Linear fractional transformations of Stieltjes functions

Jussi Behrndt\textsuperscript{1}, Seppo Hassi\textsuperscript{2}, Henk de Snoo\textsuperscript{3}, Rudi Wietsma\textsuperscript{1}, and Henrik Winkler\textsuperscript{4}

\textsuperscript{1} Institut für Numerische Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
\textsuperscript{2} Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland
\textsuperscript{3} Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands
\textsuperscript{4} Institut für Mathematik, Technische Universität Ilmenau, Postfach 10 05 65, 98684 Ilmenau, Germany

Linear fractional transformations of Stieltjes (and inverse Stieltjes) functions, which appear naturally in the extension theory of nonnegative symmetric operators with defect one in Hilbert spaces, are investigated.

1 Nevanlinna, Stieltjes, and inverse Stieltjes functions

The class of Nevanlinna functions is intimately connected with selfadjoint operators and relations in Hilbert spaces, and therefore plays a key role in spectral analysis. For instance, the set of Titchmarsh-Weyl coefficients of real trace-normed 2 × 2 canonical systems on a halfline coincide with the class of Nevanlinna functions. Recall that a Nevanlinna function, \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), is holomorphic on \( (-\infty, 0) \) and \( (0, \infty) \) without zeros there. Alternatively, \( Q \in \mathbb{S} \) if and only if \( Q, \lambda Q, \frac{Q}{\lambda} \in \mathbb{N} \) (cf. [5]).

Linear fractional transformations of Stieltjes functions

The linear fractional transformations \( Q_\tau \), \( \tau \in \mathbb{R} \cup \{ \infty \} \), of a Nevanlinna function \( Q \) are defined by

\[
Q_\tau(\lambda) = \frac{Q(\lambda) - \tau}{1 + \tau Q(\lambda)}, \quad \tau \in \mathbb{R}, \quad \text{and} \quad Q_\infty(\lambda) = -1/Q(\lambda), \quad \tau = \infty.
\]

(2)

It is not difficult to see that \( Q_\tau \) is a Nevanlinna function for all \( \tau \in \mathbb{R} \cup \{ \infty \} \). Moreover, notice that \( (Q_\tau)_\tau = Q_s \) where \( s = (1 + \tau)/(1 - \eta \tau) \) with \( \eta, \tau \in \mathbb{R} \cup \{ \infty \} \); in particular, the class of functions \( \{ Q_\tau : \tau \in \mathbb{R} \cup \{ \infty \} \} \) is stable under composition of transformations in (2).

Now assume that \( Q \) is holomorphic on \( (-\infty, 0) \) except for finitely many points, as is the case for \( Q \in \mathbb{S} \cup \mathbb{S}^{-1} \). Then the possibly improper limits of \( Q \) at \( -\infty \) and \( 0 \) exist, they are denoted by \( b \) and \( L \):

\[
b := \lim_{\lambda \to -\infty} Q(\lambda) \in \mathbb{R} \cup \{ -\infty \} \quad \text{and} \quad L := \lim_{\lambda \to 0} Q(\lambda) \in \mathbb{R} \cup \{ +\infty \}.
\]

(3)

Lemma 2.1 Let \( Q \) be a nonconstant Nevanlinna function and let \( Q_\tau \) be given by (2), \( \tau \in \mathbb{R} \cup \{ \infty \} \), then:

(i) If \( Q \) is holomorphic on \( (-\infty, 0) \), then \( b < L \) and \( Q_\tau \) has precisely one zero and one pole on \( (-\infty, 0) \) if and only if \( b < \tau < -1/L < 0 \) or \( 0 < -1/b < \tau < L \).

(ii) If \( Q \) is holomorphic on \( (-\infty, 0) \) except for one point \( a \), then \( Q_\tau \) is holomorphic on \( (-\infty, 0) \) and has no zeros on \( (-\infty, 0) \) if and only if \( -\infty < L \leq \tau \leq -1/b < 0 \) or \( 0 < -1/L \leq \tau \leq b < \infty \).

(4)
Proof. (i) Since \( Q \) is nonconstant and holomorphic on \((-∞, 0)\) the integral representation (1) yields that \( Q \) is strictly increasing on \((-∞, 0)\) and takes on all values between \( b \) and \( L \) uniquely. Hence (2) shows that \( Q_\tau \) has a zero on \((-∞, 0)\) for \( b < \tau < L \) and a pole on \((-∞, 0)\) for \( b < -1/\tau < L \). These inequalities can hold simultaneously only if \( b < 0 < L \) in which case \(-1/L \leq 0 < -1/b\). Now the assertion follows by considering the cases \( b < 0 < -1/\tau < L \) and \( b < -1/\tau < 0 < L \).

(ii) If (4) holds or \( Q_\tau \in S \cup S^{-1} \) for some \( \tau \), then \(-∞ < L < b < ∞\). Now proceed as in (i) with the interval \((L, b)\).

The next results concern the linear fractional transforms \( Q_\tau \) of a Stieltjes function.

**Proposition 2.2** Let \( Q \in S \) be a nonconstant Stieltjes function and let \( Q_\tau \) be given by (2), \( \tau \in \mathbb{R} \cup \{∞\} \). Then \( b \) and \( L \) satisfy the inequality \( 0 \leq b < L < ∞ \) (so that also \(-∞ < -1/\tau < L \leq 0 \) and the following statements hold:

(i) \( Q_\tau \in S \) if and only if \(-1/L \leq \tau \leq b\);

(ii) \( Q_\tau \in S^{-1} \) if and only if \( \tau \leq -1/b \), \( \tau \geq L \), or \( \tau = ∞\);

(iii) \( Q_\tau \) has a (unique) zero and no poles on \((-∞, 0)\) if and only if \( b < \tau < L\);

(iv) \( Q_\tau \) has a (unique) pole and no zeros on \((-∞, 0)\) if and only if \(-1/b < \tau < -1/L\).

In particular, \( Q \) (and \(-Q^{-1}\)) is the only function \( Q_\tau \) in (2) belonging to \( S \) (\( S^{-1} \), respectively) if and only if \( b = 0 \) and \( L = ∞\).

Proof. (iii) & (iv) The function \( Q_\tau \) has a (unique) zero in \((-∞, 0)\) if and only if \( b < \tau < L \), and \( Q_\tau \) has a (unique) pole in \((-∞, 0)\) if \( \tau \leq -1/b \). For \(-1/b < \tau < -1/L\) the values of \( Q_\tau \) on \((-∞, 0)\) are positive and for \( \tau \leq -1/b \), \( \tau \geq L \), and \( \tau = ∞ \) the values of \( Q_\tau \) on \((-∞, 0)\) are negative. \( (\Leftarrow) \) This implication follows with similar arguments.

The next theorem shows under which conditions a Nevanlinna function \( Q \) possesses a transformation \( Q_\tau \) in the Stieltjes or inverse Stieltjes class. In view of Lemma 2.1 only Nevanlinna functions that are holomorphic on \((-∞, 0)\), or have at most one pole on \((-∞, 0)\) and satisfy \(-∞ < L < b < ∞\), have to be considered.

Recall that a symmetric scalar function \( Q \) which is meromorphic on \( C \setminus \mathbb{R} \), is said to belong to the class of generalized Nevanlinna functions with \( κ \in \mathbb{N} \) negative squares, \( Q \in N_κ \), if its Nevanlinna kernel has \( κ \) negative squares; see, e.g. [6]. Note that, if \( Q \in S \) (\( S^{-1} \)), then \( Q(\lambda) \in N \) and, moreover, \( Q(\lambda)/λ \in N_1 \) (\( Q(\lambda)/λ \in N \) if \( λ \in N_1 \)).

**Theorem 2.3** Let \( Q \) be a nonconstant Nevanlinna function which is holomorphic on \((-∞, 0)\) except for possibly one point, in which case it is assumed that \(-∞ < L < 0 < b < ∞\). Then the following statements are equivalent:

(i) there exists \( η \in \mathbb{R} \cup \{∞\} \) such that \( Q_η \in S \), or equivalently, there exists \( η \in \mathbb{R} \cup \{∞\} \) such that \( Q_η \in S^{-1} \);

(ii) if \( Q_\tau , \tau \in \mathbb{R} \cup \{∞\} \), in (2) has a zero (pole) on \((-∞, 0)\), then it does not have a pole (zero) on \((-∞, 0)\);

(iii) if \( Q \) is holomorphic and has a zero on \((-∞, 0)\), then \(-∞ < -1/L < b < 0 \); and if \( Q \) is not holomorphic on \((-∞, 0)\), then \(-∞ < L < -1/b < 0 \);

(iv) \( λQ_\tau (λ) \in N \cup N_1 \) and \( Q_\tau (λ)/λ \in N \cup N_1 \) for all \( \tau \in \mathbb{R} \cup \{∞\} \).

Proof. (i) \( ⇒ \) (ii) This follows from Proposition 2.2.

(ii) \( ⇒ \) (iii) If \( Q \) has a zero on \((-∞, 0)\) and \(-∞ < b < 0 < L < ∞\), then \( Q_∞ \) has a pole on \((-∞, 0)\) and the corresponding limits \( L_∞ = -1/L \) and \( b_∞ = -1/b \) satisfy \(-∞ < L_∞ < 0 < b_∞ < ∞\); and conversely. On the other hand, if \( Q \) is holomorphic on \((-∞, 0)\) and \(-1/b < L \leq 0 \), then by Lemma 2.1 the transformation \( Q_η \), \( b < η < -1/L \), has a pole and a zero on \((-∞, 0)\). This contradiction together with the inequalities \(-∞ < b < 0 < L < ∞\) implies that \(-∞ < -1/L \leq b < 0 \). These inequalities are equivalent to \(-∞ < L_∞ \leq -1/b_∞ < 0 \) for the limits of \( Q_∞ \). Hence, (ii) holds.

(iii) \( ⇒ \) (i) If \( Q \) has neither a zero nor a pole on \((-∞, 0)\), then \( Q \in S \cup S^{-1} \). If \( Q \) is not holomorphic on \((-∞, 0)\) and \(-∞ < L < -1/b < 0 \), then Lemma 2.1 shows that \( Q_L \in S \cup S^{-1} \). If \( Q \) is holomorphic with a zero on \((-∞, 0)\) and \(-∞ < -1/L \leq b < 0 \), then \( Q_∞ \) has a pole and the corresponding limits \( L_∞ = -1/L < b_∞ = -1/b \) satisfy \(-∞ < L_∞ < -1/b_∞ < 0 \). Thus, \( Q_2(L_∞) \in S \cup S^{-1} \) again by Lemma 2.1.

(iv) \( ⇔ \) (iv) This follows from the fact that \( λQ(λ) \) and \( Q_∞ (λ)/λ \) belong to \( N \cup N_1 \) if and only if \( Q \) is holomorphic on \((-∞, 0)\) except for at most one point in which case it does not have a zero on \((-∞, 0)\); cf. [2, Theorem 4.5 & Remark 4.7].

References