilbert space $\mathfrak{H}$ such that $\widetilde{S} \subset S^*$ (or, equivalently $S \subset \widetilde{S}^*$). Assume that the operators T and $\widetilde{T}$ are cores of S^* and $\widetilde{S}^*$, respectively, that is, their closures coincide with S^* and $\widetilde{S}^*$. The key feature in our theory is the assumption that there exist an auxiliary (boundary) Hilbert space $\mathcal{G}$ and boundary mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ and $\widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$ such that an abstract Green's identity

$$(T f, g)_{\mathfrak{H}} - (f, \widetilde{T} g)_{\mathfrak{H}} = (\Gamma_1 f, \widetilde{\Gamma}_0 g)_{\mathcal{G}} - (\Gamma_0 f, \widetilde{\Gamma}_1 g)_{\mathcal{G}} \tag{G}$$

holds for all $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{T}$. We emphasize that (G) is not required on the (full) domain of the adjoint operators S^* and $\widetilde{S}^*$ (as is the case for ordinary boundary triples) and that (G) typically does not admit an extension onto $\operatorname{dom} S^*$ and $\operatorname{dom} \widetilde{S}^*$. We are mainly interested in the case $\dim \mathcal{G} = \infty$ (as otherwise $T = S^*$ and $\widetilde{T} = \widetilde{S}^*$, and hence ordinary boundary triples can be used). In addition to Green's identity (G) a density condition (D) or (DD) and a maximality condition (M) in Definition 2.1 is often needed for a fruitful and functioning theory. We will then study extensions of $\widetilde{S}$ and S which are restrictions of T and $\widetilde{T}$, respectively, of the form

$$\begin{aligned} A_B f &= T f, & \operatorname{dom} A_B &= \bigl\{ f \in \operatorname{dom} T : B \Gamma_1 f = \Gamma_0 f \bigr\}, \\ \widetilde{A}_{\widetilde{B}} g &= \widetilde{T} g, & \operatorname{dom} \widetilde{A}_{\widetilde{B}} &= \bigl\{ g \in \operatorname{dom} \widetilde{T} : \widetilde{B} \widetilde{\Gamma}_1 g = \widetilde{\Gamma}_0 g \bigr\}, \end{aligned}$$

where B and $\widetilde{B}$ are linear operators in $\mathcal{G}$. In the general abstract setting the goal is to show that A_B and $\widetilde{A}_{\widetilde{B}}$ are closed operators in $\mathfrak{H}$ with nonempty resolvent sets, as this ensures unique solvability or well-posedness of the abstract Robin-type boundary value problems

$$(T - \lambda) f = h, \quad B \Gamma_1 f = \Gamma_0 f, \quad \text{or} \quad (\widetilde{T} - \mu) g = k, \quad \widetilde{B} \widetilde{\Gamma}_1 g = \widetilde{\Gamma}_0 g,$$

whenever $\lambda \in \rho(A_B)$ and $\mu \in \rho(\widetilde{A}_{\widetilde{B}})$ and $h, k \in \mathfrak{H}$. After proving an abstract Birman-Schwinger principle (in a symmetrized form) we find sufficient conditions on the boundary parameter, the mapping properties of the boundary maps, and the associated Weyl functions such that the resolvents A_B and $\widetilde{A}_{\widetilde{B}}$ can be explicitly computed in terms of the resolvent of an underlying fixed extension and a perturbation term in the boundary space $\mathcal{G}$. We find it useful for reference purposes to summarize our results for general adjoint pairs in Appendix A in the special case that $S = \widetilde{S}$ is a densely defined closed symmetric operator. In this situation our results generalize those from [16, 17, 20, 23] in the sense that we impose a weaker density condition (D) on the ranges of the boundary mappings than usual.

The present paper stays mostly on an abstract operator theory level and we have decided to postpone the details of the diverse applications to future investigations and projects (with the small exceptions Example 2.8 and Example 4.10, where a strongly elliptic system on a Lipschitz domain following [64] is discussed). We only indicate here briefly as a motivation, that in the most simple situation of a Schrödinger operator $-\Delta + V$ on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with a complex potential $V \in L^\infty(\Omega)$ a natural choice for the operators $S, \widetilde{S}$ and $T, \widetilde{T}$ in $\mathfrak{H} = L^2(\Omega)$ is

$$S = -\Delta + \overline{V} \quad \text{and} \quad \widetilde{S} = -\Delta + V, \quad \operatorname{dom} S = \operatorname{dom} \widetilde{S} = H_0^2(\Omega),$$

and

$$T = -\Delta + V \quad \text{and} \quad \widetilde{T} = -\Delta + \overline{V}, \quad \operatorname{dom} T = \operatorname{dom} \widetilde{T} = H^2(\Omega),$$

where $H^2(\Omega)$ is the usual second order Sobolev spaces and $H^2_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^2(\Omega)$. It is easy to see that S and $\widetilde{S}$ form an adjoint pair, that is, $\widetilde{S} \subset S^*$, and that the closures of T and $\widetilde{T}$ coincide with S^* and $\widetilde{S}^*$, respectively. Furthermore, for $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{T}$ we have the classical Green's second identity

$$\begin{aligned}(Tf, g)_{L^2(\Omega)} - (f, \widetilde{T}g)_{L^2(\Omega)} &= \big((-\Delta + V)f, g\big)_{L^2(\Omega)} - \big(f, (-\Delta + \overline{V})g\big)_{L^2(\Omega)} \\ &= (-\Delta f, g)_{L^2(\Omega)} - (f, -\Delta g)_{L^2(\Omega)} \\ &= (\tau_D f, \tau_N g)_{L^2(\partial\Omega)} - (\tau_N f, \tau_D g)_{L^2(\partial\Omega)},\end{aligned}$$

where $\tau_D : H^2(\Omega) \to H^{3/2}(\partial\Omega)$ and $\tau_N : H^2(\Omega) \to H^{1/2}(\partial\Omega)$ denote the usual Dirichlet and Neumann trace operators. Therefore, with $\mathcal{G} = L^2(\partial\Omega)$ and the boundary mappings $\Gamma_0 = \widetilde{\Gamma}_0 = \tau_N$ and $\Gamma_1 = \widetilde{\Gamma}_1 = \tau_D$ it is clear that the abstract Green's identity (G) is satisfied. Furthermore, with this choice also the density condition (DD) and the maximality condition (M) in Definition 2.1 hold. Observe that the operators A_B and $\widetilde{A}_{\widetilde{B}}$ above are realizations of the Schrödinger operator $-\Delta + V$ and its formal adjoint $-\Delta + \overline{V}$ subject to Robin boundary conditions of the form

$$\begin{aligned} A_B f &= -\Delta f + V f, \qquad \operatorname{dom} A_B = \big\{f \in H^2(\Omega) : B\tau_D f = \tau_N f\big\}, \\ \widetilde{A}_{\widetilde{B}} g &= -\Delta g + \overline{V} g, \qquad \operatorname{dom} \widetilde{A}_{\widetilde{B}} = \big\{g \in H^2(\Omega) : \widetilde{B}\tau_D g = \tau_N g\big\},\end{aligned}$$

where B and $\widetilde{B}$ are linear (possibly unbounded) operators in $L^2(\partial\Omega)$. We refer the reader to the recent paper [12] for more details and generalizations of this specific example. We also mention that, besides this simple standard situation sketched here, many other applications to differential operators such as, e.g., Dirac operators or $2m$ order elliptic differential operators with variable coefficients can be explored.

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2. Quasi boundary triples for adjoint pairs of operators

Let throughout this section S and $\widetilde{S}$ be densely defined closed operators in a separable Hilbert space $\mathfrak{H}$ such that

$$(Sf, g) = (f, \widetilde{S}g), \qquad f \in \operatorname{dom} S,\ g \in \operatorname{dom} \widetilde{S}, \tag{2.1}$$

holds. Note that this is the same as requiring $\widetilde{S} \subset S^*$ or $S \subset \widetilde{S}^*$. We shall call a pair $\{S, \widetilde{S}\}$ of operators with this property an *adjoint pair* (sometimes also the notion *dual pair* is used in the literature). In this manuscript we are mainly interested in the situation

$$\dim\big(\operatorname{dom} S^*/\operatorname{dom} \widetilde{S}\big) = \dim\big(\operatorname{dom} \widetilde{S}^*/\operatorname{dom} S\big) = \infty, \tag{2.2}$$

although our results do not formally require this condition. Note that in the special case $S = \widetilde{S}$ the property (2.1) shows that S is a symmetric operator and (2.2) means that at least one of the defect numbers of S is infinite. In the following we shall work with operators $T \subset S^*$ and $\widetilde{T} \subset \widetilde{S}^*$ in $\mathfrak{H}$ which (are typically not closed and) satisfy

$$\overline{T} = S^* \quad \text{and} \quad \overline{\widetilde{T}} = \widetilde{S}^*,$$

or, equivalently $T^* = S$ and $\widetilde{T}^* = \widetilde{S}$. In this situation, we shall say that T and $\widetilde{T}$ are *cores* of S^* and $\widetilde{S}^*$, respectively.

Inspired by the notion of quasi boundary triples for symmetric operators in [16, 17] and typical applications in elliptic boundary value problems in the non-symmetric situation (see, e.g., [64]), we extend the definition of boundary triples for adjoint pairs in the abstract setting from [58, 73], see also [25, 26, 27, 53, 60, 61, 62]. At the same time we also formulate a slightly weaker density condition (D) than usual (that is, (DD)) and it will turn out that for many situations this is sufficient for a well functioning theory.

Definition 2.1. Let $\{S, \widetilde{S}\}$ be an adjoint pair of operators in $\mathfrak{H}$ and assume T and $\widetilde{T}$ are cores of S^* and $\widetilde{S}^*$, respectively. We shall consider *triples* of the form $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ for the adjoint pair $\{S, \widetilde{S}\}$, where $\mathcal{G}$ is a Hilbert space and

$$\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}, \qquad \widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \operatorname{dom} \widetilde{T} \to \mathcal{G},$$

are linear mappings. For such a triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ we define the additional properties

(G) the abstract *Green's identity*

$$(Tf, g)_{\mathfrak{H}} - (f, \widetilde{T}g)_{\mathfrak{H}} = (\Gamma_1 f, \widetilde{\Gamma}_0 g)_{\mathcal{G}} - (\Gamma_0 f, \widetilde{\Gamma}_1 g)_{\mathcal{G}}$$

holds for all $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{T}$,

(D) the ranges of $\Gamma_0 : \operatorname{dom} T \to \mathcal{G}$ and $\widetilde{\Gamma}_0 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$ are dense,

(DD) the ranges of $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ and $(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top : \operatorname{dom} \widetilde{T} \to \mathcal{G} \times \mathcal{G}$ are dense,

(M) the operators $A_0 := T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 := \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ satisfy

$$A_0^* = \widetilde{A}_0 \quad \text{and} \quad \widetilde{A}_0^* = A_0. \tag{2.3}$$

If the triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is such that (G), (DD), and (M) hold, then $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is said to be a *quasi boundary triple* for the adjoint pair $\{S, \widetilde{S}\}$.

In the following, we shall formulate all results under minimal assumptions on the triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$, tacitly assuming that $\{S, \widetilde{S}\}$ is an adjoint pair of operators in $\mathfrak{H}$, and T and $\widetilde{T}$ are cores of S^* and $\widetilde{S}^*$, respectively. We will refer to condition (G) as abstract *Green's identity*, condition (D) and the stronger condition (DD) as *density conditions*, and (M) is a *maximality condition.* From now on we shall also suppress the index $\mathfrak{H}$ and $\mathcal{G}$ in the notation of the scalar products in $\mathfrak{H}$ and $\mathcal{G}$, and simply use $(\cdot, \cdot)$.

Note that condition (DD) implies that $\operatorname{ran} \Gamma_0$, $\operatorname{ran} \Gamma_1$, $\operatorname{ran} \widetilde{\Gamma}_0$, and $\operatorname{ran} \widetilde{\Gamma}_1$ are all dense in $\mathcal{G}$ individually and, in particular, condition (DD) implies condition (D). Furthermore, it is clear from (M) that both restrictions $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ are closed operators in $\mathfrak{H}$. Note that, for A_0 closed, the first condition $A_0^* = \widetilde{A}_0$ in (2.3) already implies the second condition $\widetilde{A}_0^* = A_0$ and, in the same way, $\widetilde{A}_0$ closed and $\widetilde{A}_0^* = A_0$ imply the first condition $A_0^* = \widetilde{A}_0$ in (2.3). Later in Section 3 and Section 4 we will typically assume that the resolvent sets of $\rho(A_0)$ and $\rho(\widetilde{A}_0)$ are nonempty; cf. Lemma 3.1.

Remark 2.2. Note that if the triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ has property (G), then for the operators $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ one has

$$(A_0 f, g) - (f, \widetilde{A}_0 g) = (Tf, g) - (f, \widetilde{T}g) = (\Gamma_1 f, \widetilde{\Gamma}_0 g) - (\Gamma_0 f, \widetilde{\Gamma}_1 g) = 0$$

for $f \in \operatorname{dom} A_0$ and $g \in \operatorname{dom} \widetilde{A}_0$, and hence the inclusions

$$\widetilde{A}_0 \subset A_0^* \quad \text{and} \quad A_0 \subset \widetilde{A}_0^*$$

hold without further assumptions; thus property (M) for a triple is only required for the inclusions $\widetilde{A}_0 \supset A_0^*$ and $A_0 \supset \widetilde{A}_0^*$. Furthermore, if property (M) holds, this implies that the cores $T \subset S^*$ and $\widetilde{T} \subset \widetilde{S}^*$ are also extensions of $\widetilde{S}$ and S, respectively, since $A_0 \subset T$, $\widetilde{A}_0 \subset \widetilde{T}$, lead to

$$\widetilde{S} = \widetilde{T}^* \subset \widetilde{A}_0^* = A_0 \subset T \quad \text{and} \quad S = T^* \subset A_0^* = \widetilde{A}_0 \subset \widetilde{T}. \tag{2.4}$$

We point out that the operators T and $\widetilde{T}$ in Definition 2.1 are not unique and may also coincide with S^* and $\widetilde{S}^*$, respectively. However, in the special case $T = S^*$ and $\widetilde{T} = \widetilde{S}^*$ the situation simplifies and reduces to the notion of ordinary boundary triples for adjoint pairs; cf. Proposition 2.6 and, e.g., [27, 62, 58, 73]. It is not difficult to see that the mappings $\Gamma_0, \Gamma_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ are not unique, for instance, if the triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ has the property (G), then also the triple $\{\mathcal{G}, (\Gamma_1, -\Gamma_0), (\widetilde{\Gamma}_1, -\widetilde{\Gamma}_0)\}$ has the property (G). Therefore, by imposing condition (D) for Γ_1 and $\widetilde{\Gamma}_1$ and by requiring that the operators $A_1 := T \upharpoonright \ker \Gamma_1$ and $\widetilde{A}_1 := \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_1$ satisfy the analogue of property (M), that is,

$$A_1^* = \widetilde{A}_1 \quad \text{and} \quad \widetilde{A}_1^* = A_1, \tag{2.5}$$

the triple $\{\mathcal{G}, (\Gamma_1, -\Gamma_0), (\widetilde{\Gamma}_1, -\widetilde{\Gamma}_0)\}$ has the same properties (G), (D), and (M) as the original triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$. In particular, if $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is a quasi boundary triple and (2.5) holds, then $\{\mathcal{G}, (\Gamma_1, -\Gamma_0), (\widetilde{\Gamma}_1, -\widetilde{\Gamma}_0)\}$ is also a quasi boundary triple.

The next lemma shows that the density condition (DD) in Definition 2.1 can be concluded from the surjectivity of the maps Γ_0 and $\widetilde{\Gamma}_0$.

Lemma 2.3. *Assume that the triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ for the adjoint pair $\{S, \widetilde{S}\}$ has properties* (G) *and* (M). *Then the following assertions hold.*

(i) *If $\operatorname{ran} \Gamma_0$ is dense in $\mathcal{G}$ and $\operatorname{ran} \widetilde{\Gamma}_0 = \mathcal{G}$, then $\operatorname{ran} (\Gamma_0, \Gamma_1)^\top$ is dense in $\mathcal{G} \times \mathcal{G}$;*

(ii) *If $\operatorname{ran} \widetilde{\Gamma}_0$ is dense in $\mathcal{G}$ and $\operatorname{ran} \Gamma_0 = \mathcal{G}$, then $\operatorname{ran} (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top$ is dense in $\mathcal{G} \times \mathcal{G}$.*

In particular, if $\operatorname{ran} \Gamma_0 = \operatorname{ran} \widetilde{\Gamma}_0 = \mathcal{G}$, then condition (DD) *in Definition 2.1 is satisfied and $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is a quasi boundary triple for the adjoint pair $\{S, \widetilde{S}\}$.*

Proof. (i) Assume that $(\varphi, \varphi')^\top \in \mathcal{G} \times \mathcal{G}$ is orthogonal to the range of $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ and choose $g \in \operatorname{dom} \widetilde{T}$ such that $\widetilde{\Gamma}_0 g = \varphi'$. Then we have

$$0 = (\Gamma_1 f, \varphi') - (\Gamma_0 f, -\varphi) = (\Gamma_1 f, \widetilde{\Gamma}_0 g) - (\Gamma_0 f, -\varphi) \tag{2.6}$$

for all $f \in \operatorname{dom} T$, and hence the abstract Green's identity (G) becomes

$$(Tf, g) - (f, \widetilde{T} g) = (\Gamma_1 f, \widetilde{\Gamma}_0 g) - (\Gamma_0 f, \widetilde{\Gamma}_1 g) = (\Gamma_0 f, -\varphi - \widetilde{\Gamma}_1 g).$$

In particular, for $f \in \ker \Gamma_0 = \operatorname{dom} A_0$ we have

$$(A_0 f, g) - (f, \widetilde{T} g) = (Tf, g) - (f, \widetilde{T} g) = 0$$

and therefore $g \in \operatorname{dom} A_0^* = \operatorname{dom} \widetilde{A}_0 = \ker \widetilde{\Gamma}_0$, that is, $\varphi' = \widetilde{\Gamma}_0 g = 0$. Now (2.6) reduces to $0 = (\Gamma_0 f, \varphi)$ for all $f \in \operatorname{dom} T$ and as $\operatorname{ran} \Gamma_0$ is dense in $\mathcal{G}$ we conclude $\varphi = 0$. This shows that the range of $(\Gamma_0, \Gamma_1)^\top$ is dense in $\mathcal{G} \times \mathcal{G}$.

(ii) can be proved in the same way as (i). It is also clear from (i) and (ii) that under the assumption $\operatorname{ran} \Gamma_0 = \operatorname{ran} \widetilde{\Gamma}_0 = \mathcal{G}$ the ranges of both $(\Gamma_0, \Gamma_1)^\top$ and

$(\widetilde\Gamma_0,\widetilde\Gamma_1)^\top$ are dense in $\mathcal{G}\times\mathcal{G}$, that is, condition (DD) in Definition 2.1 holds, and hence $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ is a quasi boundary triple. □

We note that, in the situation of Lemma 2.3 with $\operatorname{ran}\Gamma_0=\operatorname{ran}\widetilde\Gamma_0=\mathcal{G}$, the quasi boundary triple $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ can be regarded as a *generalized boundary triple* in the context of adjoint pairs; for the case of symmetric operators this concept appeared already in [37] and we also refer to the more recent contributions [21, 33, 35].

Lemma 2.4. *Assume that the triple* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ *for the adjoint pair* $\{S,\widetilde S\}$ *has the properties* (G), (D), *and* (M). *Then*

$$\operatorname{dom} S=\ker\widetilde\Gamma_0\cap\ker\widetilde\Gamma_1 \quad \textit{and} \quad \operatorname{dom}\widetilde S=\ker\Gamma_0\cap\ker\Gamma_1. \tag{2.7}$$

Proof. We will verify the identity $\operatorname{dom} S=\ker\widetilde\Gamma_0\cap\ker\widetilde\Gamma_1$; the second identity in (2.7) can be shown in the same way. For this, let us consider some fixed $g\in\operatorname{dom} S=\operatorname{dom} T^*$. From $A_0\subset T$ and the maximality condition (M) we obtain $T^*\subset A_0^*=\widetilde A_0\subset\widetilde T$ (see (2.4)), and hence $g\in\operatorname{dom}\widetilde A_0=\ker\widetilde\Gamma_0\subset\operatorname{dom}\widetilde T$. The abstract Green's identity (G) yields

$$\begin{aligned}0&=(Tf,g)-(f,T^*g)\\&=(Tf,g)-(f,\widetilde Tg)\\&=(\Gamma_1 f,\widetilde\Gamma_0 g)-(\Gamma_0 f,\widetilde\Gamma_1 g)\\&=-(\Gamma_0 f,\widetilde\Gamma_1 g)\end{aligned}$$

for all $f\in\operatorname{dom} T$ and as $\operatorname{ran}\Gamma_0$ is dense in $\mathcal{G}$ by condition (D) it follows that also $g\in\ker\widetilde\Gamma_1$. Conversely, for $g\in\ker\widetilde\Gamma_0\cap\ker\widetilde\Gamma_1$ the abstract Green's identity implies

$$(Tf,g)-(f,\widetilde Tg)=(\Gamma_1 f,\widetilde\Gamma_0 g)-(\Gamma_0 f,\widetilde\Gamma_1 g)=0$$

for all $f\in\operatorname{dom} T$. This shows $g\in\operatorname{dom} T^*=\operatorname{dom} S$. □

Lemma 2.5. *Assume that the triple* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ *for the adjoint pair* $\{S,\widetilde S\}$ *has the properties* (G) *and* (DD). *Then the mappings*

$$\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix}:\operatorname{dom} T\to\mathcal{G}\times\mathcal{G} \quad \textit{and} \quad \begin{pmatrix}\widetilde\Gamma_0\\ \widetilde\Gamma_1\end{pmatrix}:\operatorname{dom}\widetilde T\to\mathcal{G}\times\mathcal{G} \tag{2.8}$$

are both closable with respect to the graph norm of T *and* $\widetilde T$, *respectively. In particular, the individual mappings* $\Gamma_0,\Gamma_1:\operatorname{dom} T\to\mathcal{G}$ *and* $\widetilde\Gamma_0,\widetilde\Gamma_1:\operatorname{dom}\widetilde T\to\mathcal{G}$ *are closable.*

Proof. Consider a sequence $f_n\in\operatorname{dom} T$ such that $f_n\to 0$ and $Tf_n\to 0$ as $n\to\infty$ and assume that $\Gamma_0 f_n\to\varphi$ and $\Gamma_1 f_n\to\varphi'$, $n\to\infty$, for some $\varphi,\varphi'\in\mathcal{G}$. Using (G) it follows that

$$\begin{aligned}0&=\lim_{n\to\infty}\bigl((Tf_n,g)-(f_n,\widetilde Tg)\bigr)\\&=\lim_{n\to\infty}\bigl((\Gamma_1 f_n,\widetilde\Gamma_0 g)-(\Gamma_0 f_n,\widetilde\Gamma_1 g)\bigr)\\&=(\varphi',\widetilde\Gamma_0 g)-(\varphi,\widetilde\Gamma_1 g)\end{aligned}$$

and as $\operatorname{ran}(\widetilde\Gamma_0,\widetilde\Gamma_1)^\top$ is dense in $\mathcal{G}\times\mathcal{G}$ by (DD) we conclude $\varphi=\varphi'=0$. This proves that the first mapping in (2.8) is closable and the same argument applies to the second mapping in (2.8). □

Proposition 2.6. *Let* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ *be a quasi boundary triple for the adjoint pair* $\{S,\widetilde S\}$. *Then the following are equivalent.*

(i) $T = S^*$ *and* $\widetilde{T} = \widetilde{S}^*$*;*
(ii) $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top = \mathcal{G} \times \mathcal{G}$ *and* $\operatorname{ran}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top = \mathcal{G} \times \mathcal{G}$.

Proof. (i) $\Rightarrow$ (ii): Since $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top$ and $\operatorname{ran}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top$ are both dense in $\mathcal{G} \times \mathcal{G}$ by (DD) it remains to show that both ranges are closed. We will provide the argument for $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top$; the same reasoning applies to $\operatorname{ran}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top$. Consider a sequence $(\Gamma_0 f_n, \Gamma_1 f_n)^\top$, where $f_n \in \operatorname{dom} T$ and assume that

$$\Gamma_0 f_n \to \varphi \quad \text{and} \quad \Gamma_1 f_n \to \varphi' \tag{2.9}$$

as $n \to \infty$ for some $\varphi, \varphi' \in \mathcal{G}$. Now regard (the graphs) $\widetilde{S}$ and $T = S^*$ as closed subspaces of $\mathfrak{H} \times \mathfrak{H}$ and note first that by the inclusion $\widetilde{S} \subset T = S^*$ in (2.4) there exists a closed subspace $\mathcal{V} \subset \mathfrak{H} \times \mathfrak{H}$ such that $T = \widetilde{S} \oplus \mathcal{V}$; here $\mathcal{V}$ is the orthogonal complement of $\widetilde{S}$ regarded as a subspace of T with respect to the scalar product in $\mathfrak{H} \times \mathfrak{H}$ restricted to T. Therefore, since $\operatorname{dom} \widetilde{S} = \ker \Gamma_0 \cap \ker \Gamma_1$ by Lemma 2.4 it is no restriction to assume that $f_n \in \operatorname{dom} T$ satisfy

$$\left(\begin{pmatrix} f_n \\ T f_n \end{pmatrix}, \begin{pmatrix} k \\ \widetilde{S} k \end{pmatrix} \right) = 0, \quad k \in \operatorname{dom} \widetilde{S}. \tag{2.10}$$

Let $(h, h')^\top \in \mathfrak{H} \times \mathfrak{H}$ be arbitrary and observe that there exist $\widetilde{g} \in \operatorname{dom} \widetilde{S}^*$ and $k \in \operatorname{dom} \widetilde{S}$ such that

$$\begin{pmatrix} h \\ h' \end{pmatrix} = \begin{pmatrix} -\widetilde{S}^* \widetilde{g} \\ \widetilde{g} \end{pmatrix} + \begin{pmatrix} k \\ \widetilde{S} k \end{pmatrix}.$$

Using (2.10), $\widetilde{S}^* = \widetilde{T}$, and the abtract Green's identity (G) we compute

$$\begin{aligned}
\left(\begin{pmatrix} f_n \\ T f_n \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right) &= \left(\begin{pmatrix} f_n \\ T f_n \end{pmatrix}, \begin{pmatrix} -\widetilde{S}^* \widetilde{g} \\ \widetilde{g} \end{pmatrix} + \begin{pmatrix} k \\ \widetilde{S} k \end{pmatrix} \right) \\
&= (T f_n, \widetilde{g}) - (f_n, \widetilde{S}^* \widetilde{g}) \\
&= (T f_n, \widetilde{g}) - (f_n, \widetilde{T} \widetilde{g}) \\
&= (\Gamma_1 f_n, \widetilde{\Gamma}_0 \widetilde{g}) - (\Gamma_0 f_n, \widetilde{\Gamma}_1 \widetilde{g}) \to (\varphi', \widetilde{\Gamma}_0 \widetilde{g}) - (\varphi, \widetilde{\Gamma}_1 \widetilde{g})
\end{aligned}$$

as $n \to \infty$, where (2.9) entered in the last step. It follows that $(f_n, T f_n)^\top$ is a weak Cauchy sequence in $\mathfrak{H} \times \mathfrak{H}$ and hence weakly bounded and thus bounded. This implies that there exists a weakly convergent subsequence, again denoted by $(f_n, T f_n)^\top$ with weak limit $(f, f')^\top \in \overline{T}$. By assumption $T = S^*$, and therefore T is closed, so that necessarily f and f' satisfy $f' = T f$. Now we conclude for any $g \in \operatorname{dom} \widetilde{T}$

$$\begin{aligned}
(\Gamma_1 f, \widetilde{\Gamma}_0 g) - (\Gamma_0 f, \widetilde{\Gamma}_1 g) &= (T f, g) - (f, \widetilde{T} g) \\
&= \lim_{n \to \infty} \big((T f_n, g) - (f_n, \widetilde{T} g) \big) \\
&= \lim_{n \to \infty} \big((\Gamma_1 f_n, \widetilde{\Gamma}_0 g) - (\Gamma_0 f_n, \widetilde{\Gamma}_1 g) \big) \\
&= (\varphi', \widetilde{\Gamma}_0 g) - (\varphi, \widetilde{\Gamma}_1 g),
\end{aligned}$$

that is,

$$\left(\begin{pmatrix} \Gamma_1 f - \varphi' \\ \varphi - \Gamma_0 f \end{pmatrix}, \begin{pmatrix} \widetilde{\Gamma}_0 g \\ \widetilde{\Gamma}_1 g \end{pmatrix} \right) = 0.$$

As $\operatorname{ran}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top$ is dense in $\mathcal{G} \times \mathcal{G}$ by (DD) we obtain $\varphi = \Gamma_0 f$ and $\varphi' = \Gamma_1 f$, in particular, $(\varphi, \varphi')^\top \in \operatorname{ran}(\Gamma_0, \Gamma_1)^\top$ and therefore $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top$ is closed.

(ii) $\Rightarrow$ (i): Since T and $\widetilde{T}$ are cores of S^* and $\widetilde{S}^*$, respectively, it suffices to verify that T and $\widetilde{T}$ are closed. We will provide the proof for T; the same argument can be used to show that $\widetilde{T}$ is closed. Consider a sequence f_n in $\operatorname{dom} T$ such that

$f_n \to f$ and $Tf_n \to f'$ as $n \to \infty$ for some $f, f' \in \mathfrak{H}$. Let $(\psi, \psi')^\top \in \mathcal{G} \times \mathcal{G}$ and choose $g \in \operatorname{dom} \widetilde{T}$ such that $\widetilde{\Gamma}_0 g = \psi'$ and $\widetilde{\Gamma}_1 g = -\psi$, which is possible by our assumptions. Using (G) we compute

$$\begin{aligned}\left(\begin{pmatrix}\Gamma_0 f_n \\ \Gamma_1 f_n\end{pmatrix}, \begin{pmatrix}\psi \\ \psi'\end{pmatrix}\right) &= (\Gamma_1 f_n, \widetilde{\Gamma}_0 g) - (\Gamma_0 f_n, \widetilde{\Gamma}_1 g) \\ &= (T_n f, g) - (f_n, \widetilde{T} g) \to (f', g) - (f, \widetilde{T} g)\end{aligned}$$

as $n \to \infty$. This shows that $(\Gamma_0 f_n, \Gamma_1 f_n)^\top$ is a weak Cauchy sequence in $\mathcal{G} \times \mathcal{G}$ and thus weakly bounded, and hence bounded. Therefore, there exists a weakly convergent subsequence, again denoted by $(\Gamma_0 f_n, \Gamma_1 f_n)^\top$ with weak limit $(\varphi, \varphi')^\top \in \mathcal{G} \times \mathcal{G}$. By assumption there exists $h \in \operatorname{dom} T$ such that $\Gamma_0 h = \varphi$ and $\Gamma_1 h = \varphi'$. Now it follows for $g \in \operatorname{dom} \widetilde{T}$ that

$$\begin{aligned}(f', g) - (f, \widetilde{T} g) &= \lim_{n\to\infty} \big((T f_n, g) - (f_n, \widetilde{T} g)\big) \\ &= \lim_{n\to\infty} \big((\Gamma_1 f_n, \widetilde{\Gamma}_0 g) - (\Gamma_0 f_n, \widetilde{\Gamma}_1 g)\big) \\ &= (\varphi', \widetilde{\Gamma}_0 g) - (\varphi, \widetilde{\Gamma}_1 g) \\ &= (\Gamma_1 h, \widetilde{\Gamma}_0 g) - (\Gamma_0 h, \widetilde{\Gamma}_1 g) \\ &= (Th, g) - (h, \widetilde{T} g),\end{aligned}$$

and hence $(h - f, \widetilde{T} g) = (Th - f', g)$ for all $g \in \operatorname{dom} \widetilde{T}$. This implies $h - f \in \operatorname{dom} \widetilde{T}^*$ and $\widetilde{T}^*(h - f) = Th - f'$. As $\widetilde{T}^* = \widetilde{S} \subset T$ by (2.4) and $h \in \operatorname{dom} T$ we conclude $f \in \operatorname{dom} T$ and $Tf = f'$. We have shown that T is closed. □

The next result is of a slightly different nature: it provides a method to verify that a pair of given operators T and $\widetilde{T}$ form a core of the adjoints of certain (minimal) operators S and $\widetilde{S}$, respectively. To emphasize this different point of view we shall denote the additional properties of the boundary maps here by (G'), (D') or (DD'), and (M').

Theorem 2.7. *Let $\mathfrak{H}$ and $\mathcal{G}$ be Hilbert spaces and let T and $\widetilde{T}$ be operators in $\mathfrak{H}$. Assume that*

$$\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G} \quad \text{and} \quad \widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \operatorname{dom} \widetilde{T} \to \mathcal{G} \tag{2.11}$$

are linear mappings such that

(G') *the abstract Green's identity*

$$(Tf, g) - (f, \widetilde{T} g) = (\Gamma_1 f, \widetilde{\Gamma}_0 g) - (\Gamma_0 f, \widetilde{\Gamma}_1 g)$$

holds for all $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{T}$,

(D') *the ranges of $\Gamma_0 : \operatorname{dom} T \to \mathcal{G}$ and $\widetilde{\Gamma}_0 : \operatorname{dom} \widetilde{T} \to \mathcal{G}$ are dense,*

(M') *the operators $A_0 := T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 := \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ satisfy*

$$A_0^* = \widetilde{A}_0 \quad \text{and} \quad \widetilde{A}_0^* = A_0.$$

If, in addition, $\ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1$ and $\ker \Gamma_0 \cap \ker \Gamma_1$ are dense in $\mathfrak{H}$, then the operators

$$\begin{aligned}Sf &:= \widetilde{T} f, \quad f \in \operatorname{dom} S = \ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1, \\ \widetilde{S} g &:= Tg, \quad g \in \operatorname{dom} \widetilde{S} = \ker \Gamma_0 \cap \ker \Gamma_1,\end{aligned}$$

are closed and form an adjoint pair such that T and $\widetilde{T}$ are cores of S^* and $\widetilde{S}^*$, respectively. Furthermore, if the mappings in (2.11) satisfy the conditions (G'),

(DD') *the ranges of $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ and $(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top : \operatorname{dom} \widetilde{T} \to \mathcal{G} \times \mathcal{G}$ are dense,* *and* (M'), *then* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ *is a quasi boundary triple for the adjoint pair* $\{S,\widetilde S\}$.

Proof. Observe first that by (G') and the definition of S and $\widetilde S$ we have

$$(Sf,g)-(f,\widetilde Sg)=(Tf,g)-(f,\widetilde Tg)=(\Gamma_1 f,\widetilde\Gamma_0 g)-(\Gamma_0 f,\widetilde\Gamma_1 g)=0$$

for all $f\in\operatorname{dom}S$ and $g\in\operatorname{dom}\widetilde S$, and hence $\{S,\widetilde S\}$ is an adjoint pair. We will verify the identity

$$T^*=S \tag{2.12}$$

and the same arguments can be used to prove the identity $\widetilde T^*=\widetilde S$. Note first that from $A_0\subset T$ and (M') it follows that $T^*\subset A_0^*=\widetilde A_0\subset\widetilde T$. Therefore, if $g\in\operatorname{dom}T^*$, then $g\in\operatorname{dom}\widetilde A_0=\ker\widetilde\Gamma_0\subset\operatorname{dom}\widetilde T$ and (G') implies

$$\begin{aligned}0&=(Tf,g)-(f,T^*g)\\&=(Tf,g)-(f,\widetilde Tg)\\&=(\Gamma_1 f,\widetilde\Gamma_0 g)-(\Gamma_0 f,\widetilde\Gamma_1 g)\\&=-(\Gamma_0 f,\widetilde\Gamma_1 g)\end{aligned}$$

for all $f\in\operatorname{dom}T$, and hence assumption (D') shows $\widetilde\Gamma_1 g=0$. Now it follows that $T^*g=\widetilde Tg$ and $g\in\ker\widetilde\Gamma_0\cap\ker\widetilde\Gamma_1$, that is, $T^*\subset S$. For the reverse inclusion let $g\in\ker\widetilde\Gamma_0\cap\ker\widetilde\Gamma_1$. Then it follows from (G') that

$$(Tf,g)-(f,\widetilde Tg)=0$$

holds for all $f\in\operatorname{dom}T$. This implies $g\in\operatorname{dom}T^*$ and $T^*g=\widetilde Tg$, and hence we obtain $S\subset T^*$. We have shown (2.12). It is also clear from (2.12) that the operator S is closed and $\overline T=T^{**}=S^*$ shows that T is a core for S^*. In the same way $\widetilde T^*=\widetilde S$ implies that $\widetilde S$ is closed and that $\widetilde T$ is a core for $\widetilde S^*$. Finally, note that $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ is a quasi boundary triple for the adjoint pair $\{S,\widetilde S\}$ if the conditions (G'), (DD'), and (M') hold. □

We briefly illustrate the abstract theory developed in this section for the case of strongly elliptic systems on Lipschitz domains following the presentation in [64].

Example 2.8. Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$, be a bounded Lipschitz domain with outward unit normal ν and consider a linear second order partial differential expression

$$\mathcal{P}=-\sum_{j,k=1}^n\partial_j A_{jk}\partial_k+\sum_{j=1}^n A_j\partial_j+A$$

with matrix-valued coefficient functions $A_{jk},A_j,A\in L^\infty(\Omega,\mathbb{C}^{m\times m})$ such that A_{jk}, A_j, $j,k=1,\dots,n$, are Lipschitz continuous, and its formal adjoint

$$\widetilde{\mathcal{P}}=-\sum_{j,k=1}^n\partial_j A_{kj}^*\partial_k-\sum_{j=1}^n\partial_j A_j^*+A^*.$$

We define the operators T and $\widetilde T$ in $L^2(\Omega,\mathbb{C}^{m\times m})$ by

$$\begin{aligned}Tf&=\mathcal{P}f, &\operatorname{dom}T&=\bigl\{f\in H^1(\Omega,\mathbb{C}^{m\times m}):\mathcal{P}f\in L^2(\Omega,\mathbb{C}^{m\times m})\bigr\},\\ \widetilde Tg&=\widetilde{\mathcal{P}}g, &\operatorname{dom}\widetilde T&=\bigl\{g\in H^1(\Omega,\mathbb{C}^{m\times m}):\widetilde{\mathcal{P}}g\in L^2(\Omega,\mathbb{C}^{m\times m})\bigr\}.\end{aligned}$$

Recall that the Dirichlet trace operator

$$\tau_D:H^1(\Omega,\mathbb{C}^{m\times m})\to H^{1/2}(\partial\Omega,\mathbb{C}^{m\times m}) \tag{2.13}$$

is bounded and surjective. It follows from the considerations in [64, Lemma 4.3 and Theorem 4.4] that the conormal derivatives

$$f \mapsto \sum_{j=1}^n \nu_j \tau_D(B_j f) \quad \text{and} \quad g \mapsto \sum_{j=1}^n \nu_j \tau_D(\widetilde{B}_j g), \quad f, g \in H^2(\Omega, \mathbb{C}^{m\times m}),$$

where $B_j f = \sum_{k=1}^n A_{jk}\partial_k f$ and $\widetilde{B}_j g = \sum_{k=1}^n A_{kj}^* \partial_k g + A_j^* g$, can be extended by continuity to mappings

$$\tau_N : \operatorname{dom} T \to H^{-1/2}(\partial\Omega, \mathbb{C}^{m\times m}) \quad \text{and} \quad \widetilde{\tau}_N : \operatorname{dom} \widetilde{T} \to H^{-1/2}(\partial\Omega, \mathbb{C}^{m\times m})$$

such that

$$(Tf, g) - (f, \widetilde{T}g) = \langle -\tau_N f, \tau_D g\rangle - \langle \tau_D f, -\widetilde{\tau}_N g\rangle \tag{2.14}$$

holds for all $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{T}$; here $\langle\cdot,\cdot\rangle$ denotes the usual dual pairing of $H^{1/2}(\partial\Omega, \mathbb{C}^{m\times m})$ and $H^{-1/2}(\partial\Omega, \mathbb{C}^{m\times m})$. Now choose isometric isomorphisms $\iota_\pm : H^{\pm 1/2}(\partial\Omega, \mathbb{C}^{m\times m}) \to L^2(\partial\Omega, \mathbb{C}^{m\times m})$ that are compatible with this pairing, so that (2.14) turns into

$$(Tf, g) - (f, \widetilde{T}g) = (-\iota_- \tau_N f, \iota_+ \tau_D g) - (\iota_+ \tau_D f, -\iota_- \widetilde{\tau}_N g). \tag{2.15}$$

Next, define the operators $\widetilde{S}$ and S as restrictions of T and $\widetilde{T}$ onto $\ker \tau_D \cap \ker \tau_N$ and $\ker \tau_D \cap \ker \widetilde{\tau}_N$, respectively. In a more explicit form we have

$$\begin{aligned} \widetilde{S}f &= \mathcal{P}f, \\ \operatorname{dom} \widetilde{S} &= \bigl\{ f \in H^1(\Omega, \mathbb{C}^{m\times m}) : \mathcal{P}f \in L^2(\Omega, \mathbb{C}^{m\times m}), \tau_D f = 0, \tau_N f = 0 \bigr\}, \end{aligned}$$

and

$$\begin{aligned} Sg &= \widetilde{\mathcal{P}}g, \\ \operatorname{dom} S &= \bigl\{ g \in H^1(\Omega, \mathbb{C}^{m\times m}) : \widetilde{\mathcal{P}}g \in L^2(\Omega, \mathbb{C}^{m\times m}), \tau_D g = 0, \widetilde{\tau}_N g = 0 \bigr\}. \end{aligned}$$

Let us now consider the triple

$$\bigl\{ L^2(\partial\Omega, \mathbb{C}^{m\times m}), (\iota_+\tau_D, -\iota_-\tau_N), (\iota_+\tau_D, -\iota_-\widetilde{\tau}_N) \bigr\}. \tag{2.16}$$

Observe that properties (D) and (G) hold by (2.13) and (2.15). Furthermore, in the present situation we actually have $\operatorname{ran} \iota_+\tau_D = L^2(\partial\Omega, \mathbb{C}^{m\times m})$. The operators $A_0 = T \restriction \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \restriction \ker \widetilde{\Gamma}_0$ are given by the Dirichlet realizations

$$A_0 f = \mathcal{P}f, \quad \operatorname{dom} A_0 = \bigl\{ f \in H^1(\Omega, \mathbb{C}^{m\times m}) : \mathcal{P}f \in L^2(\Omega, \mathbb{C}^{m\times m}),\ \tau_D f = 0 \bigr\}$$

and

$$\widetilde{A}_0 g = \widetilde{\mathcal{P}}g, \quad \operatorname{dom} \widetilde{A}_0 = \bigl\{ g \in H^1(\Omega, \mathbb{C}^{m\times m}) : \widetilde{\mathcal{P}}g \in L^2(\Omega, \mathbb{C}^{m\times m}),\ \tau_D g = 0 \bigr\};$$

they automatically satisfy $\widetilde{A}_0 \subset A_0^*$ and $A_0 \subset \widetilde{A}_0^*$ (see Remark 2.2). If, in addition, there exists $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 \in \rho(A_0)$ and $\bar{\lambda}_0 \in \rho(\widetilde{A}_0)$, then one has $A_0 = \widetilde{A}_0^*$ and $A_0^* = \widetilde{A}_0$ (see Lemma 3.1 below), and hence condition (M) holds. Now Theorem 2.7 implies that $\{S, \widetilde{S}\}$ form an adjoint pair and that $\overline{T} = S^*$ and $\overline{\widetilde{T}} = \widetilde{S}^*$. Note that by Lemma 2.3 the stronger density condition (DD) holds, and hence the triple (2.16) is a quasi boundary triple for the adjoint pair $\{S, \widetilde{S}\}$.

3. γ-fields and Weyl functions

In this section we introduce the notion of γ-fields and Weyl functions following the ideas in [16, 37, 33] in the setting of adjoint pairs; cf. [60, 61, 62]. In the following we consider an adjoint pair $\{S, \widetilde{S}\}$ in $\mathfrak{H}$, cores T and $\widetilde{T}$ of S^* and $\widetilde{S}^*$, respectively, and a triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ as in Definition 2.1. In addition, we assume that the operators $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ have nonempty resolvent sets $\rho(A_0)$ and $\rho(\widetilde{A}_0)$, respectively. Note that $A_0^* = \widetilde{A}_0$ and $\widetilde{A}_0^* = A_0$ in condition (M) imply $\lambda \in \rho(A_0)$ if and only if $\bar{\lambda} \in \rho(\widetilde{A}_0)$.

We provide a simple criterion for condition (M) to hold in the case that $\rho(A_0)$ and $\widetilde{\rho}(A_0)$ are nonempty.

Lemma 3.1. *Let $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ be a triple as in Definition 2.1 such that the property* (G) *holds. Let $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$, and assume that there exists $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 \in \rho(A_0)$ and $\bar{\lambda}_0 \in \rho(\widetilde{A}_0)$. Then $A_0 = \widetilde{A}_0^*$ and $A_0^* = \widetilde{A}_0$ and, in particular, condition* (M) *is satisfied.*

Proof. Since (G) holds we have $A_0 \subset \widetilde{A}_0^*$ by Remark 2.2. For λ_0 as in the assumptions one also has $\lambda_0 \in \rho(\widetilde{A}_0^*)$, and hence it follows from $A_0 - \lambda_0 \subset \widetilde{A}_0^* - \lambda_0$ that $A_0 = \widetilde{A}_0^*$ and $A_0^* = \widetilde{A}_0$ (since $\widetilde{A}_0$ is closed). □

In the following we shall introduce and collect some properties of the so-called γ-fields $\gamma, \widetilde{\gamma}$ and Weyl functions $M, \widetilde{M}$ corresponding to the triple $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$. Assume again that the resolvent sets of $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ are nonempty and recall first the direct sum decompositions

$$\begin{aligned} \operatorname{dom} T &= \operatorname{dom} A_0 \dotplus \ker(T - \lambda) = \ker \Gamma_0 \dotplus \ker(T - \lambda), \qquad \lambda \in \rho(A_0), \\ \operatorname{dom} \widetilde{T} &= \operatorname{dom} \widetilde{A}_0 \dotplus \ker(\widetilde{T} - \mu) = \ker \widetilde{\Gamma}_0 \dotplus \ker(\widetilde{T} - \mu), \qquad \mu \in \rho(\widetilde{A}_0). \end{aligned} \tag{3.1}$$

In fact, since $A_0 \subset T$ it is clear that the inclusion $\operatorname{dom} T \supset \operatorname{dom} A_0 + \ker(T - \lambda)$ holds. To verify the inclusion $\operatorname{dom} T \subset \operatorname{dom} A_0 + \ker(T - \lambda)$ consider $f \in \operatorname{dom} T$ and choose $g \in \operatorname{dom} A_0$ such that $(T - \lambda)f = (A_0 - \lambda)g$ holds; here we have used $\lambda \in \rho(A_0)$. Then $h := f - g \in \ker(T - \lambda)$, and hence $f = g + h$ with $f \in \operatorname{dom} T$ and $h \in \ker(T - \lambda)$. Note also that the sum is direct as otherwise $\lambda \in \sigma_p(A_0)$. It is clear that the direct sum decomposition of $\operatorname{dom} \widetilde{T}$ in (3.1) can be proved in the same way.

Definition 3.2. Let $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ be a triple as in Definition 2.1 and assume that the resolvent sets of $A_0 = T \upharpoonright \ker \Gamma_0$ and $\widetilde{A}_0 = \widetilde{T} \upharpoonright \ker \widetilde{\Gamma}_0$ are nonempty.

(i) The γ-fields γ and $\widetilde{\gamma}$ associated with $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ are defined by

$$\begin{aligned} \gamma(\lambda) &:= \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1}, \qquad \lambda \in \rho(A_0), \\ \widetilde{\gamma}(\mu) &:= \big(\widetilde{\Gamma}_0 \upharpoonright \ker(\widetilde{T} - \mu)\big)^{-1}, \qquad \mu \in \rho(\widetilde{A}_0), \end{aligned}$$

where the inverses of $\Gamma_0 \upharpoonright \ker(T - \lambda)$ of $\widetilde{\Gamma}_0 \upharpoonright \ker(\widetilde{T} - \mu)$ are defined on $\operatorname{ran} \Gamma_0$ and $\operatorname{ran} \widetilde{\Gamma}_0$, respectively.

(ii) The Weyl functions M and $\widetilde{M}$ associated with $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ are defined by

$$\begin{aligned} M(\lambda) &:= \Gamma_1 \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1} = \Gamma_1 \gamma(\lambda), \qquad \lambda \in \rho(A_0), \\ \widetilde{M}(\mu) &:= \widetilde{\Gamma}_1 \big(\widetilde{\Gamma}_0 \upharpoonright \ker(\widetilde{T} - \mu)\big)^{-1} = \widetilde{\Gamma}_1 \widetilde{\gamma}(\mu), \qquad \mu \in \rho(\widetilde{A}_0). \end{aligned}$$

Observe that for $f_\lambda \in \ker(T - \lambda)$, $\lambda \in \rho(A_0)$, and $g_\mu \in \ker(\widetilde{T} - \mu)$, $\mu \in \rho(\widetilde{A}_0)$, one has

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda \quad \text{and} \quad \widetilde{M}(\mu)\widetilde{\Gamma}_0 g_\mu = \widetilde{\Gamma}_1 g_\mu. \tag{3.2}$$

In the next proposition we collect some properties of the γ-fields.

Proposition 3.3. *Assume that the triple* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$ *has the properties* (G), (D), (M), *and that the resolvent set of* A_0 *or, equivalently, the resolvent set of* $\widetilde A_0$ *is nonempty. Let* γ *and* $\widetilde\gamma$ *be the* γ*-fields associated with* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde\Gamma_0,\widetilde\Gamma_1)\}$. *Then the following assertions hold for all* $\lambda\in\rho(A_0)$ *and* $\mu\in\rho(\widetilde A_0)$.

(i) $\gamma(\lambda)$ *and* $\widetilde\gamma(\mu)$ *are bounded operators from* $\mathcal{G}$ *into* $\mathfrak{H}$ *with dense domains* $\operatorname{dom}\gamma(\lambda)=\operatorname{ran}\Gamma_0$ *and* $\operatorname{dom}\widetilde\gamma(\mu)=\operatorname{ran}\widetilde\Gamma_0$, *and* $\operatorname{ran}\gamma(\lambda)=\ker(T-\lambda)$ *and* $\operatorname{ran}\widetilde\gamma(\mu)=\ker(\widetilde T-\mu)$;

(ii) *for* $\varphi\in\operatorname{ran}\Gamma_0$ *and* $\psi\in\operatorname{ran}\widetilde\Gamma_0$ *the functions* $\lambda\mapsto\gamma(\lambda)\varphi$ *and* $\mu\mapsto\widetilde\gamma(\mu)\psi$ *are holomorphic on* $\rho(A_0)$ *and* $\rho(\widetilde A_0)$, *respectively, and the relations*

$$\begin{aligned}\gamma(\lambda)&=\bigl(I+(\lambda-\nu)(A_0-\lambda)^{-1}\bigr)\gamma(\nu), &&\lambda,\nu\in\rho(A_0),\\ \widetilde\gamma(\mu)&=\bigl(I+(\mu-\omega)(\widetilde A_0-\mu)^{-1}\bigr)\widetilde\gamma(\omega), &&\mu,\omega\in\rho(\widetilde A_0),\end{aligned}\tag{3.3}$$

hold;

(iii) $\gamma(\lambda)^*$ *and* $\widetilde\gamma(\mu)^*$ *are everywhere defined bounded operators from* $\mathfrak{H}$ *to* $\mathcal{G}$ *and for all* $f,g\in\mathfrak{H}$ *one has*

$$\gamma(\lambda)^*f=\widetilde\Gamma_1(\widetilde A_0-\bar\lambda)^{-1}f\quad\textit{and}\quad\widetilde\gamma(\mu)^*g=\Gamma_1(A_0-\bar\mu)^{-1}g,$$

in particular, $\operatorname{ran}\gamma(\lambda)^*\subset\operatorname{ran}\widetilde\Gamma_1$ *and* $\operatorname{ran}\widetilde\gamma(\mu)^*\subset\operatorname{ran}\Gamma_1$.

Proof. (iii) Let $\lambda\in\rho(A_0)$, $\varphi\in\operatorname{dom}\gamma(\lambda)=\operatorname{ran}\Gamma_0$, and $f\in\mathfrak{H}$. Making use of the condition (M) we obtain $\bar\lambda\in\rho(\widetilde A_0)$, and hence there exists $g\in\operatorname{dom}\widetilde A_0=\ker\widetilde\Gamma_0$ such that $(\widetilde A_0-\bar\lambda)g=f$. By the definition of the γ-field we have $\Gamma_0\gamma(\lambda)\varphi=\varphi$ and now it follows from $\widetilde A_0\subset\widetilde T$ and the abstract Green's identity (G) that

$$\begin{aligned}(\gamma(\lambda)\varphi,f)&=\bigl(\gamma(\lambda)\varphi,(\widetilde A_0-\bar\lambda)g\bigr)\\ &=-\bigl((\lambda\gamma(\lambda)\varphi,g)-(\gamma(\lambda)\varphi,\widetilde A_0g)\bigr)\\ &=-\bigl((T\gamma(\lambda)\varphi,g)-(\gamma(\lambda)\varphi,\widetilde Tg)\bigr)\\ &=-\bigl((\Gamma_1\gamma(\lambda)\varphi,\widetilde\Gamma_0g)-(\Gamma_0\gamma(\lambda)\varphi,\widetilde\Gamma_1g)\bigr)\\ &=\bigl(\varphi,\widetilde\Gamma_1(\widetilde A_0-\bar\lambda)^{-1}f\bigr).\end{aligned}$$

Since this identity holds for all $f\in\mathfrak{H}$ and $\varphi\in\operatorname{ran}\Gamma_0$ (the latter is a dense subspace of $\mathcal{G}$ as we assume (D)) we conclude $\operatorname{dom}\gamma(\lambda)^*=\mathfrak{H}$ and $\gamma(\lambda)^*f=\widetilde\Gamma_1(\widetilde A_0-\bar\lambda)^{-1}f$ is valid for all $f\in\mathfrak{H}$. From the fact that the adjoint operator is automatically closed it follows that $\gamma(\lambda)^*$ is bounded. A similar computation leads to the identity $\widetilde\gamma(\mu)^*g=\Gamma_1(A_0-\bar\mu)^{-1}g$ for all $g\in\mathfrak{H}$ and implies that $\widetilde\gamma(\mu)^*$ is also bounded and everywhere defined on $\mathfrak{H}$.

(i) It follows from (iii) that $\gamma(\lambda)^{**}=\overline{\gamma(\lambda)}$ and $\widetilde\gamma(\mu)^{**}=\overline{\widetilde\gamma(\mu)}$ are everywhere defined and bounded operators from $\mathcal{G}$ to $\mathfrak{H}$, and hence also the operators $\gamma(\lambda)$ and $\widetilde\gamma(\mu)$ are bounded. The remaining assertions in (i) are immediate from the definition of the γ-fields.

(ii) For $\lambda,\nu\in\rho(A_0)$ we use (iii) and compute

$$\begin{aligned}\gamma(\lambda)^*-\gamma(\nu)^*&=\widetilde\Gamma_1\bigl((\widetilde A_0-\bar\lambda)^{-1}-(\widetilde A_0-\bar\nu)^{-1}\bigr)\\ &=(\bar\lambda-\bar\nu)\widetilde\Gamma_1(\widetilde A_0-\bar\nu)^{-1}(\widetilde A_0-\bar\lambda)^{-1}\\ &=(\bar\lambda-\bar\nu)\gamma(\nu)^*(\widetilde A_0-\bar\lambda)^{-1}.\end{aligned}$$

Taking adjoints and using (M) leads to

$$\overline{\gamma(\lambda)}-\overline{\gamma(\nu)}=(\lambda-\nu)(A_0-\lambda)^{-1}\overline{\gamma(\nu)},$$

and this implies the first identity in (3.3). The second identity in (3.3) can be proved in the same way. □

Now we turn to the properties of the Weyl functions.

Proposition 3.4. *Assume that the triple* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde{\Gamma}_0,\widetilde{\Gamma}_1)\}$ *has the properties* (G), (D), (M), *and that the resolvent set of* A_0 *or, equivalently, the resolvent set of* $\widetilde{A}_0$ *is nonempty. Let* $\gamma,\widetilde{\gamma}$ *and* $M,\widetilde{M}$ *be the* γ*-fields and Weyl functions, respectively, associated with* $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde{\Gamma}_0,\widetilde{\Gamma}_1)\}$. *Then the following assertions hold for all* $\lambda\in\rho(A_0)$ *and* $\mu\in\rho(\widetilde{A}_0)$.

(i) $M(\lambda)$ *and* $\widetilde{M}(\mu)$ *are operators in* $\mathcal{G}$ *with dense domains* $\operatorname{dom} M(\lambda)=\operatorname{ran}\Gamma_0$ *and* $\operatorname{dom}\widetilde{M}(\mu)=\operatorname{ran}\widetilde{\Gamma}_0$, *and* $\operatorname{ran} M(\lambda)\subset\operatorname{ran}\Gamma_1$ *and* $\operatorname{ran}\widetilde{M}(\mu)\subset\operatorname{ran}\widetilde{\Gamma}_1$;

(ii) $M(\lambda)\subset\widetilde{M}(\bar\lambda)^*$ *and* $\widetilde{M}(\bar\lambda)\subset M(\lambda)^*$ *and one has the identities*

$$\begin{aligned} M(\lambda)-\widetilde{M}(\mu)^* &= (\lambda-\bar\mu)\widetilde{\gamma}(\mu)^*\gamma(\lambda),\\ M(\lambda)^*-\widetilde{M}(\mu) &= (\bar\lambda-\mu)\gamma(\lambda)^*\widetilde{\gamma}(\mu); \end{aligned}\tag{3.4}$$

(iii) *The functions* $\lambda\mapsto M(\lambda)$ *and* $\mu\mapsto\widetilde{M}(\mu)$ *are holomorphic in the sense that they can be written as the sum of the possibly unbounded closed operators* $\widetilde{M}(\lambda_0)^*$ *and* $M(\mu_0)^*$, *respectively, where* $\lambda_0,\mu_0\in\rho(A_0)\cap\rho(\widetilde{A}_0)$ *are fixed, and a bounded holomorphic operator function:*

$$\begin{aligned} M(\lambda) &= \widetilde{M}(\lambda_0)^*+\widetilde{\gamma}(\lambda_0)^*(\lambda-\bar\lambda_0)\big(I+(\lambda-\lambda_0)(A_0-\lambda)^{-1}\big)\gamma(\lambda_0),\\ \widetilde{M}(\mu) &= M(\mu_0)^*+\gamma(\mu_0)^*(\mu-\bar\mu_0)\big(I+(\mu-\mu_0)(\widetilde{A}_0-\mu)^{-1}\big)\widetilde{\gamma}(\mu_0). \end{aligned}$$

Proof. (i) follows immediately from the definition of the Weyl functions M and $\widetilde{M}$.
(ii) Let $\varphi_\lambda\in\operatorname{ran}\Gamma_0$ and $\psi_\mu\in\operatorname{ran}\widetilde{\Gamma}_0$ and pick $f_\lambda\in\ker(T-\lambda)$ and $g_\mu\in\ker(\widetilde{T}-\mu)$ such that $\Gamma_0 f_\lambda=\varphi_\lambda$ and $\widetilde{\Gamma}_0 g_\mu=\psi_\mu$. Then we have $f_\lambda=\gamma(\lambda)\varphi_\lambda$ and $g_\mu=\widetilde{\gamma}(\mu)\psi_\mu$ and a straightforward computation leads to

$$\begin{aligned} (M(\lambda)\varphi_\lambda,\psi_\mu)-(\varphi_\lambda,\widetilde{M}(\mu)\psi_\mu) &= (M(\lambda)\Gamma_0 f_\lambda,\widetilde{\Gamma}_0 g_\mu)-(\Gamma_0 f_\lambda,\widetilde{M}(\mu)\widetilde{\Gamma}_0 g_\mu)\\ &= (\Gamma_1 f_\lambda,\widetilde{\Gamma}_0 g_\mu)-(\Gamma_0 f_\lambda,\widetilde{\Gamma}_1 g_\mu)\\ &= (Tf_\lambda,g_\mu)-(f_\lambda,\widetilde{T}g_\mu)\\ &= (\lambda f_\lambda,g_\mu)-(f_\lambda,\mu g_\mu)\\ &= \big((\lambda-\bar\mu)\gamma(\lambda)\varphi_\lambda,\widetilde{\gamma}(\mu)\psi_\mu\big). \end{aligned}\tag{3.5}$$

For $\mu=\bar\lambda$ this reduces to $(M(\lambda)\varphi_\lambda,\psi_{\bar\lambda})=(\varphi_\lambda,\widetilde{M}(\bar\lambda)\psi_{\bar\lambda})$, and hence $\widetilde{M}(\bar\lambda)\subset M(\lambda)^*$ and $M(\lambda)\subset\widetilde{M}(\bar\lambda)^*$. Furthermore, (3.5) implies the identities (3.4).
(iii) This is an immediate consequence of the formulas (3.4) and Proposition 3.3 (ii). □

4. Abstract boundary value problems

Next we introduce two families of operators in $\mathfrak{H}$ as restrictions of T and $\widetilde{T}$ via abstract boundary conditions in $\mathcal{G}$. Let $\{S,\widetilde{S}\}$ be an adjoint pair of operators in $\mathfrak{H}$ and let $\{\mathcal{G},(\Gamma_0,\Gamma_1),(\widetilde{\Gamma}_0,\widetilde{\Gamma}_1)\}$ be a triple for the adjoint pair $\{S,\widetilde{S}\}$ with linear mappings $\Gamma_0,\Gamma_1:\operatorname{dom}T\to\mathcal{G}$ and $\widetilde{\Gamma}_0,\widetilde{\Gamma}_1:\operatorname{dom}\widetilde{T}\to\mathcal{G}$ as in Definition 2.1, where T and $\widetilde{T}$ are cores of S^* and $\widetilde{S}^*$, respectively. For linear operators B and $\widetilde{B}$ in $\mathcal{G}$ we define

$$\begin{aligned} A_B f &:= Tf, &\quad \operatorname{dom} A_B &:= \big\{f\in\operatorname{dom}T : B\Gamma_1 f=\Gamma_0 f\big\},\\ \widetilde{A}_{\widetilde{B}} g &:= \widetilde{T}g, &\quad \operatorname{dom}\widetilde{A}_{\widetilde{B}} &:= \big\{g\in\operatorname{dom}\widetilde{T} : \widetilde{B}\widetilde{\Gamma}_1 g=\widetilde{\Gamma}_0 g\big\}. \end{aligned}\tag{4.1}$$

Note that the operator B may only be defined on a subspace of $\mathcal{G}$, and hence $f \in \operatorname{dom} A_B$ means that $\Gamma_1 f \in \operatorname{dom} B$ and $B\Gamma_1 f = \Gamma_0 f$; the boundary condition $\widetilde{B}\widetilde{\Gamma}_1 g = \widetilde{\Gamma}_0 g$ is understood in the same way. The goal of this section is to derive conditions on the parameters B or $\widetilde{B}$ and the mapping properties of $\Gamma_0, \Gamma_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ or the corresponding Weyl functions such that A_B or $\widetilde{A}_{\widetilde{B}}$ have nonempty resolvent sets, that is, for $h, k \in \mathfrak{H}$ and $\lambda \in \rho(A_B)$ or $\mu \in \rho(\widetilde{A}_{\widetilde{B}})$ the boundary value problem

$$(T-\lambda)f = h, \quad B\Gamma_1 f = \Gamma_0 f, \quad \text{or} \quad (\widetilde{T}-\mu)g = k, \quad \widetilde{B}\widetilde{\Gamma}_1 g = \widetilde{\Gamma}_0 g,$$

admits a unique solution $f = (A_B - \lambda)^{-1}h$ or $g = (\widetilde{A}_{\widetilde{B}} - \mu)^{-1}k$, which will be expressed in a resolvent formula involving the resolvent of $A_0, \widetilde{A}_0$ and a perturbation term consiting of the γ-fields, Weyl functions and parameters $B, \widetilde{B}$.

We start with a simple preparatory lemma that only makes use of Green's identity (G).

Lemma 4.1. *Assume that the triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *for the adjoint pair* $\{S, \widetilde{S}\}$ *has the property* (G). *Let* B, B' *be linear operators in* $\mathcal{G}$ *and assume that*

$$(B\varphi, \psi) = (\varphi, B'\psi) \tag{4.2}$$

holds for all $\varphi \in \operatorname{dom} B$ *and* $\psi \in \operatorname{dom} B'$. *Then the operators* A_B *and* $\widetilde{A}_{B'}$ *in* (4.1) *satisfy*

$$A_B \subset (\widetilde{A}_{B'})^* \quad \text{and} \quad \widetilde{A}_{B'} \subset (A_B)^*. \tag{4.3}$$

In particular, if B *is densely defined, then*

$$A_B \subset (\widetilde{A}_{B^*})^* \quad \text{and} \quad \widetilde{A}_{B^*} \subset (A_B)^*.$$

Proof. For $f \in \operatorname{dom} A_B \subset \operatorname{dom} T$ and $g \in \operatorname{dom} \widetilde{A}_{B'} \subset \operatorname{dom} \widetilde{T}$ it follows from Green's identity (G) that

$$\begin{aligned}(A_B f, g) - (f, \widetilde{A}_{B'} g) &= (Tf, g) - (f, \widetilde{T}g) \\ &= (\Gamma_1 f, \widetilde{\Gamma}_0 g) - (\Gamma_0 f, \widetilde{\Gamma}_1 g) \\ &= (\Gamma_1 f, B'\widetilde{\Gamma}_1 g) - (B\Gamma_1 f, \widetilde{\Gamma}_1 g) \\ &= 0,\end{aligned}$$

where (4.2) was used in the last step. This implies both inclusions in (4.3). □

In the next theorem we provide an abstract Birman-Schwinger principle in a symmetrized form for operators of the type

$$\begin{aligned} A_{B_1 B_2} f &:= Tf, \qquad \operatorname{dom} A_{B_1 B_2} := \{f \in \operatorname{dom} T : B_1 B_2 \Gamma_1 f = \Gamma_0 f\}, \\ \widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2} g &:= \widetilde{T} g, \qquad \operatorname{dom} \widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2} := \{g \in \operatorname{dom} \widetilde{T} : \widetilde{B}_1 \widetilde{B}_2 \widetilde{\Gamma}_1 g = \widetilde{\Gamma}_0 g\}, \end{aligned} \tag{4.4}$$

where $B_1 B_2$ and $\widetilde{B}_1 \widetilde{B}_2$ are (products of) linear operators in $\mathcal{G}$; cf. (4.1). The special case $B_2 = B$, $B_1 = I$, or $\widetilde{B}_2 = \widetilde{B}$, $\widetilde{B}_1 = I$, in which (4.4) reduces to (4.1) will be mentioned separately.

Theorem 4.2. *Consider a triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *as in Definition 2.1, assume that the resolvent sets of* A_0 *and* $\widetilde{A}_0$ *are nonempty, and let* M *and* $\widetilde{M}$ *be the associated Weyl functions. Then the following assertions hold for the operators* $A_{B_1 B_2}$ *and* $\widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2}$ *in* (4.4), *and all* $\lambda \in \rho(A_0)$ *and* $\mu \in \rho(\widetilde{A}_0)$*:*

(i) $\lambda \in \sigma_p(A_{B_1 B_2})$ *if and only if* $\ker(I - B_2 M(\lambda) B_1) \neq \{0\}$, *and in this case*

$$\ker(A_{B_1 B_2} - \lambda) = \{f_\lambda \in \ker(T - \lambda) : \Gamma_0 f_\lambda = B_1 \varphi, \varphi \in \ker(I - B_2 M(\lambda) B_1)\}. \tag{4.5}$$

(ii) $\mu \in \sigma_p(\widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2})$ *if and only if* $\ker(I - \widetilde{B}_2 \widetilde{M}(\mu) \widetilde{B}_1) \neq \{0\}$, *and in this case*

$$\ker(\widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2} - \mu) = \{g_\mu \in \ker(\widetilde{T} - \mu) : \widetilde{\Gamma}_0 g_\mu = \widetilde{B}_1 \psi, \psi \in \ker(I - \widetilde{B}_2 \widetilde{M}(\mu) \widetilde{B}_1)\}.$$

In the case $B_2 = B$, $B_1 = I$, or $\widetilde{B}_2 = \widetilde{B}$, $\widetilde{B}_1 = I$, we have the following corollary.

Corollary 4.3. *Consider a triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *as in Definition 2.1, assume that the resolvent sets of* A_0 *and* $\widetilde{A}_0$ *are nonempty, and let* M *and* $\widetilde{M}$ *be the associated Weyl functions. Then the following assertions hold for the operators* A_B *and* $\widetilde{A}_{\widetilde{B}}$ *in* (4.1)*, and all* $\lambda \in \rho(A_0)$ *and* $\mu \in \rho(\widetilde{A}_0)$*:*

(i) $\lambda \in \sigma_p(A_B)$ *if and only if* $\ker(I - BM(\lambda)) \neq \{0\}$*, and in this case*

$$\ker(A_B - \lambda) = \gamma(\lambda) \ker(I - BM(\lambda)).$$

(ii) $\mu \in \sigma_p(\widetilde{A}_{\widetilde{B}})$ *if and only if* $\ker(I - \widetilde{B}\widetilde{M}(\mu)) \neq \{0\}$*, and in this case*

$$\ker(\widetilde{A}_{\widetilde{B}} - \mu) = \widetilde{\gamma}(\mu) \ker(I - \widetilde{B}\widetilde{M}(\mu)).$$

Proof of Theorem 4.2. (i) Assume first that $\lambda \in \sigma_p(A_{B_1B_2})$ and let $f_\lambda \neq 0$ be a corresponding eigenfunction. Then $f_\lambda \in \ker(T-\lambda)$ and $\Gamma_0 f_\lambda \neq 0$ as otherwise $f_\lambda \in \operatorname{dom} A_0 \cap \ker(T - \lambda) = \ker(A_0 - \lambda) = \{0\}$. Furthermore, $f_\lambda \in \operatorname{dom} A_{B_1B_2}$ satisfies the boundary condition $\Gamma_0 f_\lambda = B_1 B_2 \Gamma_1 f_\lambda$, and hence we obtain $B_2\Gamma_1 f_\lambda \neq 0$ and $B_1B_2\Gamma_1 f_\lambda \in \operatorname{ran}\Gamma_0 = \operatorname{dom} M(\lambda)$. From the definition of the Weyl function (see (3.2)) we conclude

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda = M(\lambda) B_1 B_2 \Gamma_1 f_\lambda$$

and therefore

$$\begin{aligned} 0 &= B_2\Gamma_1 f_\lambda - B_2 M(\lambda) B_1 B_2 \Gamma_1 f_\lambda \\ &= \big(I - B_2 M(\lambda) B_1\big) B_2 \Gamma_1 f_\lambda \end{aligned}$$

that is, $B_2\Gamma_1 f_\lambda \in \ker(I - B_2M(\lambda)B_1)$ and, in particular, $\ker(I - B_2M(\lambda)B_1) \neq \{0\}$.

For the converse let us fix some $\varphi \in \ker(I - B_2M(\lambda)B_1)$, $\varphi \neq 0$, and note that $\varphi = B_2M(\lambda)B_1\varphi$ implies, in particular, $B_1\varphi \in \operatorname{dom} M(\lambda) = \operatorname{ran}\Gamma_0$ and $M(\lambda)B_1\varphi \in \operatorname{dom} B_2$. Furthermore, we have $B_1\varphi \neq 0$ and

$$B_1\varphi = B_1B_2M(\lambda)B_1\varphi. \tag{4.6}$$

Next, we choose $f_\lambda \in \ker(T - \lambda)$ such that $\Gamma_0 f_\lambda = B_1\varphi$. Since

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda = M(\lambda)B_1\varphi \in \operatorname{dom} B_2$$

we have $B_2\Gamma_1 f_\lambda = B_2M(\lambda)B_1\varphi$, and hence (4.6) implies

$$\Gamma_0 f_\lambda = B_1\varphi = B_1B_2\Gamma_1 f_\lambda.$$

The identity (4.5) follows from the above considerations. □

In the next theorem we impose abstract conditions on $f, g \in \mathfrak{H}$, the γ-fields, Weyl functions, and parameters B_1, B_2, $\widetilde{B}_1$, $\widetilde{B}_2$, such that a Krein-type formula for the inverses of $A_{B_1B_2} - \lambda$ and $\widetilde{A}_{\widetilde{B}_1\widetilde{B}_2} - \mu$ applied to f and g, respectively, becomes meaningful. These conditions will be made more explicit in Theorem 4.7 and the subsequent corollaries.

Theorem 4.4. *Assume that the triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *has the properties* (G), (D), (M)*, and that the resolvent set of* A_0 *or, equivalently, the resolvent set of* $\widetilde{A}_0$ *is nonempty. Let* $\gamma, \widetilde{\gamma}$ *and* $M, \widetilde{M}$ *be the associated* γ*-fields and Weyl functions, respectively. Then the following assertions hold for the operators* $A_{B_1B_2}$ *and* $\widetilde{A}_{\widetilde{B}_1\widetilde{B}_2}$ *in* (4.4)*, and all* $\lambda \in \rho(A_0)$ *and* $\mu \in \rho(\widetilde{A}_0)$*:*

(i) *If* $\lambda \notin \sigma_p(A_{B_1B_2})$ *and* $f \in \mathfrak{H}$ *is such that*

$$\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{dom} B_2 \quad \textit{and} \quad B_2\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{ran}(I - B_2M(\lambda)B_1), \tag{4.7}$$

then $f \in \operatorname{ran}(A_{B_1B_2} - \lambda)$ *and the Krein-type formula*

$$(A_{B_1B_2} - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f + \gamma(\lambda) B_1 \big(I - B_2M(\lambda)B_1\big)^{-1} B_2 \widetilde{\gamma}(\bar{\lambda})^* f \tag{4.8}$$

holds. In particular, if (4.7) *holds for all* $f \in \mathfrak{H}$*, that is,* $\operatorname{ran}\widetilde{\gamma}(\bar\lambda)^* \subset \operatorname{dom} B_2$ *and* $\operatorname{ran} B_2\widetilde{\gamma}(\bar\lambda)^* \subset \operatorname{ran}(I - B_2M(\lambda)B_1)$*, then* $A_{B_1B_2} - \lambda$ *is a bijective operator in* $\mathfrak{H}$*.*

(ii) *If* $\mu \notin \sigma_p(\widetilde{A}_{\widetilde{B}_1\widetilde{B}_2})$ *and* $g \in \mathfrak{H}$ *is such that*

$$\gamma(\bar\mu)^*g \in \operatorname{dom}\widetilde{B}_2 \quad \text{and} \quad \widetilde{B}_2\gamma(\bar\mu)^*g \in \operatorname{ran}(I - \widetilde{B}_2\widetilde{M}(\mu)\widetilde{B}_1), \tag{4.9}$$

then $g \in \operatorname{ran}(\widetilde{A}_{\widetilde{B}_1\widetilde{B}_2} - \mu)$ *and the Krein-type formula*

$$(\widetilde{A}_{\widetilde{B}_1\widetilde{B}_2} - \mu)^{-1}g = (\widetilde{A}_0 - \mu)^{-1}g + \widetilde{\gamma}(\mu)\widetilde{B}_1\bigl(I - \widetilde{B}_2\widetilde{M}(\mu)\widetilde{B}_1\bigr)^{-1}\widetilde{B}_2\gamma(\bar\mu)^*g$$

holds. In particular, if (4.9) *holds for all* $g \in \mathfrak{H}$*, that is,* $\operatorname{ran}\gamma(\bar\mu)^* \subset \operatorname{dom}\widetilde{B}_2$ *and* $\operatorname{ran}\widetilde{B}_2\gamma(\bar\mu)^* \in \operatorname{ran}(I - \widetilde{B}_2\widetilde{M}(\mu)\widetilde{B}_1)$*, then* $\widetilde{A}_{\widetilde{B}_1\widetilde{B}_2} - \mu$ *is a bijective operator in* $\mathfrak{H}$*.*

In the same spirit as in Corollary 4.3 we formulate the special case $B_2 = B$, $B_1 = I$, or $\widetilde{B}_2 = \widetilde{B}$, $\widetilde{B}_1 = I$, separately as a corollary.

Corollary 4.5. *Assume that the triple* $\{\mathcal{G}, (\Gamma_0,\Gamma_1), (\widetilde{\Gamma}_0,\widetilde{\Gamma}_1)\}$ *has the properties* (G), (D), (M)*, and that the resolvent set of* A_0 *or, equivalently, the resolvent set of* $\widetilde{A}_0$ *is nonempty. Let* $\gamma, \widetilde{\gamma}$ *and* $M, \widetilde{M}$ *be the associated* γ*-fields and Weyl functions, respectively. Then the following assertions hold for the operators* A_B *and* $\widetilde{A}_{\widetilde{B}}$ *in* (4.1)*, and all* $\lambda \in \rho(A_0)$ *and* $\mu \in \rho(\widetilde{A}_0)$*:*

(i) *If* $\lambda \notin \sigma_p(A_B)$ *and* $f \in \mathfrak{H}$ *is such that* $\widetilde{\gamma}(\bar\lambda)^*f \in \operatorname{dom} B$ *and* $B\widetilde{\gamma}(\bar\lambda)^*f \in \operatorname{ran}(I - BM(\lambda))$*, then* $f \in \operatorname{ran}(A_B - \lambda)$ *and*

$$(A_B - \lambda)^{-1}f = (A_0 - \lambda)^{-1}f + \gamma(\lambda)\bigl(I - BM(\lambda)\bigr)^{-1}B\widetilde{\gamma}(\bar\lambda)^*f.$$

(ii) *If* $\mu \notin \sigma_p(\widetilde{A}_{\widetilde{B}})$ *and* $g \in \mathfrak{H}$ *is such that* $\gamma(\bar\mu)^*g \in \operatorname{dom}\widetilde{B}$ *and* $\widetilde{B}\gamma(\bar\mu)^*g \in \operatorname{ran}(I - \widetilde{B}\widetilde{M}(\mu))$*, then* $g \in \operatorname{ran}(\widetilde{A}_{\widetilde{B}} - \mu)$ *and*

$$(\widetilde{A}_{\widetilde{B}} - \mu)^{-1}g = (\widetilde{A}_0 - \mu)^{-1}g + \widetilde{\gamma}(\mu)\bigl(I - \widetilde{B}\widetilde{M}(\mu)\bigr)^{-1}\widetilde{B}\gamma(\bar\mu)^*g.$$

Proof of Theorem 4.4. (i) Let $f \in \mathfrak{H}$ and observe that by the assumptions the element $h \in \mathfrak{H}$ given by

$$h := (A_0 - \lambda)^{-1}f + \gamma(\lambda)B_1\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar\lambda)^*f \tag{4.10}$$

is well defined. In fact, the inverse $(I - B_2M(\lambda)B_1)^{-1}$ exists as $\lambda \notin \sigma_p(A_{B_1B_2})$ (see Theorem 4.2 (i)), and hence (4.7) ensures that

$$\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar\lambda)^*f \in \operatorname{dom}\bigl(I - B_2M(\lambda)B_1\bigr) \subset \operatorname{dom} M(\lambda)B_1.$$

Since $\operatorname{dom} M(\lambda) = \operatorname{dom}\gamma(\lambda)$ this together with the definition of the γ-field implies

$$\gamma(\lambda)B_1\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar\lambda)^*f \in \ker(T - \lambda)$$

and, in particular, all products on the right hand side are meaningful. We claim that $h \in \operatorname{dom} A_{B_1B_2}$. First it is clear that $h \in \operatorname{dom} T$ and $\operatorname{dom} A_0 = \ker\Gamma_0$, the definition of γ and M, and Proposition 3.3 (iii) imply

$$\begin{aligned}\Gamma_0 h &= \Gamma_0(A_0 - \lambda)^{-1}f + \Gamma_0\gamma(\lambda)B_1\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar\lambda)^*f \\ &= B_1\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar\lambda)^*f\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\Gamma_1 h &= \Gamma_1(A_0 - \lambda)^{-1}f + \Gamma_1\gamma(\lambda)B_1\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar\lambda)^*f \\ &= \widetilde{\gamma}(\bar\lambda)^*f + M(\lambda)B_1\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar\lambda)^*f.\end{aligned} \tag{4.12}$$

Since $\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{dom} B_2$ and $(I - B_2 M(\lambda) B_1)^{-1} B_2 \widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{dom} B_2 M(\lambda) B_1$ we conclude from (4.12) that $\Gamma_1 h \in \operatorname{dom} B_2$ and

$$\begin{aligned} B_2 \Gamma_1 h &= B_2 \widetilde{\gamma}(\bar{\lambda})^* f + B_2 M(\lambda) B_1 \bigl(I - B_2 M(\lambda) B_1\bigr)^{-1} B_2 \widetilde{\gamma}(\bar{\lambda})^* f \\ &= \bigl((I - B_2 M(\lambda) B_1) + B_2 M(\lambda) B_1\bigr)\bigl(I - B_2 M(\lambda) B_1\bigr)^{-1} B_2 \widetilde{\gamma}(\bar{\lambda})^* f \\ &= \bigl(I - B_2 M(\lambda) B_1\bigr)^{-1} B_2 \widetilde{\gamma}(\bar{\lambda})^* f. \end{aligned}$$

The above identity shows that $B_2 \Gamma_1 h \in \operatorname{dom} B_2 M(\lambda) B_1 \subset \operatorname{dom} B_1$, and hence

$$B_1 B_2 \Gamma_1 h = B_1 \bigl(I - B_2 M(\lambda) B_1\bigr)^{-1} B_2 \widetilde{\gamma}(\bar{\lambda})^* f = \Gamma_0 h,$$

where (4.11) was used in the last step. Therefore, $h \in \operatorname{dom} A_{B_1 B_2}$ and from $A_{B_1 B_2} \subset T$, $A_0 \subset T$, and $\operatorname{ran} \gamma(\lambda) = \ker(T - \lambda)$ we conclude

$$\begin{aligned} (A_{B_1 B_2} - \lambda) h &= (T - \lambda) h \\ &= (T - \lambda)(A_0 - \lambda)^{-1} f + (T - \lambda)\gamma(\lambda) B_1 \bigl(I - B_2 M(\lambda) B_1\bigr)^{-1} B_2 \widetilde{\gamma}(\bar{\lambda})^* f \\ &= f, \end{aligned}$$

that is, $f \in \operatorname{ran}(A_{B_1 B_2} - \lambda)$. Now the Krein type formula (4.8) follows from (4.10) and $h = (A_{B_1 B_2} - \lambda)^{-1} f$. □

Note that Theorem 4.4 provides criteria such that the extensions $A_{B_1 B_2} - \lambda$ or $\widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2} - \mu$ are bijective, but here their inverses are not necessarily bounded, and hence it does not follow automatically that $A_{B_1 B_2}$ or $\widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2}$ are closed extensions with a nonempty resolvent set. The next theorem is our first result in this direction; it is an easy consequence of Theorem 4.4 and Lemma 4.1.

Theorem 4.6. *Assume that the triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *has the properties* (G), (D), (M), *and that the resolvent set of* A_0 *or, equivalently, the resolvent set of* $\widetilde{A}_0$ *is nonempty. Let* B_1, B_2, B_1', B_2' *be operators in* $\mathcal{G}$ *that satisfy*

$$(B_1 B_2 \varphi, \psi) = (\varphi, B_1' B_2' \psi), \quad \varphi \in \operatorname{dom} B_1 B_2, \ \psi \in \operatorname{dom} B_1' B_2',$$

and consider the operators $A_{B_1 B_2}$ *and* $\widetilde{A}_{B_1' B_2'}$ *in* (4.4). *If there exists some* $\lambda \in \rho(A_0)$ *such that assumption* (4.7) *holds for* λ *and all* $f \in \mathfrak{H}$ *and assumption* (4.9) *holds for* $\bar{\lambda}$ *and all* $g \in \mathfrak{H}$, *then*

$$A_{B_1 B_2} = (\widetilde{A}_{B_1' B_2'})^* \tag{4.13}$$

is a closed operator in $\mathfrak{H}$ *with nonempty resolvent set.*

Proof. It follows from Theorem 4.4 that both operators $A_{B_1 B_2} - \lambda$ and $\widetilde{A}_{B_1' B_2'} - \bar{\lambda}$ are bijective. Moreover, Lemma 4.1 implies

$$A_{B_1 B_2} - \lambda \subset (\widetilde{A}_{B_1' B_2'})^* - \lambda \tag{4.14}$$

and it is clear that $(\widetilde{A}_{B_1' B_2'})^* - \lambda$ is closed. Furthermore, $(\widetilde{A}_{B_1' B_2'})^* - \lambda$ is injective, since we have

$$\ker\bigl((\widetilde{A}_{B_1' B_2'})^* - \lambda\bigr) = \operatorname{ran}\bigl(\widetilde{A}_{B_1' B_2'} - \bar{\lambda}\bigr)^{\perp} = \{0\}.$$

Since $A_{B_1 B_2} - \lambda$ is bijective it follows that the operators in (4.14) coincide, and hence we conclude (4.13). In particular, $\lambda \in \rho(A_{B_1 B_2})$. □

Now we provide more direct and explicit criteria on the Weyl function and the parameter $B_1 B_2$ such that $A_{B_1 B_2}$ becomes a closed operator with a nonempty resolvent set. We do not formulate a variant of Theorem 4.7 for the operators $\widetilde{A}_{\widetilde{B}_1 \widetilde{B}_2}$, and we also leave it to the reader to formulate the corresponding versions of Corollary 4.8 and Corollary 4.9 below. In concrete applications for differential operators the conditions (i)-(v) below reduce to properties of the trace maps Γ_0

and Γ_1, the parameter B_1B_2 specifiying the corresponding boundary condition, and properties of the Weyl function at some point $\lambda_0 \in \rho(A_0)$.

Theorem 4.7. *Assume that the triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *has the properties* (G), (D), (M), *and that the resolvent set of* A_0 *is nonempty. Let* $\gamma, \widetilde{\gamma}$ *and* M *be the associated* γ*-fields and Weyl function, respectively. Assume that* B_1 *and* B_2 *are closable operators in* $\mathcal{G}$ *and that for some* $\lambda_0 \in \rho(A_0)$ *the following conditions hold:*

(i) $1 \in \rho(B_2\overline{M(\lambda_0)B_1})$;
(ii) $\operatorname{ran}(B_2\overline{M(\lambda_0)B_1}) \subset \operatorname{ran}\Gamma_0 \cap \operatorname{dom} B_1$;
(iii) $\operatorname{ran}(B_1 \upharpoonright \operatorname{ran}\Gamma_0) \subset \operatorname{ran}\Gamma_0$;
(iv) $\operatorname{ran}(B_2 \upharpoonright \operatorname{ran}\Gamma_1) \subset \operatorname{ran}\Gamma_0$;
(v) $\operatorname{ran}(\Gamma_1 \upharpoonright \ker\Gamma_0) \subset \operatorname{dom} B_1B_2$.

Then $A_{B_1B_2}$ *in* (4.4) *is a closed operator with a nonempty resolvent set and for all* $\lambda \in \rho(A_0) \cap \rho(A_{B_1B_2})$ *the Krein-type resolvent formula*

$$(A_{B_1B_2} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)B_1\bigl(I - B_2M(\lambda)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar{\lambda})^* \tag{4.15}$$

is valid.

Proof. We verify the inclusion

$$\operatorname{ran}\bigl(B_2\widetilde{\gamma}(\bar{\lambda}_0)^*\bigr) \subset \operatorname{ran}\bigl(I - B_2M(\lambda_0)B_1\bigr). \tag{4.16}$$

In fact, consider some $\psi \in \operatorname{ran}(B_2\widetilde{\gamma}(\bar{\lambda}_0)^*)$. Then

$$\psi = B_2\widetilde{\gamma}(\bar{\lambda}_0)^* f = B_2\Gamma_1(A_0 - \lambda_0)^{-1} f$$

for some $f \in \mathfrak{H}$ by Proposition 3.3 (iii) and from $\operatorname{dom} A_0 = \ker\Gamma_0$ and conditions (iv)–(v) we obtain $\psi \in \operatorname{ran}\Gamma_0 \cap \operatorname{dom} B_1$. By condition (i)

$$\varphi := \bigl(I - B_2\overline{M(\lambda_0)B_1}\bigr)^{-1}\psi \tag{4.17}$$

is well defined and $\varphi - \psi = B_2\overline{M(\lambda_0)B_1}\varphi \in \operatorname{ran}\Gamma_0 \cap \operatorname{dom} B_1$ by (ii). Hence also $\varphi \in \operatorname{ran}\Gamma_0 \cap \operatorname{dom} B_1$ and (iii) implies $B_1\varphi \in \operatorname{ran}\Gamma_0 = \operatorname{dom} M(\lambda_0)$. Therefore, $B_2\overline{M(\lambda_0)B_1}\varphi = B_2M(\lambda_0)B_1\varphi$ and together with (4.17) we conclude

$$\bigl(I - B_2M(\lambda_0)B_1\bigr)\varphi = \psi,$$

which shows (4.16).

It is clear from condition (i) and Theorem 4.2 (i) that $\lambda_0 \notin \sigma_p(A_{B_1B_2})$ and by the above observation we can apply Theorem 4.4 (i) for $\lambda_0 \in \rho(A_0)$. More precisely, for any $f \in \mathfrak{H}$ we have $\widetilde{\gamma}(\bar{\lambda}_0)^* f \in \operatorname{dom} B_2$ by condition (v) and $B_2\widetilde{\gamma}(\bar{\lambda}_0)^* f \in \operatorname{ran}(I - B_2M(\lambda_0)B_1)$ was shown in (4.16). Hence (4.7) is valid for all $f \in \mathfrak{H}$ and

$$(A_{B_1B_2} - \lambda_0)^{-1} f = (A_0 - \lambda_0)^{-1} f + \gamma(\lambda_0)B_1\bigl(I - B_2M(\lambda_0)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar{\lambda}_0)^* f \tag{4.18}$$

holds. Moreover, as B_2 is closable and $\widetilde{\gamma}(\bar{\lambda}_0)^*$ is everywhere defined and bounded (see Proposition 3.3 (iii)) it follows that $B_2\widetilde{\gamma}(\bar{\lambda}_0)^*$ is closable, and hence closed and everywhere defined, and thus bounded. Similarly, condition (i) and the assumption that B_1 is closable imply that $B_1(I - B_2\overline{M(\lambda_0)B_1})^{-1}$ is everywhere defined and bounded, and hence the restriction $B_1(I - B_2M(\lambda_0)B_1)^{-1}$ is also bounded. Furthermore, $\gamma(\lambda_0)$ is a bounded operator by Proposition 3.3 (i). Summing up we have shown that

$$\gamma(\lambda_0)B_1\bigl(I - B_2M(\lambda_0)B_1\bigr)^{-1}B_2\widetilde{\gamma}(\bar{\lambda}_0)^*$$

is a bounded and everywhere defined operator. The same is true for $(A_0 - \lambda_0)^{-1}$ and from (4.18) we conclude that $(A_{B_1B_2} - \lambda_0)^{-1}$ is a bounded and everywhere defined operator, and hence closed. This implies that $A_{B_1B_2}$ is closed and $\lambda_0 \in \rho(A_{B_1B_2})$.

Now consider $\lambda \in \rho(A_{B_1B_2}) \cap \rho(A_0)$. As above we have for any $f \in \mathfrak{H}$ that $\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{dom} B_2$ by condition (v). We claim that

$$B_2\widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{ran}\bigl(I - B_2 M(\lambda) B_1\bigr). \tag{4.19}$$

For this we consider $k = (A_{B_1B_2} - \lambda)^{-1} f$ and $h = (A_0 - \lambda)^{-1} f$. Note that $B_1B_2\Gamma_1 k = \Gamma_0 k$ and, in particular, $\Gamma_1 k \in \operatorname{dom} B_2$. Moreover, $\Gamma_0 h = 0$ and from $(T - \lambda)(k - h) = 0$ we conclude $M(\lambda)\Gamma_0(k - h) = \Gamma_1(k - h)$. Therefore,

$$M(\lambda)B_1B_2\Gamma_1 k = M(\lambda)\Gamma_0 k = M(\lambda)\Gamma_0(k - h) = \Gamma_1(k - h). \tag{4.20}$$

As $\Gamma_1 k \in \operatorname{dom} B_2$ and $\Gamma_1 h = \Gamma_1(A_0 - \lambda)^{-1} f = \widetilde{\gamma}(\bar{\lambda})^* f \in \operatorname{dom} B_2$ by (v) we see that $M(\lambda)B_1B_2\Gamma_1 k \in \operatorname{dom} B_2$, and hence the element

$$B_2 M(\lambda) B_1 B_2 \Gamma_1 k$$

is well defined. Now we use (4.20) and $\Gamma_1 h = \widetilde{\gamma}(\bar{\lambda})^* f$ and compute

$$\bigl(I - B_2 M(\lambda) B_1\bigr) B_2 \Gamma_1 k = B_2\Gamma_1 k - B_2\Gamma_1(k - h) = B_2\Gamma_1 h = B_2\widetilde{\gamma}(\bar{\lambda})^* f,$$

which shows (4.19). Therefore, both conditions in (4.7) are satisfied for all $\lambda \in \rho(A_{B_1B_2}) \cap \rho(A_0)$ and $f \in \mathfrak{H}$, and hence the Krein-type resolvent formula (4.15) follows from Theorem 4.4. □

In the special case $B_1 = I$ and $B_2 = B$ one obtains the following statement.

Corollary 4.8. *Assume that the triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *has the properties* (G), (D), (M), *and that the resolvent set of* A_0 *or, equivalently, the resolvent set of* $\widetilde{A}_0$ *is nonempty. Let* $\gamma, \widetilde{\gamma}$ *and* $M, \widetilde{M}$ *be the associated* γ*-fields and Weyl functions, respectively. Assume that* B *is a closable operator in* $\mathcal{G}$ *and that for some* $\lambda_0 \in \rho(A_0)$ *the following conditions hold:*

(i) $1 \in \rho(B\overline{M(\lambda_0)})$;
(ii) $\operatorname{ran}(B\overline{M(\lambda_0)}) \subset \operatorname{ran}\Gamma_0$;
(iii) $\operatorname{ran}(B \upharpoonright \operatorname{ran}\Gamma_1) \subset \operatorname{ran}\Gamma_0$;
(iv) $\operatorname{ran}(\Gamma_1 \upharpoonright \ker\Gamma_0) \subset \operatorname{dom} B$.

Then A_B *in* (4.1) *is a closed operator with a nonempty resolvent set and for all* $\lambda \in \rho(A_0) \cap \rho(A_B)$ *the Krein-type resolvent formula*

$$(A_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\bigl(I - BM(\lambda)\bigr)^{-1} B\widetilde{\gamma}(\bar{\lambda})^* \tag{4.21}$$

is valid.

In the next corollary the special case $\operatorname{ran}\Gamma_0 = \operatorname{ran}\widetilde{\Gamma}_0 = \mathcal{G}$ is considered. Recall from Lemma 2.3, that in this situation $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ is a quasi boundary triple (or even generalized boundary triple) if also (G) and (M) are required.

Corollary 4.9. *Assume that the triple* $\{\mathcal{G}, (\Gamma_0, \Gamma_1), (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)\}$ *has the properties* (G), $\operatorname{ran}\Gamma_0 = \operatorname{ran}\widetilde{\Gamma}_0 = \mathcal{G}$, (M), *and that the resolvent set of* A_0 *or, equivalently, the resolvent set of* $\widetilde{A}_0$ *is nonempty. Assume that* B_1 *and* B_2 *are closable operators in* $\mathcal{G}$ *and that for some* $\lambda_0 \in \rho(A_0)$ *the following conditions hold:*

(i) $1 \in \rho(B_2\overline{M(\lambda_0)B_1})$;
(ii) $\operatorname{ran}(B_2\overline{M(\lambda_0)B_1}) \subset \operatorname{dom} B_1$;
(iii) $\operatorname{ran}(\Gamma_1 \upharpoonright \ker\Gamma_0) \subset \operatorname{dom} B_1B_2$.

Then $A_{B_1B_2}$ *in* (4.4) *is a closed operator with a nonempty resolvent set and for all* $\lambda \in \rho(A_0) \cap \rho(A_{B_1B_2})$ *the Krein-type resolvent formula* (4.15) *is valid. In the special case* $B_1 = I$ *and* $B_2 = B$ *the conditions (i)–(iii) reduce to*

(i) $1 \in \rho(BM(\lambda_0))$;
(ii) $\operatorname{ran}(\Gamma_1 \upharpoonright \ker\Gamma_0) \subset \operatorname{dom} B$;

and A_B in (4.1) *is a closed operator with a nonempty resolvent set and for all $\lambda\in\rho(A_0)\cap\rho(A_B)$ the Krein-type resolvent formula* (4.21) *is valid.*

We briefly return to the setting in Example 2.8 at the end of Section 2.

Example 4.10. Consider the triple in Example 2.8 and assume that the resolvent sets of the Dirichlet realizations A_0 and $\widetilde A_0$ are nonempty. Then (2.16) is a quasi boundary triple for the adjoint pair $\{S,\widetilde S\}$ and the associated γ-fields and Weyl functions are given by

$$\begin{aligned}\gamma(\lambda)\varphi &= f_\lambda(\varphi), &\qquad \varphi\in L^2(\partial\Omega,\mathbb{C}^{m\times m}),\ \ \lambda\in\rho(A_0),\\ \widetilde\gamma(\mu)\psi &= g_\mu(\psi), &\qquad \psi\in L^2(\partial\Omega,\mathbb{C}^{m\times m}),\ \ \mu\in\rho(\widetilde A_0),\end{aligned}$$

and

$$\begin{aligned}M(\lambda)\varphi &= -\iota_-\tau_N f_\lambda(\varphi), &\qquad \varphi\in L^2(\partial\Omega,\mathbb{C}^{m\times m}),\ \ \lambda\in\rho(A_0),\\ \widetilde M(\mu)\psi &= -\iota_-\widetilde\tau_N g_\mu(\psi), &\qquad \psi\in L^2(\partial\Omega,\mathbb{C}^{m\times m}),\ \ \mu\in\rho(\widetilde A_0),\end{aligned}\tag{4.22}$$

where $f_\lambda(\varphi),g_\mu(\psi)\in H^1(\Omega,\mathbb{C}^{m\times m})$ are the unique solutions of the boundary value problems

$$\mathcal{P}f_\lambda(\varphi)=\lambda f_\lambda(\varphi),\quad \iota_+\tau_D f_\lambda(\varphi)=\varphi$$

and

$$\widetilde{\mathcal{P}}g_\mu(\psi)=\mu g_\mu(\psi),\quad \iota_+\tau_D g_\mu(\psi)=\psi,$$

respectively. Note that

$$\begin{aligned}-\iota_-^{-1}M(\lambda)\iota_+ &: H^{1/2}(\partial\Omega,\mathbb{C}^{m\times m})\to H^{-1/2}(\partial\Omega,\mathbb{C}^{m\times m}),\quad \lambda\in\rho(A_0),\\ -\iota_-^{-1}\widetilde M(\mu)\iota_+ &: H^{1/2}(\partial\Omega,\mathbb{C}^{m\times m})\to H^{-1/2}(\partial\Omega,\mathbb{C}^{m\times m}),\quad \mu\in\rho(\widetilde A_0),\end{aligned}$$

are the Dirichlet-to-Neumann maps corresponding to the differential expressions $\mathcal{P}-\lambda$ and $\widetilde{\mathcal{P}}-\mu$, respectively. Note also that $M(\lambda)$ and $\widetilde M(\mu)$ in (4.22) are defined on $L^2(\partial\Omega,\mathbb{C}^{m\times m})$, and hence Proposition 3.4 (ii) implies that $M(\lambda)$ and $\widetilde M(\mu)$ are bounded operators in $L^2(\partial\Omega,\mathbb{C}^{m\times m})$ for all $\lambda\in\rho(A_0)$ and $\mu\in\rho(\widetilde A_0)$, respectively.

It follows from Corollary 4.9 that if, for instance, B is an everywhere defined bounded operator in $L^2(\partial\Omega,\mathbb{C}^{m\times m})$ such that $1\in\rho(BM(\lambda_0))$ for some $\lambda_0\in\rho(A_0)$ (a situation that appears, e.g., when $M(\eta)\to 0$ for $\eta\to-\infty$; cf. [19, 20] for symmetric second order elliptic equations), then

$$\begin{aligned}A_Bf &= \mathcal{P}f,\\ \operatorname{dom}A_B &= \bigl\{f\in H^1(\Omega,\mathbb{C}^{m\times m}):\mathcal{P}f\in L^2(\Omega,\mathbb{C}^{m\times m}),\ \iota_+\tau_Df+B\iota_-\tau_Nf=0\bigr\},\end{aligned}$$

is a closed operator in $L^2(\Omega,\mathbb{C}^{m\times m})$ with a nonempty resolvent set; it is clear that for $\lambda\in\rho(A_B)$ and $h\in L^2(\Omega,\mathbb{C}^{m\times m})$ the unique $H^1(\Omega,\mathbb{C}^{m\times m})$-solution of the boundary value problem

$$(\mathcal{P}-\lambda)f=h,\qquad \iota_+\tau_Df+B\iota_-\tau_Nf=0,$$

is given by $f=(A_B-\lambda)^{-1}h$. Furthermore, if $\lambda\in\rho(A_0)\cap\rho(A_B)$, then the solution can be expressed via the Krein-type resolvent formula (4.21). We leave it to the reader to formulate a variant of this observation for Robin-type realizations of the adjoint differential expression $\widetilde{\mathcal{P}}$.

Appendix A. The special case $S=\widetilde S$

We provide a summary of our results in the special situation that S is a densely defined closed symmetric operator, that is, $\{S,S\}$ is an adjoint pair. In this case one can choose $T=\widetilde T$ and $\Gamma_0=\widetilde\Gamma_0$, $\Gamma_1=\widetilde\Gamma_1$. The results below are known from [16, 17, 20, 23] for the special case that $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is a quasi boundary triple, but still remain valid under some of the weaker assumptions $(\mathfrak{G})$, $(\mathfrak{D})$, or $(\mathfrak{M})$; cf. Definition 2.1, which reduces to the following:

Definition A.1. Let S be a densely defined closed symmetric operator in $\mathfrak{H}$ and assume that T is a core of S^*. We shall consider *triples* of the form $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ for S, where $\mathcal{G}$ is a Hilbert space and

$$\Gamma_0,\Gamma_1 : \operatorname{dom} T \to \mathcal{G}$$

are linear mappings. For such a triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ we define the additional properties

$(\mathfrak{G})$ the abstract Green's identity

$$(Tf,g)_{\mathfrak{H}} - (f,Tg)_{\mathfrak{H}} = (\Gamma_1 f,\Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f,\Gamma_1 g)_{\mathcal{G}}$$

holds for all $f,g \in \operatorname{dom} T$,

$(\mathfrak{D})$ the range of $\Gamma_0 : \operatorname{dom} T \to \mathcal{G}$ is dense,

$(\mathfrak{D}\mathfrak{D})$ the range of $(\Gamma_0,\Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G}\times\mathcal{G}$ is dense,

$(\mathfrak{M})$ the operator $A_0 := T \restriction \ker\Gamma_0$ is self-adjoint in $\mathfrak{H}$.

If $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is such that $(\mathfrak{G})$, $(\mathfrak{D}\mathfrak{D})$, and $(\mathfrak{M})$ hold, then $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is said to be a *quasi boundary triple* for S.

Note, that in the present situation the natural counterpart of the maximality condition (M) in Definition 2.1 is the requirement in $(\mathfrak{M})$ that $A_0 = T \restriction \ker\Gamma_0$ is self-adjoint in $\mathfrak{H}$. We also mention that A_0 is symmetric whenever $(\mathfrak{G})$ holds; cf. Remark 2.2.

Lemma A.2. *Assume that the triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ has the properties $(\mathfrak{G})$ and $(\mathfrak{M})$. If $\operatorname{ran}\Gamma_0 = \mathcal{G}$, then $\operatorname{ran}(\Gamma_0,\Gamma_1)^\top$ is dense in $\mathcal{G}\times\mathcal{G}$ and $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is a quasi boundary triple for S.*

The statement in Lemma A.2 is known from [37, Lemma 6.1]; in this situation the quasi boundary triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is even a so-called generalized boundary triple in the sense of [37, Section 6], see also [33, 35].

Lemma A.3. *Assume that the triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ has the properties $(\mathfrak{G})$, $(\mathfrak{D})$, and $(\mathfrak{M})$. Then*

$$\operatorname{dom} S = \ker\Gamma_0 \cap \ker\Gamma_1.$$

Lemma A.4. *Assume that the triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ has the properties $(\mathfrak{G})$ and $(\mathfrak{D}\mathfrak{D})$. Then the mapping*

$$\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix} : \operatorname{dom} T \to \mathcal{G}\times\mathcal{G}$$

is closable with respect to the graph norm of T. In particular, the individual mappings $\Gamma_0,\Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are closable.

Proposition 2.6 is known for quasi boundary triples from [16, Theorem 2.3], where also a variant of Theorem 2.7 is contained in the symmetric setting. Here we only state the symmetric version of Proposition 2.6, which leads to an ordinary boundary triple; cf. [15, 36]

Proposition A.5. *Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a quasi boundary triple for S. Then the following are equivalent.*

(i) $T = S^*$,

(ii) $\operatorname{ran}(\Gamma_0,\Gamma_1)^\top = \mathcal{G}\times\mathcal{G}$.

The γ-field and Weyl function corresponding to a triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ are introduced in the same way as in Section 3 using the decomposition

$$\operatorname{dom} T = \operatorname{dom} A_0 \dotplus \ker(T-\lambda) = \ker\Gamma_0 \dotplus \ker(T-\lambda)$$

for $\lambda \in \rho(A_0)$ as follows:

$$\gamma(\lambda) := \bigl(\Gamma_0 \upharpoonright \ker(T-\lambda)\bigr)^{-1} \quad \text{and} \quad M(\lambda) := \Gamma_1\bigl(\Gamma_0 \upharpoonright \ker(T-\lambda)\bigr)^{-1}.$$

In the symmetric situation Proposition 3.3 and Proposition 3.4 reduce to the following statements.

Proposition A.6. *Assume that the triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ has the properties $(\mathfrak{G})$, $(\mathfrak{D})$, $(\mathfrak{M})$, and let γ be the corresponding γ-field. Then the following assertions hold for all $\lambda \in \rho(A_0)$.*

(i) *$\gamma(\lambda)$ is a bounded operator from $\mathcal{G}$ into $\mathfrak{H}$ with dense domain $\operatorname{dom}\gamma(\lambda) = \operatorname{ran}\Gamma_0$ and $\operatorname{ran}\gamma(\lambda) = \ker(T-\lambda)$;*

(ii) *for $\varphi \in \operatorname{ran}\Gamma_0$ the function $\lambda \mapsto \gamma(\lambda)\varphi$ is holomorphic on $\rho(A_0)$ and*

$$\gamma(\lambda) = \bigl(I + (\lambda-\nu)(A_0-\lambda)^{-1}\bigr)\gamma(\nu), \qquad \lambda, \nu \in \rho(A_0),$$

holds;

(iii) *$\gamma(\lambda)^*$ is an everywhere defined bounded operator from $\mathfrak{H}$ to $\mathcal{G}$ and for all $f \in \mathfrak{H}$ one has*

$$\gamma(\lambda)^* f = \Gamma_1(A_0-\bar{\lambda})^{-1}f,$$

in particular, $\operatorname{ran}\gamma(\lambda)^* \subset \operatorname{ran}\Gamma_1$.

Proposition A.7. *Assume that the triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ has the properties $(\mathfrak{G})$, $(\mathfrak{D})$, $(\mathfrak{M})$, and let γ and M be the corresponding γ-field and Weyl function, respectively. Then the following assertions hold for all $\lambda, \mu \in \rho(A_0)$.*

(i) *$M(\lambda)$ is an operator in $\mathcal{G}$ with dense domain $\operatorname{dom} M(\lambda) = \operatorname{ran}\Gamma_0$ and $\operatorname{ran} M(\lambda) \subset \operatorname{ran}\Gamma_1$;*

(ii) *$M(\lambda) \subset M(\bar{\lambda})^*$ and one has the identities*

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda)$$

(iii) *The function $\lambda \mapsto M(\lambda)$ is holomorphic in the sense that it can be written as the sum of the possibly unbounded closed operator $M(\lambda_0)^*$, where $\lambda_0 \in \rho(A_0)$ is fixed, and a bounded holomorphic operator function:*

$$M(\lambda) = M(\lambda_0)^* + \gamma(\lambda_0)^*(\lambda-\bar{\lambda}_0)\bigl(I + (\lambda-\lambda_0)(A_0-\lambda)^{-1}\bigr)\gamma(\lambda_0).$$

Next we state the abstract Birman-Schwinger principle in Theorem 4.2 in a symmetrized form for the operator

$$A_{B_1B_2}f := Tf, \qquad \operatorname{dom} A_{B_1B_2} := \bigl\{f \in \operatorname{dom} T : B_1B_2\Gamma_1 f = \Gamma_0 f\bigr\}, \tag{A.1}$$

where B_1B_2 is a (product of) linear operators in $\mathcal{G}$. The special case $B_2 = B$, $B_1 = I$ is left to the reader.

Theorem A.8. *Consider a triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ as in Definition A.1, assume that the resolvent set of A_0 is nonempty, and let M be the associated Weyl function. Furthermore, let $A_{B_1B_2}$ be the operator in* (A.1) *and let $\lambda \in \rho(A_0)$. Then $\lambda \in \sigma_p(A_{B_1B_2})$ if and only if $\ker(I - B_2M(\lambda)B_1) \neq \{0\}$, and in this case*

$$\ker(A_{B_1B_2}-\lambda) = \bigl\{f_\lambda \in \ker(T-\lambda) : \Gamma_0 f_\lambda = B_1\varphi, \varphi \in \ker(I - B_2M(\lambda)B_1)\bigr\}.$$

Theorem 4.4 has the following form in the symmetric case.

Theorem A.9. *Assume that the triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ has the properties $(\mathfrak{G})$, $(\mathfrak{D})$, $(\mathfrak{M})$, and let γ and M be the associated γ-field and Weyl function, respectively. Furthermore, let $A_{B_1B_2}$ be the operator in* (A.1) *and let $\lambda \in \rho(A_0)$. If $\lambda \notin \sigma_p(A_{B_1B_2})$ and $f \in \mathfrak{H}$ is such that*

$$\gamma(\bar{\lambda})^* f \in \operatorname{dom} B_2 \quad \textit{and} \quad B_2\gamma(\bar{\lambda})^* f \in \operatorname{ran}(I - B_2M(\lambda)B_1), \tag{A.2}$$

then $f \in \operatorname{ran}(A_{B_1B_2} - \lambda)$ *and the Krein-type formula*

$$(A_{B_1B_2} - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f + \gamma(\lambda) B_1 \big(I - B_2 M(\lambda) B_1\big)^{-1} B_2 \gamma(\bar\lambda)^* f$$

holds. In particular, if (A.2) *holds for all* $f \in \mathfrak{H}$*, that is,* $\operatorname{ran}\gamma(\bar\lambda)^* \subset \operatorname{dom} B_2$ *and* $\operatorname{ran} B_2\gamma(\bar\lambda)^* \subset \operatorname{ran}(I - B_2 M(\lambda) B_1)$*, then* $A_{B_1B_2} - \lambda$ *is a bijective operator in* $\mathfrak{H}$*.*

In the special case that B_1B_2 is a symmetric operator and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple the next result is known from [23, Theorem 2.2 and Remark 2.5].

Theorem A.10. *Assume that the triple* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *has the properties* $(\mathfrak{G})$, $(\mathfrak{D})$, $(\mathfrak{M})$*, and let* γ *and* M *be the associated* γ*-field and Weyl function, respectively. Assume that* B_1 *and* B_2 *are closable operators in* $\mathcal{G}$ *and that for some* $\lambda_0 \in \rho(A_0)$ *the following conditions hold:*

(i) $1 \in \rho(B_2\overline{M(\lambda_0)B_1})$;
(ii) $\operatorname{ran}(B_2\overline{M(\lambda_0)B_1}) \subset \operatorname{ran}\Gamma_0 \cap \operatorname{dom} B_1$;
(iii) $\operatorname{ran}(B_1 \upharpoonright \operatorname{ran}\Gamma_0) \subset \operatorname{ran}\Gamma_0$;
(iv) $\operatorname{ran}(B_2 \upharpoonright \operatorname{ran}\Gamma_1) \subset \operatorname{ran}\Gamma_0$;
(v) $\operatorname{ran}(\Gamma_1 \upharpoonright \ker\Gamma_0) \subset \operatorname{dom} B_1B_2$.

Then $A_{B_1B_2}$ *in* (A.1) *is a closed operator with a nonempty resolvent set and for all* $\lambda \in \rho(A_0) \cap \rho(A_{B_1B_2})$ *the Krein-type resolvent formula*

$$(A_{B_1B_2} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda) B_1 \big(I - B_2 M(\lambda) B_1\big)^{-1} B_2 \gamma(\bar\lambda)^*$$

is valid. If, in addition, the parameter B_1B_2 *is a symmetric operator in* $\mathcal{G}$ *and the conditions (i)-(v) hold for some* $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ *or some* $\lambda_\pm \in \mathbb{C}^\pm$*, then* $A_{B_1B_2}$ *is self-adjoint in* $\mathfrak{H}$*.*

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Institut für Angewandte Mathematik, Technische Universität Graz, Steyrergasse 30, A-8010 Graz, Austria

Email address: behrndt@tugraz.at