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Two-dimensional Dirac operators with singular interactions supported on closed curves

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A B S T R A C T

We study the two-dimensional Dirac operator with a class of interface conditions along a smooth closed curve, which model the so-called electrostatic and Lorentz scalar interactions of constant strengths, and we provide a rigorous description of their self-adjoint realizations and their qualitative spectral properties. We are able to cover in a uniform way all so-called critical combinations of coupling constants, for which there is a loss of regularity in the operator domain. For the case of a non-zero mass term, this results in an additional point in the essential spectrum, which reflects the creation of an infinite number of eigenvalues in the central gap, and the position of this point can be made arbitrary by a suitable choice of the parameters. The analysis is based on a combination of the extension theory of symmetric operators

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with a detailed study of boundary integral operators viewed as periodic pseudodifferential operators.

1. Introduction

1.1. Motivations and state of the art

In the present paper we study the self-adjointness of Dirac operators in two dimensions with a special type of transmission conditions along a smooth curve. The interest in such operators appeared originally in numerous works discussing quantum-mechanical Hamiltonians with interactions supported by zero measure sets such as points or hypersurfaces, see, e.g., [1,10,20]. Due to the singular nature of the interactions, special approaches are required to define and analyze the operators rigorously. For Schrödinger operators with such singular interactions, the quadratic form approach is an efficient tool, which uses in an essential way the semiboundedness of these operators [12]. For Dirac operators, the lack of semiboundedness imposes the use of other methods, such as suitable resolvent formulas or a definition through interface conditions, which involves much heavier analytical techniques. The case of one-dimensional Dirac operators with point interactions is well-studied, see [1,15,23,30]. However, the higher dimensional situations were only considered quite recently, mostly for three-dimensional Dirac operators with interactions supported by surfaces, see [3–7,9,19,24,28,29], and the recent contribution [31] is devoted to a particular problem in two dimensions. In the above works, it was observed that there are critical combinations of parameters (interaction strengths) for which the standard elliptic regularity fails, and the self-adjoint realization of the operator shows a loss of regularity in the operator domain. In some of these critical cases (for purely electrostatic critical interactions) in the three-dimensional setting the essential self-adjointness of the operators on the standard domain was shown and it was noted that the spectral properties can differ from what was observed for the non-critical case [9,29]; for general critical combinations of the parameters a systematic analysis is missing.

In the present paper we provide a complete treatment of the problem in two dimensions. Our main advance is that we show the self-adjointness of the resulting operators and describe the spectral properties for *all* possible combinations of parameters, which include all critical cases. For this we use a systematic approach combining some tools of the operator extension theory with pseudodifferential techniques for the analysis of matrix-valued singular integral operators. This is partly inspired by the recent paper [14] dealing with special transmission problems for Laplacians and which we expect to be of use for higher-dimensional operators as well. In particular, our work answers fully the question of [28, Open Problem 11] in dimension two. The main novelty of the results is that the Dirac operator with a critical interface condition along a smooth compact curve has infinitely many eigenvalues in the gap of the essential spectrum, while the point at which the eigenvalues accumulate can be controlled by a suitable choice of parameters. Such effects were not observed previously for Dirac operators with singular interactions.

Let us now introduce the problem setting in greater detail. To set the stage, let Σ be a smooth planar loop, i.e. a closed non-self-intersecting C^∞-smooth curve in $\mathbb{R}^2$. It splits $\mathbb{R}^2$ into a bounded domain Ω_+ and an unbounded domain Ω_-, and we denote by $\nu = (\nu_1, \nu_2)$ the unit normal to Σ pointing outwards of Ω_+. For a function f defined on $\mathbb{R}^2$ we will often use the notation $f_\pm := f \upharpoonright \Omega_\pm$, where $\upharpoonright \Omega_\pm$ stands for the restriction to $\Omega_\pm$. If a function f has suitably defined Dirichlet traces on both sides of Σ, we define the distribution $\delta_\Sigma f$ by

$$\langle \delta_\Sigma f, \varphi \rangle := \int_\Sigma \frac{1}{2} \big(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_- \big) \varphi \, \mathrm{d}s, \quad \varphi \in C_0^\infty(\mathbb{R}^2),$$

where $\mathcal{T}_\pm^D f_\pm$ denotes the Dirichlet trace of $f_\pm$ at Σ and $\mathrm{d}s$ is the integration with respect to the arc-length. We are going to study Dirac operators $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ given by the formal differential expression

$$D_{\eta,\tau} := -\mathrm{i}\big(\sigma_1 \partial_1 + \sigma_2 \partial_2\big) + m\sigma_3 + (\eta\sigma_0 + \tau\sigma_3)\delta_\Sigma,$$

where σ_0 is the identity matrix in $\mathbb{C}^{2\times 2}$, $\sigma_1, \sigma_2, \sigma_3$ are the $\mathbb{C}^{2\times 2}$-valued Pauli spin matrices defined in (1.3) below, and $m, \eta, \tau \in \mathbb{R}$. Following the standard language [37] of relativistic quantum mechanics, one may interpret η and τ as the strengths of the electrostatic and Lorentz scalar interactions on Σ, respectively, while the parameter m is usually interpreted as the mass. Integration by parts shows that if the distribution $D_{\eta,\tau} f$ is generated by an L^2-function, then the function f has to fulfill (at least formally) the transmission condition

$$-\mathrm{i}\,(\sigma_1\nu_1 + \sigma_2\nu_2)\,(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-) = \frac{1}{2}(\eta\sigma_0 + \tau\sigma_3)(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-). \tag{1.1}$$

Our goal is to make this observation rigorous and to show that there is a unique reasonably defined self-adjoint operator $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ for this transmission condition and then to study its qualitative spectral properties.

In our approach we consider $A_{\eta,\tau}$ as an extension of a suitably chosen symmetric operator and make use of the standard machinery of boundary triples [8,13,16,17] in order to reformulate the main questions in terms of integral operators on Σ. We note that a similar idea was used in [6,9,15,30]. The second main ingredient is the periodic pseudodifferential calculus, which is heavily used for a detailed study of various integral operators arising in this construction; cf. [3–7,9,29] for closely related objects in the three-dimensional case.

1.2. Main results

Let us pass to the formulation and discussion of the main results of this paper. To define the operator $A_{\eta,\tau}$ rigorously, we introduce for an open set $\Omega \subset \mathbb{R}^2$

$$H(\sigma,\Omega) = \big\{ f \in L^2(\Omega;\mathbb{C}^2) : (\sigma_1\partial_1 + \sigma_2\partial_2) f \in L^2(\Omega;\mathbb{C}^2) \big\}.$$

One can show that functions $f_\pm$ in $H(\sigma,\Omega_\pm)$ admit Dirichlet traces $\mathcal{T}_\pm^D f_\pm$ in $H^{-\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. With these notations in hand we define, following (1.1), for $\eta,\tau \in \mathbb{R}$ the operator $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ by

$$\begin{aligned} A_{\eta,\tau} f &:= \big(-\mathrm{i}(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3 \big) f_+ \oplus \big(-\mathrm{i}(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3 \big) f_-, \\ \operatorname{dom} A_{\eta,\tau} &:= \Big\{ f = f_+ \oplus f_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) : \\ &\qquad -\mathrm{i}\,(\sigma_1\nu_1 + \sigma_2\nu_2)\big(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-\big) = \frac{1}{2}\,(\eta\sigma_0 + \tau\sigma_3)\big(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\big) \Big\}. \end{aligned} \tag{1.2}$$

It turns out that the value $\eta^2 - \tau^2$ plays a special role. More precisely, if $\eta^2 - \tau^2 = 4$ we will say that we are in a *critical* case, while all the cases with $\eta^2 - \tau^2 \neq 4$ will be referred to as *non-critical* ones. We also remark that for some combinations of coupling constants the boundary condition in (1.2) leads to a so-called decoupling, i.e. the operator $A_{\eta,\tau}$ becomes the direct sum of two operators acting in $\Omega_\pm$, see Lemma 4.1 below.

It appears that the non-critical case is easier to deal with, and the results for $A_{\eta,\tau}$ are summarized as follows:

Theorem 1.1 *(Non-critical case). Let $\eta,\tau \in \mathbb{R}$ with $\eta^2 - \tau^2 \neq 4$. Then $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with* $\operatorname{dom} A_{\eta,\tau} \subset H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$, *its essential spectrum is given by*

$$\operatorname{spec}_{\text{ess}} A_{\eta,\tau} = (-\infty, -|m|] \cup [|m|, +\infty),$$

while the discrete spectrum in $(-|m|, |m|)$ is finite.

The proof of Theorem 1.1 is given in Section 4.2. There, also some additional properties of $A_{\eta,\tau}$ like a Krein-type resolvent formula, an abstract version of the Birman-Schwinger principle, and some symmetry relations in the point spectrum of $A_{\eta,\tau}$ are shown. Similar results are known in the three-dimensional case, see [7].

Our main results in the critical case $\eta^2 - \tau^2 = 4$ are collected in the following theorem.

Theorem 1.2 *(Critical case). Let $\eta,\tau \in \mathbb{R}$ with $\eta^2 - \tau^2 = 4$. Then $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$, while its restriction onto* $\operatorname{dom} A_{\eta,\tau} \cap H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$ *is only essentially self-adjoint, and* $\operatorname{dom} A_{\eta,\tau} \not\subset H^s(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$ *for any $s > 0$. The essential spectrum is*

$$\operatorname{spec}_{\text{ess}} A_{\eta,\tau} = \big(-\infty, -|m| \big] \cup \Big\{ -\frac{\tau}{\eta} m \Big\} \cup \big[|m|, +\infty \big).$$

Theorem 1.2 is the main result of this paper, and it is proved in Section 4.3. There, also a Krein type resolvent formula, a Birman Schwinger principle, and several symmetry relations in the point spectrum of $A_{\eta,\tau}$ are shown. Some analogs in three dimensions are only known in the case of purely electrostatic interactions, i.e. when $\eta = \pm 2$ and $\tau = 0$, see [9,29]. The additional point $-\frac{\tau}{\eta}m$ of the essential spectrum can take any value in the gap $(-|m|, |m|)$ under a suitable choice of η and τ, and this effect was not observed in previous works. Several papers addressed the question of presence of a non-empty essential spectrum for Dirac operators in bounded domains with various boundary conditions, see, e.g., [11,22,35], and our results can also be regarded as a contribution in this direction.

By a minor modification of the argument, one can deal with an interaction supported on several loops. Let $N \geq 1$ and consider a family of non-intersecting smooth loops $\Sigma_1, \dots, \Sigma_N$ with unit normals ν_j, $j \in \{1, \dots, N\}$. We set $\Sigma := \bigcup_{j=1}^N \Sigma_j$, and for any $f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma)$ we denote its Dirichlet traces on the two sides of Σ_j as $\mathcal{T}^D_{\pm,j} f$, where $-$ corresponds to the side to which ν_j is directed. In addition, consider a family of pairs of real parameters $\mathcal{P} := ((\eta_j, \tau_j))_{j\in\{1,\dots,N\}}$, $\eta_j, \tau_j \in \mathbb{R}$, and define the associated operator $A_{\Sigma,\mathcal{P}}$ by

$$A_{\Sigma,\mathcal{P}} f := \big(- \mathrm{i}(\sigma_1 \partial_1 + \sigma_2 \partial_2) + m\sigma_3 \big) f \qquad \text{in } \mathbb{R}^2 \setminus \Sigma,$$

$$\operatorname{dom} A_{\Sigma,\mathcal{P}} := \Big\{ f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma) : -\mathrm{i}\,(\sigma_1\nu_1 + \sigma_2\nu_2)\big(\mathcal{T}^D_{+,j} f - \mathcal{T}^D_{-,j} f\big) = \frac{1}{2}\,(\eta_j\sigma_0 + \tau_j\sigma_3)\big(\mathcal{T}^D_{+,j} f + \mathcal{T}^D_{-,j} f\big),\ j = 1, \dots, N \Big\}.$$

Then the preceding results can be extended as follows:

Theorem 1.3 *(Interaction supported on several loops). Let* $\mathcal{J}_{\text{crit}} := \{j : \eta_j^2 - \tau_j^2 = 4\}$. *Then the following is true:*

(i) *If* $\mathcal{J}_{\text{crit}} = \emptyset$, *then* $A_{\Sigma,\mathcal{P}}$ *is self-adjoint in* $L^2(\mathbb{R}^2; \mathbb{C}^2)$ *with* $\operatorname{dom} A_{\Sigma,\mathcal{P}} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$, *the essential spectrum of* $A_{\Sigma,\mathcal{P}}$ *is*

$$\operatorname{spec}_{\text{ess}} A_{\Sigma,\mathcal{P}} = (-\infty, -|m|] \cup [|m|, \infty),$$

and the discrete spectrum of $A_{\Sigma,\mathcal{P}}$ *in* $(-|m|, |m|)$ *is finite.*

(ii) *If* $\mathcal{J}_{\text{crit}} \neq \emptyset$, *then* $A_{\Sigma,\mathcal{P}}$ *is self-adjoint and the restriction of* $A_{\Sigma,\mathcal{P}}$ *onto the set* $\operatorname{dom} A_{\Sigma,\mathcal{P}} \cap H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ *is essentially self-adjoint in* $L^2(\mathbb{R}^2; \mathbb{C}^2)$, *but one has* $\operatorname{dom} A_{\Sigma,\mathcal{P}} \not\subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ *for any* $s > 0$. *The essential spectrum of* $A_{\Sigma,\mathcal{P}}$ *is*

$$\operatorname{spec}_{\text{ess}} A_{\Sigma,\mathcal{P}} = \big(-\infty, -|m|\big] \cup \Big\{ -\frac{\tau_j}{\eta_j} m : j \in \mathcal{J}_{\text{crit}} \Big\} \cup \big[|m|, +\infty\big).$$

In particular, one easily observes that if Σ has N connected components, then for any finite set $\Xi \subset (-|m|, |m|)$ with $\#\Xi \leq N$ it is possible to find a combination of parameters

$\mathcal{P}$ such that the essential spectrum of $A_{\Sigma,\mathcal{P}}$ in $(-|m|,|m|)$ coincides with Ξ. Necessary modifications for the proof of Theorem 1.3 are sketched in Subsection 4.4.

1.3. Structure of the paper

Let us shortly describe the structure of the paper. First, in Section 2 we recall some facts on periodic pseudodifferential operators and boundary triples. With that we study then in Section 3 integral operators, which are associated to the Green function corresponding to the free Dirac operator in $\mathbb{R}^2$, and construct a boundary triple which is suitable to study the properties of $A_{\eta,\tau}$. The two sections 2 and 3 occupy an important portion of the text, which is due to the big number of tools from various domains which are put together and which are rarely (if at all) used simultaneously. We believe that the construction can be of use for other two-dimensional boundary value problems with the help of the boundary triple machinery. Finally, Section 4 is devoted to the proofs of the main results of this paper, Theorems 1.1–1.3.

1.4. Notations

We use the convention $0 \notin \mathbb{N}$ and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote the 2×2 identity matrix by σ_0 and the 2×2 Pauli spin matrices by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

Recall that they fulfill

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}\sigma_0, \quad j,k \in \{1,2,3\}. \tag{1.4}$$

For $x = (x_1, x_2) \in \mathbb{R}^2$ we write $\sigma \cdot x := \sigma_1 x_1 + \sigma_2 x_2$ and, similarly, $\sigma \cdot \nabla := \sigma_1 \partial_1 + \sigma_2 \partial_2$.

Next, $\Sigma \subset \mathbb{R}^2$ is always a C^∞-loop of length $\ell > 0$, which splits $\mathbb{R}^2$ into a bounded domain Ω_+ and an unbounded domain Ω_- with common boundary Σ. By ν we denote the unit normal vector field at Σ which points outwards of Ω_+, and $\mathbf{t}$ denotes the unit tangent vector at Σ. If $\gamma : [0,\ell] \to \mathbb{R}^2$ is an arc length parametrization of Σ with positive orientation, then $\mathbf{t} = \gamma'$ and $\nu = (\gamma_2', -\gamma_1')$. We sometimes identify the vector $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2) \in \mathbb{R}^2$ with the complex number $T = \mathbf{t}_1 + \mathrm{i}\mathbf{t}_2$.

If Ω is a measurable set, we write, as usual, $L^2(\Omega)$ for the classical L^2-spaces and $L^2(\Omega; \mathbb{C}^2) := L^2(\Omega) \otimes \mathbb{C}^2$. If $\Omega = \Sigma$, then $L^2(\Sigma)$ is based on the inner product in which the integrals are taken with respect to the arc-length. By $H^s(\Omega)$ we denote the Sobolev spaces of order $s \in \mathbb{R}$ on Ω, and the Sobolev spaces on the curve Σ are reviewed in Section 2.1.

Next, we denote $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Then $C^\infty(\mathbb{T})$ can be identified with the space of all 1-periodic $C^\infty(\mathbb{R})$-functions. For $\alpha \in \mathbb{R}$ we denote the set of periodic pseudodifferential operators of order α on $\mathbb{T}$ by Ψ^α and the set of periodic pseudodifferential operators of order α on Σ by Ψ^α_Σ (see Definitions 2.1 and 2.3 below).

For a linear operator A in a Hilbert space $\mathcal{H}$ we write $\operatorname{dom} A$, $\operatorname{ran} A$, and $\ker A$ for its domain, range, and kernel, respectively. The identity operator is often denoted by $\mathbb{1}$. If A is self-adjoint, then we denote by $\operatorname{res} A, \operatorname{spec} A, \operatorname{spec}_{\mathrm{p}} A$, and $\operatorname{spec}_{\mathrm{ess}} A$ its resolvent set, spectrum, point, and essential spectrum, respectively. If A is self-adjoint and semi-bounded from below, then $\mathcal{N}(A, z)$ is the number of eigenvalues smaller than z taking multiplicities into account. For $z > \inf \operatorname{spec}_{\mathrm{ess}} A$ this is understood as $\mathcal{N}(A, z) = \infty$.

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2. Preliminaries

In this section we provide some preliminary material from functional analysis and operator theory. First, in Section 2.1 we recall the definition and some properties of periodic pseudodifferential operators on smooth curves and some special integral operators of this form. Furthermore, in Section 2.2 the concept of boundary triples is briefly reviewed.

2.1. Sobolev spaces and periodic pseudodifferential operators on closed curves

In this section some properties of periodic pseudodifferential operators on closed curves are discussed along the lines of [34, Chapters 5 and 7]. Special realizations of such operators will play an important role in the analysis of Dirac operators with singular interactions later.

Throughout this section $\Sigma \subset \mathbb{R}^2$ is a C^∞-smooth loop of length ℓ and let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. By $\gamma : \ell\mathbb{T} \to \Sigma$ we denote a fixed arc-length parametrization of Σ, i.e. a C^∞-function with $|\gamma'(\cdot)| \equiv 1$ and $\gamma(\ell\mathbb{T}) = \Sigma$. First, we recall the construction of Sobolev spaces of periodic functions on a loop. For a distribution[1] $f \in \mathcal{D}'(\mathbb{T}) := C^\infty(\mathbb{T})'$ we write

$$\widehat{f}(n) := \langle f, e_{-n} \rangle_{\mathcal{D}'(\mathbb{T}),\mathcal{D}(\mathbb{T})} \in \mathbb{C}, \quad e_n(t) = e^{2\pi n i t}, \qquad n \in \mathbb{Z},$$

[1] In [34] the notation $\mathcal{D}'_1(\mathbb{R})$ is used instead of $\mathcal{D}'(\mathbb{T})$. The subindex 1 means the 1-periodicity.

for its Fourier coefficients. Recall that a distribution $f \in \mathcal{D}'(\mathbb{T})$ can be reconstructed from its Fourier coefficients by

$$f = \sum\nolimits_{n\in\mathbb{Z}} \widehat{f}(n) e_n, \tag{2.1}$$

where the series converges in $\mathcal{D}'(\mathbb{T})$, see [34, Theorem 5.2.1]. For any two distributions $f, g \in \mathcal{D}'(\mathbb{T})$ we denote by $f \star g$ their convolution which is defined (via its Fourier coefficients) by $\widehat{f \star g}(n) = \widehat{f}(n)\,\widehat{g}(n)$, $n \in \mathbb{Z}$. In particular, for $f, g \in L^1(\mathbb{T})$ one has

$$f \star g = \int_{\mathbb{T}} f(s) g(\cdot - s)\,\mathrm{d}s.$$

For convenience we set $\underline{n} := |n|$ for $n \in \mathbb{Z} \setminus \{0\}$ and $\underline{n} := 1$ for $n = 0$. Then for $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ consists of the distributions $f \in \mathcal{D}'(\mathbb{T})$ with

$$\|f\|^2_{H^s(\mathbb{T})} := \sum\nolimits_{n\in\mathbb{Z}} \underline{n}^{2s} \big|\widehat{f}(n)\big|^2 < \infty.$$

The set $H^s(\mathbb{T})$ endowed with the above norm and induced scalar product becomes a Hilbert space. If $s < t$, then $H^t(\mathbb{T})$ is compactly embedded into $H^s(\mathbb{T})$.

The Sobolev spaces H^s on $\mathbb{T}$ can be translated to Sobolev spaces on Σ. For that we define on $\mathcal{D}'(\Sigma) := C^\infty(\Sigma)'$ the linear map

$$U : \mathcal{D}'(\Sigma) \to \mathcal{D}'(\mathbb{T}), \quad (Uf)(\varphi) = f\left(\ell^{-1}\varphi(\ell^{-1}\gamma^{-1}(\cdot))\right), \quad \varphi \in C^\infty(\mathbb{T}). \tag{2.2}$$

It is not difficult to verify that

$$Uf(t) = f(\gamma(\ell t)), \quad f \in L^1(\Sigma),\ t \in \mathbb{T}; \tag{2.3}$$

this property will often be used. For $s \in \mathbb{R}$ we define the Hilbert space

$$H^s(\Sigma) := \big\{ f \in \mathcal{D}'(\Sigma) : Uf \in H^s(\mathbb{T}) \big\}, \quad \|f\|_{H^s(\Sigma)} := \|Uf\|_{H^s(\mathbb{T})}, \quad f \in H^s(\Sigma).$$

By construction, the induced map $U : H^s(\Sigma) \to H^s(\mathbb{T})$ is unitary for any $s \in \mathbb{R}$. It is easily seen that $C^\infty(\Sigma)$ is dense in $H^s(\Sigma)$ for all $s \in \mathbb{R}$.

Next, we recall the definition of periodic pseudodifferential operators on $\mathbb{T}$ and Σ. Define first the linear operator ω acting on mappings $F : \mathbb{Z} \to \mathbb{C}$ by

$$(\omega F)(n) := F(n+1) - F(n), \quad n \in \mathbb{Z}. \tag{2.4}$$

Definition 2.1. A linear operator H acting on $C^\infty(\mathbb{T})$ is called a *periodic pseudodifferential operator of order* $\alpha \in \mathbb{R}$, if there exists a function $h : \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ with $h(\cdot, n) \in C^\infty(\mathbb{T})$ for each $n \in \mathbb{Z}$ and

$$Hu(t) = \sum\nolimits_{n\in\mathbb{Z}} h(t,n)\,\widehat{u}(n)\,e_n(t), \qquad u \in C^\infty(\mathbb{T}), \tag{2.5}$$

and for all $k, l \in \mathbb{N}_0$ there exist constants $c_{k,l} > 0$ such that

$$\Big|\frac{\partial^k}{\partial t^k}\,\omega_n^l h(t,n)\Big| \le c_{k,l}\,\underline{n}^{\alpha-l}, \qquad n \in \mathbb{Z},$$

where ω_n means the application of ω to the second argument of h. The class of all periodic pseudodifferential operators of order α is denoted by Ψ^α, and we set $\Psi^{-\infty} := \cap_{\alpha\in\mathbb{R}}\Psi^\alpha$.

One has the obvious inclusions $\Psi^\alpha \subset \Psi^\beta$ for $\alpha < \beta$. Moreover, in the spirit of (2.1) the periodic pseudodifferential operator H is determined by its Fourier coefficients

$$\widehat{Hu}(m) = \sum\nolimits_{n\in\mathbb{Z}} \widehat{u}(n)\big\langle h(\cdot,n)e_n, e_{-m}\big\rangle_{\mathcal{D}'(\mathbb{T}),\mathcal{D}(\mathbb{T})}.$$

In particular, if h is independent of t, then we simply have $\widehat{Hu}(n) = h(n)\widehat{u}(n)$. The following properties of periodic pseudodifferential operators can be found in [34, Theorem 7.3.1 and Theorem 7.8.1].

Proposition 2.2.

(i) *Let $H \in \Psi^\alpha$. Then for any $s \in \mathbb{R}$ the operator H uniquely extends to a bounded operator $H^s(\mathbb{T}) \to H^{s-\alpha}(\mathbb{T})$; this extension will be denoted by the same symbol H.*
(ii) *For any $H \in \Psi^\alpha$ and $G \in \Psi^\beta$ one has $H + G \in \Psi^{\max\{\alpha,\beta\}}$, $HG \in \Psi^{\alpha+\beta}$, and $HG - GH \in \Psi^{\alpha+\beta-1}$.*

It is now straightforward to define periodic pseudodifferential operators on Σ.

Definition 2.3. A linear map $H : C^\infty(\Sigma) \to \mathcal{D}'(\Sigma)$ is called a periodic pseudodifferential operator of order $\alpha \in \mathbb{R}$ on Σ, if there exists a periodic pseudodifferential operator H_0 of order α on $\mathbb{T}$ such that $H = U^{-1}H_0U$. We denote by Ψ^α_Σ the linear space of all periodic pseudodifferential operators of order $\alpha \in \mathbb{R}$ on Σ and set $\Psi^{-\infty}_\Sigma := \cap_{\alpha\in\mathbb{R}}\Psi^\alpha_\Sigma$.

In view of Proposition 2.2 and the fact that U is unitary it is clear that each $H \in \Psi^\alpha_\Sigma$ induces a unique bounded operator $H : H^s(\Sigma) \to H^{s-\alpha}(\Sigma)$.

In what follows we discuss several special periodic pseudodifferential operators which will play an important role in the main part of this paper. First, let $c_0 > 0$ be a constant and consider the operator

$$L^\alpha u(t) = \sum_{n\in\mathbb{Z}} \big(c_0^2 + |n|\big)^{\frac{\alpha}{2}}\,\widehat{u}(n)\,e_n(t), \quad u \in C^\infty(\mathbb{T}), \quad \alpha \in \mathbb{R}, \tag{2.6}$$

on $C^\infty(\mathbb{T})$. Note that the Fourier coefficients of $L^\alpha u$ are $\widehat{L^\alpha u}(n) = (c_0^2 + |n|)^{\frac{\alpha}{2}}\,\widehat{u}(n)$ for $n \in \mathbb{Z}$. One can show that $L^\alpha \in \Psi^{\frac{\alpha}{2}}$ and hence L^α induces an isomorphism from $H^s(\mathbb{T})$

to $H^{s-\frac{\alpha}{2}}(\mathbb{T})$ for any $s \in \mathbb{R}$. The operator $L = L^1$ will be of particular importance in the following.

Using the operator U from (2.2) we introduce

$$\Lambda^\alpha := U^{-1} L^\alpha U \in \Psi_\Sigma^{\frac{\alpha}{2}}, \quad \alpha \in \mathbb{R}, \tag{2.7}$$

and conclude that $\Lambda^\alpha : H^s(\Sigma) \to H^{s-\frac{\alpha}{2}}(\Sigma)$ is an isomorphism for any $\alpha, s \in \mathbb{R}$, and $\Lambda^\alpha \Lambda^\beta = \Lambda^{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{R}$. We note that the realization of $\Lambda = \Lambda^1$ for $s = \frac{1}{2}$ is viewed as an unbounded self-adjoint operator in $L^2(\Sigma)$ satisfying $\Lambda \geq c_0$. In particular, by varying c_0 we get that Λ is a uniformly positive operator and that its lower bound can be arbitrarily large.

The following lemma, in which the adjoint of a formally symmetric periodic pseudodifferential operator is described and that can be proved with standard manipulations for distributions, will be useful later.

Lemma 2.4. *For $H \in \Psi_\Sigma^\alpha$ consider the linear operator in $L^2(\Sigma)$ defined by $H_\infty u = Hu$ on $C^\infty(\Sigma)$. If H_∞ is symmetric, then its adjoint H_∞^* is given by*

$$H_\infty^* f = Hf, \quad \operatorname{dom} H_\infty^* = \big\{ f \in L^2(\Sigma) : Hf \in L^2(\Sigma) \big\}.$$

Various integral operators on $\mathbb{T}$ are in fact periodic pseudodifferential operators, which allows us to deduce their mapping properties from the general theory, and which can be translated to integral operators on Σ using the map U from (2.2). The following proposition is borrowed from [34, Theorem 7.6.1]; recall that ω is given by (2.4).

Proposition 2.5. *Let $\alpha \in \mathbb{R}$ and $\kappa \in \mathcal{D}'(\mathbb{T})$ such that for any $j \in \mathbb{N}_0$ there exists $c_j > 0$ with $\left|\omega^j \widehat{\kappa}(n)\right| \leq c_j \underline{n}^{\alpha - j}$ for all $n \in \mathbb{Z}$. Let $h \in C^\infty(\mathbb{T}^2)$. Then the operator H defined by*

$$(Hu)(t) := \Big(\kappa \star \big(h(t, \cdot) u \big) \Big)(t), \quad u \in C^\infty(\mathbb{T}), \tag{2.8}$$

belongs to $H \in \Psi^\alpha$. In particular, the integral operator acting as

$$Hu(t) := \int_{\mathbb{T}} h(t,s) u(s) \, \mathrm{d}s, \quad u \in C^\infty(\mathbb{T}),$$

belongs to $\Psi^{-\infty}$.

In the following proposition we discuss a class of integral operators that appear quite frequently in our applications.

Proposition 2.6. *Let $a : \mathbb{T}^2 \to \mathbb{C}$ and $\rho : \mathbb{T} \to \mathbb{C}$ be C^∞-functions, where ρ is injective with $\rho'(t) \neq 0$ for all $t \in \mathbb{T}$. For $m \in \mathbb{N}_0$ set $\kappa_m(z) := z^m \log|z|$ for $z \in \mathbb{C} \setminus \{0\}$ and define an integral operator H_m by*

$$H_m u(t) := \int_{\mathbb{T}} \kappa_m\big(\rho(t)-\rho(s)\big)\, a(t,s)\, u(s)\, \mathrm{d}s, \quad u \in C^\infty(\mathbb{T}).$$

Then $H_m \in \Psi^{-m-1}$. Furthermore, in the special case $a \equiv 1$ and $m = 0$ one has

$$\mathbb{1} + 2LH_0L \in \Psi^{-1}, \tag{2.9}$$

where the operator L is defined by (2.6).

Proof. First, we treat the case $m = 0$. We introduce an auxiliary function $\chi_0 : \mathbb{T} \to \mathbb{R}$ by $\chi_0(t) := \log|\sin(\pi t)|$, then its Fourier coefficients are

$$\widehat{\chi_0}(n) = \begin{cases} -\log 2, & n = 0, \\ -\frac{1}{2|n|}, & n \neq 0, \end{cases} \tag{2.10}$$

see [34, Example 5.6.1]. Next, one has

$$\log|\rho(t)-\rho(s)| = \log\big|\sin\big(\pi(t-s)\big)\big| + a_0(t,s), \tag{2.11}$$
$$a_0(t,s) = \log\left|\frac{\rho(t)-\rho(s)}{\sin(\pi(t-s))}\right|, \quad t \neq s, \quad \text{and} \quad a_0(t,t) = \log\left(\frac{|\rho'(t)|}{\pi}\right).$$

Using Taylor expansions one sees that there exist smooth functions f_1 and f_2 such that

$$\frac{1}{\sin(\pi(t-s))} = \frac{1}{\pi(t-s)} f_1(t,s) \quad \text{and} \quad \rho(t)-\rho(s) = (t-s) f_2(t,s),$$

and since ρ is injective, we have $(\rho(t)-\rho(s))/\sin(\pi(t-s)) \neq 0$. One concludes that $a_0 : \mathbb{T}^2 \to \mathbb{C}$ is a C^∞-function. Now we decompose $H_0 = C_0 + D_0$, where

$$C_0 u(t) = \int_{\mathbb{T}} \chi_0(t-s)\, a(t,s)\, u(s)\, \mathrm{d}s = \big(\chi_0 \star (a(t,\cdot)u)\big)(t),$$
$$D_0 u(t) = \int_{\mathbb{T}} a_0(t,s)\, a(t,s)\, u(s)\, \mathrm{d}s.$$

It follows from (2.10) and Proposition 2.5 that $C_0 \in \Psi^{-1}$ and $D_0 \in \Psi^{-\infty}$. Therefore $H_0 \in \Psi^{-1}$ by Proposition 2.2.

To show (2.9) consider $LH_0L = LC_0L + LD_0L$ and note that the second summand is in $\Psi^{-\infty}$. Furthermore, for $a \equiv 1$ the Fourier coefficients of C_0Lu are given by

$$\widehat{C_0Lu}(n) = \widehat{\chi_0}(n)\widehat{Lu}(n) = \widehat{\chi_0}(n)(c_0^2 + |n|)^{\frac{1}{2}}\widehat{u}(n),$$

and using (2.10) one finds

$$\widehat{LC_0Lu}(n) = \big(c_0^2 + |n|\big)^{\frac{1}{2}} \widehat{\chi_0}(n)(c_0^2 + |n|)^{\frac{1}{2}} \widehat{u}(n) = b(n)\widehat{u}(n)$$

with

$$b(n) = \big(c_0^2 + |n|\big)\widehat{\chi_0}(n) = \begin{cases} -c_0^2 \log 2, & n = 0, \\ -\frac{1}{2} - \frac{c_0^2}{2|n|}, & n \neq 0, \end{cases}$$

which shows that the action of the operator $K := \mathbb{1} + 2LC_0L$ is determined by

$$\widehat{Ku}(n) = k(n)\widehat{u}(n) \quad \text{with} \quad k(n) = \begin{cases} 1 - 2c_0^2 \log 2, & n = 0, \\ -\frac{c_0^2}{|n|}, & n \neq 0. \end{cases}$$

Proposition 2.5 implies $K \in \Psi^{-1}$.

For $m \geq 1$ we represent $\rho(t) - \rho(s) = (e^{-2\pi \mathrm{i}(t-s)} - 1)\, a_1(t,s)$ with

$$a_1(t,s) = \frac{\rho(t) - \rho(s)}{e^{-2\pi \mathrm{i}(t-s)} - 1}, \quad t \neq s, \quad \text{and} \quad a_1(t,t) = \frac{\rho'(t)}{-2\pi \mathrm{i}},$$

and note that $a_1 \in C^\infty(\mathbb{T}^2)$. Then using the decomposition (2.11) we write

$$\begin{aligned}(\rho(t) - \rho(s))^m \log(|\rho(t) - \rho(s)|) = \big(e^{-2\pi \mathrm{i}(t-s)} - 1\big)^m \log(|\sin(\pi(t-s))|) a_1(t,s)^m \\ + \big(e^{-2\pi \mathrm{i}(t-s)} - 1\big)^m a_0(t,s) a_1(t,s)^m.\end{aligned}$$

This shows that $H_m = C_m + D_m$, where C_m and D_m are integral operators

$$C_m u(t) = \int_{\mathbb{T}} \big(e^{-2\pi \mathrm{i}(t-s)} - 1\big)^m \log(|\sin(\pi(t-s))|) a_1(t,s)^m \, a(t,s) u(s)\, \mathrm{d}s,$$

$$D_m u(t) = \int_{\mathbb{T}} \big(e^{-2\pi \mathrm{i}(t-s)} - 1\big)^m a_0(t,s) a_1(t,s)^m a(t,s)\, u(s)\, \mathrm{d}s.$$

The integral kernel of D_m is smooth, which implies by Proposition 2.5 that $D_m \in \Psi^{-\infty}$. It remains to show that $C_m \in \Psi^{-(m+1)}$. For that consider the function

$$\chi_m : \mathbb{T} \to \mathbb{C}, \quad \chi_m(t) := \big(e^{-2\pi \mathrm{i} t} - 1\big)^m \log\big|\sin(\pi t)\big|$$

and remark that $\widehat{\chi_m}(n) = \big(\omega^m \widehat{\chi_0}\big)(n)$. With the help of equation (2.10) it follows that $|\omega^j \widehat{\chi_m}(n)| = |\omega^{m+j} \widehat{\chi_0}(n)| \leq c_j \underline{n}^{-m-1-j}$. By Proposition 2.5 this yields $C_m \in \Psi^{-(m+1)}$, which completes the proof of this proposition. □

Next, recall that the *Hilbert transform* T_0 on $\mathbb{T}$ is defined by

$$T_0 u(t) := \mathrm{i} \text{ p.v.} \int_{\mathbb{T}} \cot\big(\pi(t-s)\big) u(s) \mathrm{d}s = (\kappa \star u)(t), \quad \kappa = \mathrm{i}\, \text{p.v.} \cot(\pi \cdot), \tag{2.12}$$

where p.v. means the principal value of the integral. By [34, Section 5.7] for the distribution κ one has $\widehat{\kappa}(n) = \operatorname{sgn} n$. It follows that $\widehat{T_0^2 u}(n) = (1 - \delta_{0,n})\widehat{u}(n)$, and

$$T_0 \in \Psi^0, \quad T_0^2 - \mathbb{1} \in \Psi^{-\infty}. \tag{2.13}$$

In the following assume that $a \in C^\infty(\mathbb{T}^2)$. Then the operator

$$(T_1 u)(t) = \mathrm{i}\ \text{p.v.} \int_{\mathbb{T}} \cot\big(\pi(t-s)\big)\, a(s,t)\, u(s)\, \mathrm{d}s$$

satisfies for $a_0(t) := a(t,t)$ the relation

$$T_1 - a_0 T_0 \in \Psi^{-\infty}, \tag{2.14}$$

see Section 7.6.2 in [34]. Since the commutator $T_2 := a_0 T_0 - T_0 a_0$, which acts as

$$T_2 u(t) = \mathrm{i}\ \text{p.v.} \int_{\mathbb{T}} \cot\big(\pi(t-s)\big)\, \big(a(t,t) - a(s,s)\big)\, u(s)\, \mathrm{d}s,$$

has a C^∞-smooth integral kernel, the principal value can be dropped, as the integral is convergent, and Proposition 2.5 implies that $T_2 \in \Psi^{-\infty}$. Hence, we also have

$$T_1 - T_0 a_0 \in \Psi^{-\infty}. \tag{2.15}$$

Corollary 2.7. *Let $\rho : \mathbb{T} \to \mathbb{C}$ be C^∞-smooth and injective with $\rho'(t) \neq 0$ for all $t \in \mathbb{T}$. Then the operator C given by*

$$Cu(t) = \frac{\mathrm{i}}{\pi}\ \text{p.v.} \int_{\mathbb{T}} \frac{u(s)}{\rho(t) - \rho(s)}\, \mathrm{d}s, \quad u \in C^\infty(\mathbb{T}),$$

satisfies

$$C - \frac{1}{\rho'} T_0 \in \Psi^{-\infty} \qquad \text{and} \qquad C - T_0 \frac{1}{\rho'} \in \Psi^{-\infty}. \tag{2.16}$$

Proof. We write

$$\frac{1}{\pi} \frac{1}{\rho(t) - \rho(s)} = \cot\big(\pi(t-s)\big) a(t,s) \quad \text{with} \quad a(t,s) = \frac{1}{\pi} \frac{\tan\big(\pi(t-s)\big)}{\rho(t) - \rho(s)}, \quad t \neq s,$$

and $a(t,t) = 1/\rho'(t)$. Then $a \in C^\infty(\mathbb{T}^2)$ and $a_0(t) = a(t,t) = 1/\rho'(t)$. Thus (2.16) follows from (2.14) and (2.15). □

Finally we recall the definition of the *Cauchy transform* C_Σ *on* Σ. We identify $\mathbb{R}^2$ with $\mathbb{C}$ by

$$\mathbb{R}^2 \ni x = (x_1, x_2) \sim x_1 + \mathrm{i}x_2 =: \xi \in \mathbb{C}, \quad \mathbb{R}^2 \ni y = (y_1, y_2) \sim y_1 + \mathrm{i}y_2 =: \zeta \in \mathbb{C},$$

then C_Σ is defined by

$$C_\Sigma u(\xi) := \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_\Sigma \frac{u(\zeta)}{\xi - \zeta}\mathrm{d}\zeta, \quad u \in C^\infty(\Sigma), \quad \xi \in \Sigma. \tag{2.17}$$

With an arc-length parametrization γ of Σ and $x = \gamma(t), y = \gamma(s)$ one has

$$C_\Sigma u\big(\gamma(t)\big) = \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{\big(\gamma_1'(s) + \mathrm{i}\gamma_2'(s)\big)u\big(\gamma(s)\big)}{\big(\gamma_1(t) + \mathrm{i}\gamma_2(t)\big) - \big(\gamma_1(s) + \mathrm{i}\gamma_2(s)\big)}\,\mathrm{d}s.$$

Recall that for the tangent vector field $\mathbf{t}$ at Σ and $y = \gamma(s) \in \Sigma$ we use the notation $T(y) := \mathbf{t}_1(y) + \mathrm{i}\mathbf{t}_2(y) = \gamma_1'(s) + \mathrm{i}\gamma_2'(s)$. We shall also view $y \mapsto T(y)$ as a function on Σ or $s \mapsto T(\gamma(s))$ as a function on the interval $[0, \ell]$. The same holds for the function $\overline{T}(y) := \mathbf{t}_1(y) - \mathrm{i}\mathbf{t}_2(y) = \gamma_1'(s) - \mathrm{i}\gamma_2'(s)$, and we will also denote the corresponding multiplication operators by T and $\overline{T}$. With this we see for $u \in C^\infty(\Sigma)$ and $x = \gamma(t) \in \Sigma$ that

$$\begin{aligned}(C_\Sigma \overline{T}u)(x) &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{\big(\gamma_1'(s) + \mathrm{i}\gamma_2'(s)\big)\big(\gamma_1'(s) - \mathrm{i}\gamma_2'(s)\big)u\big(\gamma(s)\big)}{\big(\gamma_1(t) + \mathrm{i}\gamma_2(t)\big) - \big(\gamma_1(s) + \mathrm{i}\gamma_2(s)\big)}\,\mathrm{d}s \\ &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_\Sigma \frac{u(y)}{(x_1 + \mathrm{i}x_2) - (y_1 + \mathrm{i}y_2)}\,\mathrm{d}s(y).\end{aligned} \tag{2.18}$$

We also consider the formal dual C_Σ' of C_Σ in $L^2(\Sigma)$, which acts as

$$C_\Sigma' u(\gamma(t)) = \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{\big(\gamma_1'(t) - \mathrm{i}\gamma_2'(t)\big)u\big(\gamma(s)\big)}{\big(\gamma_1(t) - \mathrm{i}\gamma_2(t)\big) - \big(\gamma_1(s) - \mathrm{i}\gamma_2(s)\big)}\,\mathrm{d}s \tag{2.19}$$

for $u \in C^\infty(\Sigma)$ and $x = \gamma(t) \in \Sigma$. Note that C_Σ' is the operator which satisfies $(C_\Sigma u, v)_{L^2(\Sigma)} = (u, C_\Sigma' v)_{L^2(\Sigma)}$ for all $u, v \in C^\infty(\Sigma)$. Similarly as in (2.18) we have

$$\begin{aligned}(TC_\Sigma' u)(x) &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{\big(\gamma_1'(t) + \mathrm{i}\gamma_2'(t)\big)\big(\gamma_1'(t) - \mathrm{i}\gamma_2'(t)\big)u\big(\gamma(s)\big)}{\big(\gamma_1(t) - \mathrm{i}\gamma_2(t)\big) - \big(\gamma_1(s) - \mathrm{i}\gamma_2(s)\big)}\,\mathrm{d}s \\ &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_\Sigma \frac{u(y)}{(x_1 - \mathrm{i}x_2) - (y_1 - \mathrm{i}y_2)}\,\mathrm{d}s(y).\end{aligned} \tag{2.20}$$

In the following proposition we summarize the basic properties of C_Σ and C'_Σ which are needed for our further considerations. They basically follow directly from (2.18), (2.20), Corollary 2.7, and (2.13).

Proposition 2.8. *Let C_Σ and C'_Σ be defined by* (2.17) *and* (2.19), *let U be given by* (2.2), *and let the Hilbert transform T_0 be defined by* (2.12). *Then the following is true:*

(i) $C_\Sigma - U^{-1}T_0U \in \Psi_\Sigma^{-\infty}$ *and, in particular,* $C_\Sigma \in \Psi_\Sigma^0$.
(ii) $C'_\Sigma - U^{-1}T_0U \in \Psi_\Sigma^{-\infty}$ *and, in particular,* $C'_\Sigma \in \Psi_\Sigma^0$.

Furthermore, one has $C'_\Sigma C_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty}$ *and* $C_\Sigma C'_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty}$.

Proof. Let us prove (i). Denote by T and $\overline{T}$ the multiplication operators by the functions $s \mapsto T(\gamma(s)) = \gamma_1'(s) + \mathrm{i}\gamma_2'(s)$ and $s \mapsto \overline{T}(\gamma(s)) = \gamma_1'(s) - \mathrm{i}\gamma_2'(s)$ respectively. Clearly, they both belong to Ψ_Σ^0, see [34, Section 7.2]. Hence (i) is equivalent to

$$C_\Sigma\overline{T} - U^{-1}T_0U\overline{T} = C_\Sigma\overline{T} - U^{-1}T_0\overline{T}\big(\gamma(\ell\cdot)\big)U \in \Psi_\Sigma^{-\infty},$$

which in turn is equivalent, by definition, to $UC_\Sigma\overline{T}U^{-1} - T_0\overline{T}(\gamma(\ell\cdot)) \in \Psi^{-\infty}$. For $v \in C^\infty(\mathbb{T})$ and $t \in \mathbb{T}$, we compute $\big(UC_\Sigma\overline{T}U^{-1}v\big)(t)$. Note that for $x = (x_1, x_2) \in \Sigma$ and $w(x) := (U^{-1}v)(x)$, (2.3) and (2.18) give

$$\begin{aligned}(C_\Sigma\overline{T}w)(x) &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{w(\gamma(s))}{(x_1 + \mathrm{i}x_2) - \big(\gamma_1(s) + \mathrm{i}\gamma_2(s)\big)}\,\mathrm{d}s \\ &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{v(\ell^{-1}s)}{(x_1 + \mathrm{i}x_2) - \big(\gamma_1(s) + \mathrm{i}\gamma_2(s)\big)}\,\mathrm{d}s.\end{aligned}$$

Hence, a change of variable yields

$$(UC_\Sigma\overline{T}U^{-1}v)(t) = \ell\frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_{\mathbb{T}} \frac{v(s)}{\rho(t) - \rho(s)}\,\mathrm{d}s$$

with $\rho(t) := \gamma_1(\ell t) + \mathrm{i}\gamma_2(\ell t)$. For all $t \in \mathbb{T}$ we have $\rho'(t) = \ell T\big(\gamma(\ell t)\big) \neq 0$ and $1/\rho'(t) = \ell^{-1}\overline{T}\big(\gamma(\ell t)\big)$, and Corollary 2.7 gives $\ell^{-1}UC_\Sigma\overline{T}U^{-1} - \ell^{-1}T_0\overline{T}(\ell\cdot) \in \Psi^{-\infty}$, which completes the proof of (i). Item (ii) is proved in a similar fashion and the last statement is a consequence of (i), (ii), and (2.13). This can be seen by the equivalences

$$T_0^2 - \mathbb{1} \in \Psi^{-\infty} \quad \text{iff} \quad UC'_\Sigma U^{-1}UC_\Sigma U^{-1} - \mathbb{1} \in \Psi^{-\infty} \quad \text{iff} \quad C'_\Sigma C_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty},$$

and a similar argument shows $C_\Sigma C'_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty}$. This completes the proof. □

2.2. Boundary triples and their Weyl functions

We recall some basic facts about boundary triples following the first chapter of the paper [13], in which the proofs for all statements of this subsection can be found. We also refer the reader to [16,17] and the monographs [8,18] for more details and applications. Throughout this abstract section $\mathcal{H}$ stands for a separable Hilbert space.

Definition 2.9. Let S be a closed densely defined symmetric operator in $\mathcal{H}$. A *boundary triple* for S^* is a triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space $\mathcal{G}$ and two linear maps $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ satisfying the following two conditions:

(i) For all $f, g \in \operatorname{dom} S^*$ there holds $(S^* f, g)_{\mathcal{H}} - (f, S^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$.
(ii) The map $\operatorname{dom} S^* \ni f \mapsto (\Gamma_0 f, \Gamma_1 f) \in \mathcal{G} \times \mathcal{G}$ is surjective.

A boundary triple for S^* exists if and only if S admits self-adjoint extensions in $\mathcal{H}$. From now on we assume that this is satisfied and pick a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. This induces a number of additional objects. First, the operator $B_0 := S^* \upharpoonright \ker \Gamma_0$ is self-adjoint, and for any $z \in \operatorname{res} B_0$ one has the direct sum decomposition

$$\operatorname{dom} S^* = \operatorname{dom} B_0 \dot{+} \ker(S^* - z) = \ker \Gamma_0 \dot{+} \ker(S^* - z), \tag{2.21}$$

showing that $\Gamma_0 \upharpoonright \ker(S^* - z)$ is bijective. This allows to define the γ-field G and the Weyl function M associated to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ by

$$\begin{aligned} \operatorname{res} B_0 \ni z \mapsto G_z &:= \big(\Gamma_0 \upharpoonright \ker(S^* - z)\big)^{-1} : \mathcal{G} \to \mathcal{H}, \\ \operatorname{res} B_0 \ni z \mapsto M_z &:= \Gamma_1 G_z : \mathcal{G} \to \mathcal{G}. \end{aligned}$$

For $z \in \operatorname{res} B_0$ the operators G_z and M_z are bounded, and $z \mapsto G_z$ and $z \mapsto M_z$ are holomorphic. Their adjoints are given by $G_z^* = \Gamma_1 (B_0 - \overline{z})^{-1}$ and $M_z^* = M_{\bar{z}}$.

Let $\mathcal{G}_\Pi$ be a closed subspace of $\mathcal{G}$ viewed as a Hilbert space when endowed with the induced inner product. In addition, let $\Pi : \mathcal{G} \to \mathcal{G}_\Pi$ be the orthogonal projection, then $\Pi^* : \mathcal{G}_\Pi \to \mathcal{G}$ is the canonical embedding. Finally, let Θ be a linear operator in $\mathcal{G}_\Pi$. We will be interested in the operator $B_{\Pi,\Theta}$ defined as the restriction of S^* onto the set

$$\operatorname{dom} B_{\Pi,\Theta} = \big\{ f \in \operatorname{dom} S^* : \Pi \Gamma_1 f = \Theta \Pi \Gamma_0 f,\ (\mathbb{1} - \Pi^* \Pi) \Gamma_0 f = 0 \big\},$$

where the boundary condition $\Pi \Gamma_1 f = \Theta \Pi \Gamma_0 f$ in $\operatorname{dom} B_{\Pi,\Theta}$ also contains the condition $\Pi \Gamma_0 f \in \operatorname{dom} \Theta$. A number of properties of $B_{\Pi,\Theta}$ appear to be encoded in Θ. The most important of them for our purposes are summarized in the following theorem:

Theorem 2.10. *The operator $B_{\Pi,\Theta}$ is (essentially) self-adjoint in $\mathcal{H}$ if and only if Θ is (essentially) self-adjoint in $\mathcal{G}_\Pi$. Furthermore, if Θ is self-adjoint and $z \in \operatorname{res} B_0$, then the following assertions hold:*

(i) *$z \in \operatorname{spec} B_{\Pi,\Theta}$ if and only if $0 \in \operatorname{spec}(\Theta - \Pi M_z \Pi^*)$.*
(ii) *$z \in \operatorname{spec_p} B_{\Pi,\Theta}$ if and only if $0 \in \operatorname{spec_p}(\Theta - \Pi M_z \Pi^*)$, and in that case the eigenspaces are related by*

$$\ker(B_{\Pi,\Theta} - z) = G_z \Pi^* \ker(\Theta - \Pi M_z \Pi^*).$$

(iii) *$z \in \operatorname{spec_{ess}} B_{\Pi,\Theta}$ if and only if $0 \in \operatorname{spec_{ess}}(\Theta - \Pi M_z \Pi^*)$.*
(iii) *For all $z \in \operatorname{res} B_{\Pi,\Theta} \cap \operatorname{res} B_0$ one has*

$$(B_{\Pi,\Theta} - z)^{-1} = (B_0 - z)^{-1} + G_z \Pi^* (\Theta - \Pi M_z \Pi^*)^{-1} \Pi G_{\bar z}^*.$$

Finally we recall a special approach for the construction of boundary triples using abstract trace maps developed in [32] and [33], see also [13, Section 1.4.2]. Let B be a self-adjoint operator in the Hilbert space $\mathcal{H}$, let $\mathcal{G}$ be another Hilbert space, and assume that $\mathcal{T} : \operatorname{dom} B \to \mathcal{G}$ is a *surjective* linear operator which is bounded with respect to the graph norm of B and such that $\ker \mathcal{T}$ is a dense subspace of the initial Hilbert space $\mathcal{H}$. Then $S := B \upharpoonright \ker \mathcal{T}$ is a densely defined closed symmetric operator. Next, define for any $z \in \operatorname{res} B$ the injective operator

$$G_z := \big(\mathcal{T}(B - \bar z)^{-1}\big)^*, \tag{2.22}$$

which is bounded from $\mathcal{G}$ to $\mathcal{H}$. Then one has $\operatorname{ran} G_z = \ker(S^* - z)$ for $z \in \operatorname{res} B$ and (2.21) leads to the direct sum decomposition

$$\operatorname{dom} S^* = \operatorname{dom} B \dot{+} \operatorname{ran} G_z, \quad z \in \operatorname{res} B, \tag{2.23}$$

which shows that for all $f \in \operatorname{dom} S^*$ there exist unique $f_z \in \operatorname{dom} B$ and $\xi \in \mathcal{G}$ such that $f = f_z + G_z \xi$; one can show that the component ξ is independent of the choice of z. Having these notations in hand we can formulate now the following proposition:

Proposition 2.11. *Let $\zeta \in \operatorname{res} B$ be fixed and define the mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ for $f = f_\zeta + G_\zeta \xi = f_{\bar\zeta} + G_{\bar\zeta}\xi \in \operatorname{dom} S^*$ by*

$$\Gamma_0 f := \xi \quad \text{and} \quad \Gamma_1 f := \frac{1}{2}\mathcal{T}(f_\zeta + f_{\bar\zeta}).$$

Then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triple for S^* with $S^* \upharpoonright \ker \Gamma_0 = B$. Moreover, the γ-field and the Weyl function are given by (2.22) and $M_z = \mathcal{T}(G_z - \frac{1}{2}(G_\zeta + G_{\bar\zeta}))$.

3. The free Dirac operator and a boundary triple for its singular perturbations

In this section we first recall the definition of the free Dirac operator in $\mathbb{R}^2$, a minimal and a maximal realization of the Dirac operator in $\mathbb{R}^2 \setminus \Sigma$, and we introduce and study some families of integral operators which will play an important role in our analysis in Section 4. Afterwards, we define a boundary triple which is useful in the treatment of Dirac operators with singular δ-interactions.

3.1. Dirac operators and associated integral operators

For $m \in \mathbb{R}$ the free Dirac operator in $\mathbb{R}^2$ is defined by

$$A_0 f = -\mathrm{i}\,\sigma \cdot \nabla f + m\sigma_3 f, \quad \mathrm{dom}\, A_0 = H^1(\mathbb{R}^2;\mathbb{C}^2), \tag{3.1}$$

where $\sigma := (\sigma_1, \sigma_2)$ and σ_3 are the $\mathbb{C}^{2\times 2}$-valued Pauli spin matrices in (1.3). It is well-known that A_0 is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with purely essential spectrum,

$$\mathrm{spec}\, A_0 = \mathrm{spec}_{\mathrm{ess}}\, A_0 = \big(-\infty, -|m|\big] \cup \big[|m|, +\infty\big).$$

With (1.4) one gets $A_0^2 = (-\Delta + m^2)\sigma_0$, where $-\Delta$ is the free Laplacian defined on $H^2(\mathbb{R}^2)$, and this implies for $z \in \mathrm{res}(A_0)$

$$\begin{aligned}(A_0 - z)^{-1} f(x) &= (A_0 + z)(-\Delta + m^2 - z^2)^{-1} f(x)\\ &= (A_0 + z)\left[\frac{1}{2\pi}\int_{\mathbb{R}^2} K_0\big(\sqrt{m^2 - z^2}|\cdot - y|\big) f(y)\,\mathrm{d}y\right](x)\\ &= \int_{\mathbb{R}^2} \phi_z(x-y) f(y)\,\mathrm{d}y, \quad f \in L^2(\mathbb{R}^2;\mathbb{C}^2),\end{aligned}$$

where

$$\phi_z(x) = \mathrm{i}\frac{\sqrt{m^2 - z^2}}{2\pi} K_1\Big(\sqrt{m^2 - z^2}|x|\Big)\Big(\sigma \cdot \frac{x}{|x|}\Big) + \frac{1}{2\pi} K_0\Big(\sqrt{m^2 - z^2}|x|\Big)\big(m\sigma_3 + z\sigma_0\big); \tag{3.2}$$

here K_j stands for the modified Bessel function of second kind of order j, and we take the principal square root function, i.e. for $z \in \mathbb{C} \setminus [0, \infty)$ the number $\sqrt{z}$ is determined by $\mathrm{Re}\sqrt{z} > 0$.

Next we introduce a symmetric operator which is suitable for our purposes. More precisely, denote by S the restriction of A_0 to the functions vanishing at Σ, i.e.

$$Sf = (-\mathrm{i}\sigma \cdot \nabla + m\sigma_3) f, \quad \mathrm{dom}\, S = H_0^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2). \tag{3.3}$$

Then the operator $A_{\eta,\tau}$ defined in (1.2) is an extension of S. The standard theory implies that the adjoint S^* is the maximal realization of the same differential expression in $\mathbb{R}^2 \setminus \Sigma$,

$$\begin{aligned} S^* f &= (-\mathrm{i}\sigma \cdot \nabla + m\sigma_3) f_+ \oplus (-\mathrm{i}\sigma \cdot \nabla + m\sigma_3) f_-,\\ \mathrm{dom}\, S^* &= \big\{ f = f_+ \oplus f_- \in L^2(\Omega_+;\mathbb{C}^2) \oplus L^2(\Omega_-;\mathbb{C}^2) :\ f_\pm \in H(\sigma, \Omega_\pm)\big\},\end{aligned} \tag{3.4}$$

and we recall that

$$H(\sigma,\Omega_\pm)=\big\{f_\pm\in L^2(\Omega_\pm;\mathbb{C}^2):(-\mathrm{i}\sigma\cdot\nabla+m\sigma_3)f_\pm\in L^2(\Omega_\pm;\mathbb{C}^2)\big\},\tag{3.5}$$

which becomes a Hilbert space if endowed with the norm

$$\|f_\pm\|^2_{H(\sigma,\Omega_\pm)}:=\|f\|^2_{L^2(\Omega_\pm;\mathbb{C}^2)}+\big\|(-\mathrm{i}\sigma\cdot\nabla+m\sigma_3)f_\pm\big\|^2_{L^2(\Omega_\pm;\mathbb{C}^2)}.$$

For our further considerations, it is useful to extend the Dirichlet trace operator onto $H(\sigma,\Omega_\pm)$. In the following lemma we summarize several known results; we refer to [11, Lemma 2.3 and Lemma 2.4] for compact proofs:

Lemma 3.1. *The trace map $\mathcal{T}^D_{\pm,0}:H^1(\Omega_\pm;\mathbb{C}^2)\to H^{\frac12}(\Sigma;\mathbb{C}^2)$, $\mathcal{T}^D_{\pm,0}f=f|_\Sigma$, extends uniquely to a bounded linear operator $\mathcal{T}^D_\pm:H(\sigma,\Omega_\pm)\to H^{-\frac12}(\Sigma;\mathbb{C}^2)$. Moreover, if one has $\mathcal{T}^D_\pm f\in H^{\frac12}(\Sigma;\mathbb{C}^2)$ for $f\in H(\sigma,\Omega_\pm)$, then $f\in H^1(\Omega_\pm;\mathbb{C}^2)$.*

Now we introduce some families of integral operators corresponding to the Green function ϕ_z from (3.2). Let us denote the Dirichlet trace operator on $H^1(\mathbb{R}^2;\mathbb{C}^2)$ by $\mathcal{T}^D:H^1(\mathbb{R}^2;\mathbb{C}^2)\to H^{\frac12}(\Sigma;\mathbb{C}^2)$. It is well-known that $\mathcal{T}^D$ is bounded, surjective, and $\ker\mathcal{T}^D=H^1_0(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$; cf. [25, Theorems 3.37 and 3.40]. For $z\in\operatorname{res}A_0$ we first consider the bounded operator

$$\Phi'_z:=\mathcal{T}^D(A_0-\bar z)^{-1}:L^2(\mathbb{R}^2;\mathbb{C}^2)\to H^{\frac12}(\Sigma;\mathbb{C}^2)\tag{3.6}$$

and its anti-dual

$$\Phi_z:=\big(\mathcal{T}^D(A_0-\overline{z})^{-1}\big)':H^{-\frac12}(\Sigma;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2).\tag{3.7}$$

Using that Φ_z is defined as the anti-dual of Φ'_z one finds, in a similar way as in [6, Proposition 3.4], that Φ_z acts on $\varphi\in L^2(\Sigma;\mathbb{C}^2)$ as

$$\Phi_z\varphi(x)=\int_\Sigma\phi_z(x-y)\varphi(y)\,\mathrm{d}s(y)\quad\text{for a.e. }x\in\mathbb{R}^2\setminus\Sigma.$$

Moreover, similarly as in [9, Proposition 4.4] or [29, Proposition 2.21] one gets that $\operatorname{ran}\Phi_z\subset\ker(S^*-z)\subset H(\sigma,\mathbb{R}^2\setminus\Sigma)$. In fact, we will see later in Proposition 3.5 that Φ_z is closely related to the γ-field for a boundary triple for S^* and hence Φ_z is a bounded bijective operator from $H^{-\frac12}(\Sigma;\mathbb{C}^2)$ onto $\ker(S^*-z)$.

We will also need a family of boundary integral operators with integral kernel ϕ_z. For this purpose we first expose the structure of the Green function ϕ_z in more detail:

Lemma 3.2. *Let $z\in\operatorname{res}A_0$ and consider the function ϕ_z in (3.2). Then there exist scalar analytic functions g_1,g_2,g_3, and g_4 and a constant $c_1<0$ such that*

$$\begin{aligned}\phi_z(x) = \frac{\mathrm{i}}{2\pi}\,\sigma\cdot\frac{x}{|x|^2} - \frac{1}{2\pi}\Big(\log|x| + \log\sqrt{m^2-z^2} + c_1\Big)(m\sigma_3 + z\sigma_0)\\ + \frac{\mathrm{i}}{2\pi}(m^2-z^2)\Big[g_1\big((m^2-z^2)|x|^2\big)\big(\log\sqrt{m^2-z^2} + \log|x|\big)\\ + g_2\big((m^2-z^2)|x|^2\big)\Big](\sigma\cdot x)\\ + \frac{1}{2\pi}(m^2-z^2)|x|^2\Big[g_3\big((m^2-z^2)|x|^2\big)\big(\log\sqrt{m^2-z^2} + \log|x|\big)\\ + g_4\big((m^2-z^2)|x|^2\big)\Big](m\sigma_3 + z\sigma_0).\end{aligned} \tag{3.8}$$

In particular, there exist C^∞-smooth matrix valued functions f_1 and f_2 such that

$$\phi_z(x) = \frac{\mathrm{i}}{2\pi}\begin{pmatrix} 0 & \dfrac{1}{x_1+\mathrm{i}x_2} \\ \dfrac{1}{x_1-\mathrm{i}x_2} & 0\end{pmatrix} + f_1(x)\log|x| + f_2(x). \tag{3.9}$$

Proof. In order to prove the claimed results, let us recall the series representations of K_j from, e.g., §10.25.2, 10.31.1, and 10.31.2 in [27], which read

$$\begin{aligned} I_\mu(t) &= \frac{t^\mu}{2^\mu}\sum_{k=0}^\infty \frac{t^{2k}}{4^k k!\Gamma(\mu+k+1)}, \quad \mu\in\{0,1\},\\ K_1(t) &= \frac{1}{t} + (\log t - \log 2)I_1(t) - \frac{t}{4}\sum_{k=0}^\infty\big(\psi(k+1)+\psi(k+2)\big)\frac{t^{2k}}{4^k k!(k+1)!},\\ K_0(t) &= -(\log t - \log 2 + \gamma)I_0(t) + \sum_{k=1}^\infty\sum_{j=1}^k \frac{1}{j}\frac{t^{2k}}{4^k(k!)^2}, \end{aligned}$$

with $\psi(t) = \Gamma'(t)/\Gamma(t)$ and $\gamma = -\psi(1) < \log 2$. This implies first that $I_0(t) = 1 + t^2 h_0(t^2)$ and $I_1(t) = t h_1(t^2)$ with some analytic functions h_0 and h_1. Furthermore, with some analytic functions k_0 and k_1 we have

$$\begin{aligned} K_1(t) &= \frac{1}{t} + t h_1(t^2)\log t + t\big(k_1(t^2) - h_1(t^2)\log 2\big),\\ K_0(t) &= -\log t - c_1 - t^2 h_0(t^2)\log t - c_1 t^2 h_0(t^2) + t^2 k_0(t^2) \end{aligned}$$

with $c_1 := \gamma - \log 2 < 0$. This can be rewritten in a simplified form as

$$K_1(t) = \frac{1}{t} + t g_1(t^2)\log t + t g_2(t^2), \qquad K_0(t) = -\log t - c_1 + t^2 g_3(t^2)\log t + t^2 g_4(t^2),$$

where g_1, g_2, g_3, and g_4 are analytic functions and $c_1 < 0$. Using this now in the explicit expression for ϕ_z from (3.2) one immediately gets (3.8). The representation (3.9) follows from (3.8) after noting that

$$\frac{\mathrm{i}}{2\pi}\,\sigma\cdot\frac{x}{|x|^2}=\frac{\mathrm{i}}{2\pi}\begin{pmatrix}0 & \dfrac{1}{x_1+\mathrm{i}x_2}\\ \dfrac{1}{x_1-\mathrm{i}x_2} & 0\end{pmatrix}. \qquad \square$$

For $z\in\operatorname{res}A_0$ we introduce the operator

$$\mathcal{C}_z\varphi(x):=\mathrm{p.v.}\int_\Sigma \phi_z(x-y)\varphi(y)\mathrm{d}s(y),\quad \varphi\in C^\infty(\Sigma;\mathbb{C}^2),\ x\in\Sigma. \tag{3.10}$$

The basic properties of $\mathcal{C}_z$ are stated in the following proposition. For the formulation of the result, recall the definition of the operator Λ from (2.7) and of the Cauchy transform C_Σ and its dual C'_Σ from (2.17) and (2.19), respectively.

Proposition 3.3. *Let $z\in\operatorname{res}A_0$ and consider the operator $\mathcal{C}_z$ in (3.10). Then $\mathcal{C}_z\in\Psi^0_\Sigma$ and, in particular, $\mathcal{C}_z$ gives rise to a bounded operator in $H^s(\Sigma;\mathbb{C}^2)$ for any $s\in\mathbb{R}$. The realization in $L^2(\Sigma;\mathbb{C}^2)$ satisfies $\mathcal{C}_z^*=\mathcal{C}_{\bar z}$. Moreover, if $\mathbf{t}=(\mathbf{t}_1,\mathbf{t}_2)$ is the tangent vector field at Σ and $T=\mathbf{t}_1+\mathrm{i}\mathbf{t}_2$, $\overline{T}=\mathbf{t}_1-\mathrm{i}\mathbf{t}_2$, then with some $\Psi\in\Psi^{-1}_\Sigma$ one has*

$$\Lambda\mathcal{C}_z\Lambda=\frac12\begin{pmatrix}0 & \Lambda C_\Sigma\overline{T}\Lambda\\ \Lambda T C'_\Sigma\Lambda & 0\end{pmatrix}+\frac{\ell}{4\pi}\begin{pmatrix}(z+m)\mathbb{1} & 0\\ 0 & (z-m)\mathbb{1}\end{pmatrix}+\Psi. \tag{3.11}$$

Proof. We make use of (3.8) to decompose ϕ_z in the form $\phi_z(x)=\chi_1(x)+\chi_2(x)+\chi_3(x)$, where

$$\chi_1(x)=\frac{\mathrm{i}}{2\pi}\begin{pmatrix}0 & \dfrac{1}{x_1+\mathrm{i}x_2}\\ \dfrac{1}{x_1-\mathrm{i}x_2} & 0\end{pmatrix},\quad \chi_2(x)=-\frac{1}{2\pi}\begin{pmatrix}z+m & 0\\ 0 & z-m\end{pmatrix}\log|x|,$$
$$\chi_3(x)=\big[h_1\big(|x|^2\big)\log|x|+h_2\big(|x|^2\big)\big]\,(\sigma\cdot x)$$
$$\qquad+\big[|x|^2h_3\big(|x|^2\big)\log|x|+h_4\big(|x|^2\big)\big]\,(m\sigma_3+z\sigma_0),$$

and h_j are analytic functions. Now use the decomposition $\mathcal{C}_z=P_1+P_2+P_3$,

$$(P_1\varphi)(x)=\mathrm{p.v.}\int_\Sigma\chi_1(x-y)\varphi(y)\,\mathrm{d}s(y),\quad (P_j\varphi)(x)=\int_\Sigma\chi_j(x-y)\varphi(y)\,\mathrm{d}s(y),\ j=2,3;$$

we have removed the principal value in the expressions for P_2 and P_3 as the integral kernels are sufficiently regular and the integrals converge, see, e.g., [21, Proposition 3.10].

Let us discuss the operator P_1 first. With the help of (2.18) and (2.20) we obtain

$$P_1=\frac12\begin{pmatrix}0 & C_\Sigma\overline{T}\\ TC'_\Sigma & 0\end{pmatrix} \tag{3.12}$$

and since $T,\overline{T}\in\Psi^0_\Sigma$ we conclude from Proposition 2.8 that $P_1\in\Psi^0_\Sigma$.

Next, we claim that the integral operator P_2 admits the representation

$$P_2 = \frac{\ell}{4\pi}\begin{pmatrix}(z+m)\Lambda^{-2} & 0\\ 0 & (z-m)\Lambda^{-2}\end{pmatrix} + \Psi_1 \tag{3.13}$$

with some $\Psi_1 \in \Psi_\Sigma^{-2}$; due to $\Lambda^{-2} = U^{-1}L^{-2}U \in \Psi_\Sigma^{-1}$ this implies $P_2 \in \Psi_\Sigma^{-1}$. In fact, using the parametrization $\gamma : [0,\ell] \to \mathbb{R}^2$ of Σ we find

$$(UP_2 f)(t) = -\frac{\ell}{2\pi}\begin{pmatrix}z+m & 0\\ 0 & z-m\end{pmatrix}\int_{\mathbb{T}} \log\bigl|\gamma(\ell t)-\gamma(\ell s)\bigr|\, f\bigl(\gamma(\ell s)\bigr)\,\mathrm{d}s$$

for $f \in C^\infty(\Sigma)$. Therefore, with $f = U^{-1}u$ and $\rho(\cdot) = \gamma_1(\ell\cdot)+\mathrm{i}\gamma_2(\ell\cdot) \equiv \gamma(\ell\cdot)$ we conclude

$$\begin{aligned}(UP_2U^{-1}u)(t) &= -\frac{\ell}{2\pi}\begin{pmatrix}z+m & 0\\ 0 & z-m\end{pmatrix}\int_{\mathbb{T}} \log\bigl|\rho(t)-\rho(s)\bigr|\, u(s)\,\mathrm{d}s\\ &= -\frac{\ell}{2\pi}\begin{pmatrix}z+m & 0\\ 0 & z-m\end{pmatrix} H_0 u(t)\end{aligned}$$

with H_0 as in Proposition 2.6. Now it follows from Proposition 2.6 (with $m = 0$, $a \equiv 1$, and ρ as above) that $H_0 \in \Psi^{-1}$ and $\mathbb{1} + 2LH_0L \in \Psi^{-1}$. Furthermore, Proposition 2.2 (ii) and $L^{-1} \in \Psi^{-\frac{1}{2}}$ yield $\frac{1}{2}L^{-2} + H_0 \in \Psi^{-2}$ and hence

$$\begin{aligned}&-\frac{\ell}{4\pi}\begin{pmatrix}(z+m)L^{-2} & 0\\ 0 & (z-m)L^{-2}\end{pmatrix} + UP_2U^{-1} \in \Psi^{-2},\\ &-\frac{\ell}{4\pi}\begin{pmatrix}(z+m)\Lambda^{-2} & 0\\ 0 & (z-m)\Lambda^{-2}\end{pmatrix} + P_2 \in \Psi_\Sigma^{-2},\end{aligned}$$

which leads to (3.13).

It will be shown now that $P_3 \in \Psi_\Sigma^{-2}$. Indeed, setting again $\rho(\cdot) = \gamma_1(\ell\cdot)+\mathrm{i}\gamma_2(\ell\cdot) \equiv \gamma(\ell\cdot)$ we see that χ_3 decomposes as

$$\chi_3(\rho(t)-\rho(s)) = \log|\rho(t)-\rho(s)| a_1(t,s)\begin{pmatrix}0 & \overline{\rho(t)-\rho(s)}\\ \rho(t)-\rho(s) & 0\end{pmatrix} + a_2(t,s)$$

with the C^∞-smooth matrix-valued functions

$$\begin{aligned}a_1(t,s) &:= h_1\bigl(|\rho(t)-\rho(s)|^2\bigr)\sigma_0\\ &\quad + h_3\bigl(|\rho(t)-\rho(s)|^2\bigr)(m\sigma_3 + z\sigma_0)\begin{pmatrix}0 & \overline{\rho(t)-\rho(s)}\\ \rho(t)-\rho(s) & 0\end{pmatrix},\\ a_2(t,s) &:= h_2\bigl(|\rho(t)-\rho(s)|^2\bigr)]\begin{pmatrix}0 & \overline{\rho(t)-\rho(s)}\\ \rho(t)-\rho(s) & 0\end{pmatrix}\\ &\quad + h_4\bigl(|\rho(t)-\rho(s)|^2\bigr)\,(m\sigma_3 + z\sigma_0).\end{aligned}$$

Hence, it follows as above in the proof of (3.13) with Proposition 2.6 applied in the case $m = 1$ that $UP_3U^{-1} = H_1 \in \Psi^{-2}$, so that $P_3 \in \Psi_\Sigma^{-2}$. Together with (3.12) and (3.13) this implies first $\mathcal{C}_z \in \Psi_\Sigma^0$ and in a second step, together with Proposition 2.2 (i) and $\Lambda \in \Psi_\Sigma^{\frac{1}{2}}$, that also (3.11) is true.

Finally, since $\phi_z(y-x)^* = \phi_{\bar{z}}(x-y)$, we find that the realization of $\mathcal{C}_z$ in $L^2(\Sigma;\mathbb{C}^2)$ satisfies $\mathcal{C}_z^* = \mathcal{C}_{\bar{z}}$. Hence, all claims have been shown. □

Finally, we prove a result on how Φ_z and $\mathcal{C}_z$ are related to each other by taking traces. Recall that $\mathcal{T}_\pm^D$ is the Dirichlet trace operator on $H(\sigma,\Omega_\pm)$, see Lemma 3.1, and that $\mathcal{T}_\pm^D \Phi_z \varphi$ is well-defined for $\varphi \in H^{-1/2}(\Sigma;\mathbb{C}^2)$, as $\operatorname{ran}\Phi_z \subset \ker(S^* - z) \subset H(\sigma,\mathbb{R}^2 \setminus \Sigma)$.

Proposition 3.4. *For $\varphi \in H^{-\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ one has $\mathcal{T}_\pm^D \Phi_z \varphi = \mp \frac{\mathrm{i}}{2}(\sigma\cdot\nu)\varphi + \mathcal{C}_z\varphi$.*

Proof. It suffices to prove the equality for $\varphi \in C^\infty(\Sigma;\mathbb{C}^2)$; it is then extended by continuity to all $\varphi \in H^{-\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. The assertion essentially follows from the classical Plemelj-Sokhotskii formula, see, e.g., [34, Theorem 4.1.1], which states that the holomorphic function

$$\mathbb{C}\setminus\Sigma \ni \xi \mapsto \Phi(\xi) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\varphi(\zeta)}{\zeta-\xi}\,\mathrm{d}\zeta$$

satisfies

$$\mathcal{T}_\pm^D \Phi(\xi) = \frac{1}{2\pi\mathrm{i}}\,\mathrm{p.v.}\int_\Sigma \frac{\varphi(\zeta)}{\zeta-\xi}\,\mathrm{d}\zeta \pm \frac{1}{2}\varphi(\xi), \quad \xi\in\Sigma. \tag{3.14}$$

In order to use it, recall that by (3.9) we can write $\phi_z(x) = \chi_1(x) + \widetilde{\chi}_2(x)$ with

$$\chi_1(x) = -\frac{1}{2\pi\mathrm{i}}\begin{pmatrix} 0 & \dfrac{1}{x_1+\mathrm{i}x_2} \\ \dfrac{1}{x_1-\mathrm{i}x_2} & 0\end{pmatrix} \quad\text{and}\quad \widetilde{\chi}_2(x) = f_1(x)\log|x| + f_2(x),$$

where f_1 and f_2 are C^∞-smooth matrix functions. We decompose $\Phi_z = \Psi_1 + \Psi_2$ and $\mathcal{C}_z = P_1 + P_2$ with

$$\Psi_1\varphi(x) = \int_\Sigma \chi_1(x-y)\,\varphi(y)\,\mathrm{d}s(y) \qquad \Psi_2\varphi(x) = \int_\Sigma \widetilde{\chi}_2(x-y)\,\varphi(y)\,\mathrm{d}s(y),$$

$$P_1\varphi(x) = \mathrm{p.v.}\int_\Sigma \chi_1(x-y)\,\varphi(y)\,\mathrm{d}s(y), \qquad P_2\varphi(x) = \int_\Sigma \widetilde{\chi}_2(x-y)\,\varphi(y)\,\mathrm{d}s(y).$$

As in the proof of Proposition 3.3 we have removed the principal value from the expression for P_2, since the integral converges. One sees easily that $\Psi_2\varphi$ is continuous on $\mathbb{R}^2$, and its value on Σ coincides with $P_2\varphi$, i.e.

$$\mathcal{T}_{\pm}^{D}\Psi_2\varphi = P_2\varphi. \tag{3.15}$$

In order to find the relation between $\Psi_1\varphi$ and $P_1\varphi$, we write the normal vector field as a complex number $N = \nu_1 + \mathrm{i}\nu_2 = \gamma_2' - \mathrm{i}\gamma_1'$ and note that $\mathrm{d}(y_1 + \mathrm{i}y_2) = \mathrm{i}N(y)\,\mathrm{d}s(y)$. With $\varphi = (\varphi_1, \varphi_2)$ a computation leads to

$$\Psi_1\varphi(x) = \begin{pmatrix} \dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_\Sigma \frac{-\mathrm{i}\overline{N(y)}\varphi_2(y)}{(y_1+\mathrm{i}y_2)-(x_1+\mathrm{i}x_2)}\,\mathrm{d}(y_1+\mathrm{i}y_2) \\ \hline -\dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_\Sigma \frac{-\mathrm{i}\overline{N(y)\varphi_1(y)}}{(y_1+\mathrm{i}y_2)-(x_1+\mathrm{i}x_2)}\,\mathrm{d}(y_1+\mathrm{i}y_2) \end{pmatrix}.$$

Applying now (3.14) to each component of this vector we find that

$$\begin{aligned} \mathcal{T}_{\pm}^{D}\Psi_1\varphi(x) &= \begin{pmatrix} -\dfrac{1}{2\pi\mathrm{i}}\,\mathrm{p.v.}\displaystyle\int_\Sigma \frac{\varphi_2(y)}{(x_1+\mathrm{i}x_2)-(y_1+\mathrm{i}y_2)}\,\mathrm{d}s(y) \\ -\dfrac{1}{2\pi\mathrm{i}}\,\mathrm{p.v.}\displaystyle\int_\Sigma \frac{\varphi_1(y)}{(x_1-\mathrm{i}x_2)-(y_1-\mathrm{i}y_2)}\,\mathrm{d}s(y) \end{pmatrix} \mp \frac{\mathrm{i}}{2}\begin{pmatrix} \overline{N(x)}\varphi_2(x) \\ N(x)\varphi_1(x) \end{pmatrix} \\ &= P_1\varphi(x) \mp \frac{\mathrm{i}}{2}\,(\sigma\cdot\nu(x))\,\varphi(x). \end{aligned}$$

A combination of this and (3.15) leads to the claim of this proposition. □

3.2. A boundary triple for Dirac operators with singular interactions supported on a loop

We now follow the strategy from Section 2.2 to introduce a boundary triple which is suitable to study our main operator $A_{\eta,\tau}$. The construction will heavily use the results of Section 3.1. The final formulas are closely related to those of [9] for the three dimensional case.

Recall that the free Dirac operator A_0, its symmetric restriction S as well as the adjoint S^* were defined in (3.1), (3.3), and (3.4). Moreover, $\mathcal{T}_{\pm}^{D}$ is the Dirichlet trace operator defined on $\mathrm{dom}\, S^*$ from Lemma 3.1, the integral operators Φ_z and $\mathcal{C}_z$ are introduced for $z \in \mathrm{res}\, A_0$ in (3.7) and (3.10), respectively. The operator $\Lambda \in \Psi_\Sigma^{\frac{1}{2}}$ is given by (2.7) and will sometimes be viewed as an isomorphism from $L^2(\Sigma;\mathbb{C}^2)$ to $H^{-\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ or from $H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ to $L^2(\Sigma;\mathbb{C}^2)$, and is also regarded as an unbounded strictly positive self-adjoint operator in $L^2(\Sigma;\mathbb{C}^2)$.

Proposition 3.5. *Let $\zeta \in \mathrm{res}\, A_0$ be fixed. Define $\Gamma_0, \Gamma_1 : \mathrm{dom}\, S^* \to L^2(\Sigma;\mathbb{C}^2)$ by*

$$\begin{aligned} \Gamma_0 f &= \mathrm{i}\Lambda^{-1}(\sigma\cdot\nu)\big(\mathcal{T}_{+}^{D} f_+ - \mathcal{T}_{-}^{D} f_-\big), \\ \Gamma_1 f &= \frac{1}{2}\Lambda\Big((\mathcal{T}_{+}^{D} f_+ + \mathcal{T}_{-}^{D} f_-) - (\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big), \qquad f = f_+ \oplus f_- \in \mathrm{dom}\, S^*. \end{aligned} \tag{3.16}$$

Then $\{L^2(\Sigma;\mathbb{C}^2),\Gamma_0,\Gamma_1\}$ *is a boundary triple for* S^* *such that* $A_0 = S^* \upharpoonright \ker\Gamma_0$. *Moreover, the corresponding* γ*-field and Weyl function are*

$$\operatorname{res} A_0 \ni z \mapsto G_z = \Phi_z\Lambda \quad \textit{and} \quad \operatorname{res} A_0 \ni z \mapsto M_z = \Lambda\Big(\mathcal{C}_z - \frac{1}{2}\big(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\big)\Big)\Lambda.$$

Proof. Recall that the Dirichlet trace operator $\mathcal{T}^D : H^1(\mathbb{R}^2;\mathbb{C}^2) \to H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ is bounded and surjective with $\ker\mathcal{T}^D = H^1_0(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$. Hence,

$$\mathcal{T} := \Lambda\mathcal{T}^D : H^1(\mathbb{R}^2;\mathbb{C}^2) = \operatorname{dom} A_0 \to L^2(\Sigma;\mathbb{C}^2)$$

is bounded and surjective with $\ker\mathcal{T} = \operatorname{dom} S$. Following the constructions in Section 2.2 for $B = A_0$ we consider $\mathcal{T}(A_0-\bar z)^{-1} = \Lambda\mathcal{T}^D(A_0-\bar z)^{-1} = \Lambda\Phi'_z$ for $z \in \operatorname{res} A_0$ with Φ'_z given by (3.6), so that the operator G_z from (2.22) in the present context is given by

$$G_z = \Phi_z\Lambda. \tag{3.17}$$

Let $\zeta \in \operatorname{res} A_0$ be fixed. Then, by (2.23) any function $f \in \operatorname{dom} S^*$ can be written as $f = f_\zeta + G_\zeta\xi = f_{\bar\zeta} + G_{\bar\zeta}\xi$ for some $\xi \in L^2(\Sigma;\mathbb{C}^2)$ and $f_\zeta, f_{\bar\zeta} \in H^1(\mathbb{R}^2;\mathbb{C}^2)$, and according to Proposition 2.11

$$\Gamma_0 f = \xi \quad \text{and} \quad \Gamma_1 f = \frac{1}{2}\big(\mathcal{T}f_\zeta + \mathcal{T}f_{\bar\zeta}\big)$$

defines a boundary triple for S^* such that $A_0 = S^* \upharpoonright \ker\Gamma_0$.

Next we show that the above boundary maps coincide with the more explicit representations of Γ_0 and Γ_1 stated in the proposition. Let $f = f_\zeta + G_\zeta\xi = f_\zeta + \Phi_\zeta\Lambda\xi$ with $\xi \in L^2(\Sigma;\mathbb{C}^2)$ and $f_\zeta \in H^1(\mathbb{R}^2;\mathbb{C}^2)$ be fixed. Using that the jump of the trace of $f_\zeta \in H^1(\mathbb{R}^2;\mathbb{C}^2)$ at Σ is zero and the trace formula from Proposition 3.4 we find

$$\begin{aligned}\mathcal{T}^D_+ f_+ - \mathcal{T}^D_- f_- &= \mathcal{T}^D_+\big(\Phi_\zeta\Lambda\xi\big)_+ - \mathcal{T}^D_-\big(\Phi_\zeta\Lambda\xi\big)_- \\ &= -\frac{\mathrm{i}}{2}(\sigma\cdot\nu)\Lambda\xi + \mathcal{C}_\zeta\Lambda\xi - \frac{\mathrm{i}}{2}(\sigma\cdot\nu)\Lambda\xi - \mathcal{C}_\zeta\Lambda\xi = -\mathrm{i}(\sigma\cdot\nu)\Lambda\xi.\end{aligned}$$

Hence, $\Gamma_0 f = \xi = \mathrm{i}\Lambda^{-1}(\sigma\cdot\nu)(\mathcal{T}^D_+ f_+ - \mathcal{T}^D_- f_-)$, which is the claimed formula for $\Gamma_0 f$. Employing again Proposition 3.4 we find

$$\begin{aligned}\mathcal{T}^D f_\zeta &= \frac{1}{2}\big(\mathcal{T}^D_+(f-\Phi_\zeta\Lambda\xi)_+ + \mathcal{T}^D_-(f-\Phi_\zeta\Lambda\xi)_-\big) \\ &= \frac{1}{2}\Big(\mathcal{T}^D_+ f_+ - \mathcal{C}_\zeta\Lambda\xi + \frac{\mathrm{i}}{2}(\sigma\cdot\nu)\Lambda\xi + \mathcal{T}^D_- f_- - \mathcal{C}_\zeta\Lambda\xi - \frac{\mathrm{i}}{2}(\sigma\cdot\nu)\Lambda\xi\Big) \\ &= \frac{1}{2}\Big(\mathcal{T}^D_+ f_+ + \mathcal{T}^D_- f_-\Big) - \mathcal{C}_\zeta\Lambda\Gamma_0 f\end{aligned}$$

and analogously $\mathcal{T}^D f_{\bar\zeta} = \frac{1}{2}\big(\mathcal{T}^D_+ f_+ + \mathcal{T}^D_- f_-\big) - \mathcal{C}_{\bar\zeta}\Lambda\Gamma_0 f$. By summing up the last two formulae we find

$$\Gamma_1 f = \frac{1}{2}\big(\mathcal{T} f_\zeta + \mathcal{T} f_{\bar\zeta}\big) = \frac{1}{2}\Lambda\big(\mathcal{T}^D f_\zeta + \mathcal{T}^D f_{\bar\zeta}\big) = \frac{1}{2}\Lambda\Big((\mathcal{T}^D_+ f_+ + \mathcal{T}^D_- f_-) - (\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big),$$

which is the claimed formula for Γ_1 in (3.16).

Finally, the claimed representation of the γ-field follows from Proposition 2.11 and the equality (3.17). Using again Proposition 3.4, we can simplify the formula for the Weyl function M_z from Proposition 2.11 and get for $\varphi \in L^2(\Sigma;\mathbb{C}^2)$

$$\begin{aligned} M_z\varphi &= \Lambda \mathcal{T}^D_+ \left(\Phi_z - \frac{1}{2}(\Phi_\zeta + \Phi_{\bar\zeta})\right)\Lambda\varphi \\ &= \Lambda\Big(\mathcal{C}_z - \frac{\mathrm{i}}{2}(\sigma\cdot\nu) - \frac{1}{2}\Big(\mathcal{C}_\zeta - \frac{\mathrm{i}}{2}(\sigma\cdot\nu) + \mathcal{C}_{\bar\zeta} - \frac{\mathrm{i}}{2}(\sigma\cdot\nu)\Big)\Big)\Lambda\varphi \\ &= \Lambda\Big(\mathcal{C}_z - \frac{1}{2}\left(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\right)\Big)\Lambda\varphi. \end{aligned}$$

Remark that in the above computation we used the well-known regularization property $(G_z - \frac{1}{2}(G_\zeta + G_{\bar\zeta}))\varphi \in \operatorname{dom} A_0 = H^1(\mathbb{R}^2;\mathbb{C}^2)$, which holds automatically by the abstract theory (see the formula for the Weyl function in Proposition 2.11), and hence $\mathcal{T}^D$ and $\mathcal{T}^D_+$ lead to the same trace in the second equality above. Therefore, all claimed statements have been shown. $\square$

Finally, we state an auxiliary regularity result that will be used later.

Lemma 3.6. *Let $f \in \operatorname{dom} S^*$. Then $f \in H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$ if and only if $\Gamma_0 f \in H^1(\Sigma;\mathbb{C}^2)$.*

Proof. First, if $f = f_+ \oplus f_- \in H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$, then one has $\mathcal{T}^D_\pm f_\pm \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ implying $\mathcal{T}^D_+ f_+ - \mathcal{T}^D_- f_- \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. As $\sigma\cdot\nu$ is a C^∞-matrix function it follows that $\mathrm{i}(\sigma\cdot\nu)\big(\mathcal{T}^D_+ f_+ - \mathcal{T}^D_- f_-\big) \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. Using that Λ is a bijection from $H^s(\Sigma)$ to $H^{s-\frac{1}{2}}(\Sigma)$ for all $s\in\mathbb{R}$, this yields

$$\Gamma_0 f = \mathrm{i}\Lambda^{-1}(\sigma\cdot\nu)\big(\mathcal{T}^D_+ f_+ - \mathcal{T}^D_- f_-\big) \in H^1(\Sigma;\mathbb{C}^2).$$

Conversely, let $f = f_+\oplus f_- \in \operatorname{dom} S^*$ with $\Gamma_0 f \in H^1(\Sigma;\mathbb{C}^2)$. As $\Lambda : H^1(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ is bijective and the C^∞-matrix function $\sigma\cdot\nu$ is invertible we conclude from the definition of Γ_0 that

$$\mathcal{T}^D_+ f_+ - \mathcal{T}^D_- f_- \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2). \tag{3.18}$$

By Proposition 3.3 the operators $\mathcal{C}_\zeta$ and $\mathcal{C}_{\bar\zeta}$ are bounded in $H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$, which gives $(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. In addition, $\Gamma_1 f \in L^2(\Sigma;\mathbb{C}^2)$ implies $\Lambda^{-1}\Gamma_1 \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. With the definition of Γ_1 this yields

$$\frac{1}{2}\big(\mathcal{T}^D_+ f_+ + \mathcal{T}^D_- f_-\big) = \Lambda^{-1}\Gamma_1 f + \frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2).$$

Hence, together with (3.18) this implies $\mathcal{T}_\pm^D f_\pm \in H^{\frac{1}{2}}(\Sigma; \mathbb{C}^2)$. Finally, Lemma 3.1 shows $f_\pm \in H^1(\Omega_\pm; \mathbb{C}^2)$. $\square$

3.3. Some basic properties of the self-adjoint extensions

In this subsection we prove two results which are valid for the essential and discrete spectra of a large class of self-adjoint extensions of S defined in (3.3) and which are independent of the preceding construction of a boundary triple. These properties will be used later for a more detailed spectral analysis of $A_{\eta,\tau}$.

For the essential spectrum we have the following result, which can be proved using a singular Weyl sequence constructed in a similar way as in [9, Theorem 5.7 (i)]:

Proposition 3.7. *For any self-adjoint extension A of S one has the inclusion*

$$\big(-\infty, -|m| \big] \cup \big[|m|, +\infty \big) \subset \operatorname{spec}_{\mathrm{ess}} A.$$

Some information about the discrete spectrum can be obtained under an additional regularity assumption:

Proposition 3.8. *Let A be a self-adjoint extension of the symmetric operator S in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ satisfying the inclusion $\operatorname{dom} A \subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ for some $s > 0$. Then the spectrum of A in $(-|m|, |m|)$ is purely discrete and finite.*

Proof. It is sufficient to show that A^2 has at most finitely many eigenvalues in $(-\infty, m^2)$. For that, consider the quadratic form

$$a[f, f] = \int_{\mathbb{R}^2} |Af|^2 \,\mathrm{d}x, \quad \operatorname{dom} a = \operatorname{dom} A.$$

Since A is self-adjoint and hence closed, also the densely defined nonnegative form a is closed. The self-adjoint operator associated to a via the first representation theorem is A^2. Next, take $0 < r < R$ with r chosen sufficiently large, such that the open ball $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$ contains $\overline{\Omega_+}$ in its interior, and choose $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^2)$ which satisfy

$$0 \le \varphi_1, \varphi_2 \le 1, \quad \varphi_1^2 + \varphi_2^2 = 1, \quad \varphi_1 = 1 \text{ in } B_r, \quad \varphi_2 = 1 \text{ in } \mathbb{R}^2 \setminus B_R.$$

Let $f \in \operatorname{dom} A$ be fixed. Then one has $\varphi_j f \in \operatorname{dom} A$ and $A(\varphi_j f) = \varphi_j A f - \mathrm{i}\sigma \cdot (\nabla \varphi_j) f$. In particular, we note that $\varphi_2 f \in H(\sigma, \Omega_-)$ with $\mathcal{T}_-^D f = 0 \in H^{\frac{1}{2}}(\Sigma; \mathbb{C}^2)$. Thus, it follows from Lemma 3.1 that $\varphi_2 f \in H^1(\Omega_-; \mathbb{C}^2)$.

Next, we remark that $\nabla \varphi_j$ is supported in $\overline{B_R} \setminus B_r$. Hence, we have for $j \in \{1, 2\}$

$$a[\varphi_j f, \varphi_j f] = \int_{\mathbb{R}^2} \big(\varphi_j^2 |Af|^2 + |\mathrm{i}\sigma \cdot (\nabla \varphi_j) f|^2 \big) \,\mathrm{d}x + \mathcal{I}_j,$$

$$\mathcal{I}_j := \int_{B_R \setminus B_r} 2\,\mathrm{Re}\left(\varphi_j(-\mathrm{i}\sigma\cdot\nabla + m\sigma_3)f, -\mathrm{i}\sigma\cdot(\nabla\varphi_j)f\right)_{\mathbb{C}^2} \mathrm{d}x$$
$$= \int_{B_R \setminus B_r} \mathrm{Re}\left((-\mathrm{i}\sigma\cdot\nabla + m\sigma_3)f, -\mathrm{i}\sigma\cdot\nabla(\varphi_j^2)f\right)_{\mathbb{C}^2} \mathrm{d}x.$$

From $\varphi_1^2 + \varphi_2^2 = 1$ we obtain $\nabla(\varphi_1^2) = -\nabla(\varphi_2^2)$ and hence $\mathcal{I}_1 = -\mathcal{I}_2$. Moreover, using (1.4) one verifies $|\mathrm{i}\sigma\cdot(\nabla\varphi_j)f|^2 = |\nabla\varphi_j|^2|f|^2$ for $j \in \{1,2\}$. Therefore, it follows that

$$a[\varphi_1 f, \varphi_1 f] + a[\varphi_2 f, \varphi_2 f] = \int_{\mathbb{R}^2} (\varphi_1^2 + \varphi_2^2)|Af|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} \left(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2)|f|^2\right)\mathrm{d}x$$
$$= \int_{\mathbb{R}^2} |Af|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} V|f|^2\,\mathrm{d}x,$$

where we have used the abbreviation $V := |\nabla\varphi_1|^2 + |\nabla\varphi_2|^2$ in the last step; note that V is supported in $\overline{B_R} \setminus B_r$. This leads to

$$a[f,f] = a[\varphi_1 f, \varphi_1 f] - \int_{\mathbb{R}^2} V|\varphi_1 f|^2\,\mathrm{d}x + a[\varphi_2 f, \varphi_2 f] - \int_{\mathbb{R}^2} V|\varphi_2 f|^2\,\mathrm{d}x. \tag{3.19}$$

In the following we will often restrict functions in $\mathrm{dom}\, a$ to B_R or $\mathbb{R}^2 \setminus \overline{B_r}$ and view them as elements in $L^2(B_R;\mathbb{C}^2)$ or $L^2(\mathbb{R}^2 \setminus \overline{B_r};\mathbb{C}^2)$, or we will extend L^2-functions on B_R or $\mathbb{R}^2 \setminus \overline{B_r}$ by zero onto $\mathbb{R}^2$ and view them as elements in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. We find it convenient to use the same letter for the original and the restricted or extended function.

Let a_1 be the quadratic form in $L^2(B_R;\mathbb{C}^2)$ defined by

$$\mathrm{dom}\, a_1 = \left\{g \in \mathrm{dom}\, a : \mathrm{supp}\, g \subset \overline{B_R}\right\}, \quad a_1[g,g] = a[g,g] - \int_{B_R} V|g|^2\,\mathrm{d}x.$$

As V is bounded and a is nonnegative it follows that a_1 is semibounded from below. It is also clear that a_1 is densely defined in $L^2(B_R;\mathbb{C}^2)$. To see that a_1 is closed consider $g_n \in \mathrm{dom}\, a_1$ such that $g_n \to g$ in $L^2(B_R;\mathbb{C}^2)$ for $n \to \infty$ and $a_1(g_n - g_m, g_n - g_m) \to 0$ for $n, m \to \infty$. Since V is bounded it follows that the zero extensions of g_n and g satisfy $g_n \to g$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ for $n \to \infty$ and $a(g_n - g_m, g_n - g_m) \to 0$ for $n, m \to \infty$. As a is closed we conclude $g \in \mathrm{dom}\, a$ and $a(g_n - g, g_n - g) \to 0$ for $n \to \infty$. Furthermore, as $\mathrm{supp}\, g \subset \overline{B_R}$ we have $g \in \mathrm{dom}\, a_1$ and $a_1(g_n - g, g_n - g) \to 0$ for $n \to \infty$, thus a_1 is closed. Let A_1 be the self-adjoint operator in $L^2(B_R;\mathbb{C}^2)$ corresponding to a_1. Then A_1 has a compact resolvent since the form domain $\mathrm{dom}\, a_1 \subset H^s(B_R \setminus \Sigma;\mathbb{C}^2)$ is compactly embedded in $L^2(B_R;\mathbb{C}^2)$ for $s > 0$. Hence, the number of eigenvalues $\mathcal{N}(A_1, m^2)$ of A_1 below m^2 is finite, that is, $\mathcal{N}(A_1, m^2) < \infty$.

Next, let a_2 be the quadratic form in $L^2(\mathbb{R}^2 \setminus \overline{B_r}; \mathbb{C}^2)$ defined by

$$\operatorname{dom} a_2 = H_0^1\big(\mathbb{R}^2 \setminus \overline{B_r}; \mathbb{C}^2\big), \quad a_2[g,g] = a[g,g] - \int\limits_{\mathbb{R}^2 \setminus \overline{B_r}} V|g|^2 \,\mathrm{d}x.$$

As above it is clear that a_2 is densely defined and semibounded from below. Using integration by parts and (1.4) one sees for $g \in C_0^\infty(\mathbb{R}^2 \setminus \overline{B_r}; \mathbb{C}^2)$ that

$$\begin{aligned} a[g,g] &= \int\limits_{\mathbb{R}^2 \setminus \overline{B_r}} \big(g, (-\mathrm{i}\sigma \cdot \nabla + m\sigma_3)^2 g\big)_{\mathbb{C}^2} \,\mathrm{d}x \\ &= \int\limits_{\mathbb{R}^2 \setminus \overline{B_r}} \big(g, (-\Delta + m^2) g\big)_{\mathbb{C}^2} \,\mathrm{d}x = \int\limits_{\mathbb{R}^2 \setminus \overline{B_r}} \big(|\nabla g|^2 + m^2 |g|^2\big) \,\mathrm{d}x, \end{aligned}$$

which then extends by density to all $g \in H_0^1\big(\mathbb{R}^2 \setminus \overline{B_r}; \mathbb{C}^2\big)$. Therefore, the form a_2 is closed and the self-adjoint operator associated to a_2 is $A_2 = -\Delta^D + m^2 - V$, where $-\Delta^D$ denotes the Dirichlet Laplacian in $\mathbb{R}^2 \setminus \overline{B_r}$.

Let us prove that $\mathcal{N}(A_2, m^2) < \infty$. Recall that V is bounded and that its support is contained in $\overline{B_R}$. Consider the following closed sesquilinear forms a_3 in $L^2(B_R \setminus \overline{B_r})$ and a_4 in $L^2(\mathbb{R}^2 \setminus \overline{B_R})$,

$$\begin{aligned} a_3[g,g] &= \int\limits_{B_R \setminus \overline{B_r}} \big(|\nabla g|^2 + (m^2 - V)|g|^2\big) \,\mathrm{d}x, \\ \operatorname{dom} a_3 &= \big\{ g \in H^1(B_R \setminus \overline{B_r}; \mathbb{C}^2) : \, g = 0 \text{ on } \partial B_r \big\}, \\ a_4[g,g] &= \int\limits_{\mathbb{R}^2 \setminus \overline{B_R}} \big(|\nabla g|^2 + m^2 |g|^2\big) \,\mathrm{d}x, \qquad \operatorname{dom} a_4 = H^1(\mathbb{R}^2 \setminus \overline{B_R}). \end{aligned}$$

For $g \in \operatorname{dom} a_2$ one has $f_3 := g \restriction B_R \setminus \overline{B_r} \in \operatorname{dom} a_3$, $f_4 := g \restriction \mathbb{R}^2 \setminus \overline{B_R} \in \operatorname{dom} a_4$, and $a_2(g,g) = a_3(f_3, f_3) + a_4(f_4, f_4)$. Therefore, if the self-adjoint operator in $L^2(B_R \setminus \overline{B_r})$ generated by a_3 is denoted by A_3 and A_4 is the self-adjoint operator in $L^2(\mathbb{R}^2 \setminus \overline{B_R})$ generated by a_4, then it follows by the min-max principle that the eigenvalues of a_2 are bounded from below by the respective eigenvalues of $A_3 \oplus A_4$. In particular, this implies $\mathcal{N}(A_2, m^2) \le \mathcal{N}(A_3, m^2) + \mathcal{N}(A_4, m^2)$. One clearly has $\mathcal{N}(A_4, m^2) = 0$. On the other hand, the operator A_3 is semibounded from below and has a compact resolvent, hence, $\mathcal{N}(A_3, m^2) < \infty$. This implies $\mathcal{N}(A_2, m^2) < \infty$.

Now we consider $J : L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(B_R; \mathbb{C}^2) \oplus L^2(\mathbb{R}^2 \setminus \overline{B_r}; \mathbb{C}^2)$, $Jf = \varphi_1 f \oplus \varphi_2 f$. Due to the properties of φ_1 and φ_2 we get that J is an isometry. The above considerations show that $J(\operatorname{dom} a) \subset \operatorname{dom} a_1 \oplus \operatorname{dom} a_2$, and with the equality (3.19) we obtain

$$\frac{a[f,f]}{\|f\|^2_{L^2(\mathbb{R}^2; \mathbb{C}^2)}} = \frac{(a_1 \oplus a_2)[Jf, Jf]}{\|Jf\|^2_{L^2(B_R; \mathbb{C}^2) \oplus L^2(\mathbb{R}^2 \setminus \overline{B_r}; \mathbb{C}^2)}}.$$

It follows from the min-max principle that

$$\mathcal{N}(A^2, m^2) \le \mathcal{N}(A_1 \oplus A_2, m^2) = \mathcal{N}(A_1, m^2) + \mathcal{N}(A_2, m^2).$$

As we have seen above, the quantity on the right hand side is finite and hence $\mathcal{N}(A^2, m^2) < \infty$. This completes the proof. □

4. Dirac operators with singular interactions

In this section we study the Dirac operator $A_{\eta,\tau}$ introduced in (1.2) and we prove the main results of this paper. First, in Section 4.1 we show how $A_{\eta,\tau}$ is related to the boundary triple $\{L^2(\Sigma;\mathbb{C}^2), \Gamma_0, \Gamma_1\}$ from Proposition 3.5. Then, in Section 4.2, we verify the self-adjointness of $A_{\eta,\tau}$ for non-critical interaction strengths, i.e. when $\eta^2 - \tau^2 \neq 4$, and investigate the spectral properties of $A_{\eta,\tau}$ in this setting. In Section 4.3 we study the self-adjointness and the spectral properties of $A_{\eta,\tau}$ in the case of critical interaction strengths. Finally, in Section 4.4 we provide a sketch of the proof of Theorem 1.3.

4.1. Definition of $A_{\eta,\tau}$ via the boundary triple

Recall the definition of the space $H(\sigma, \Omega_\pm)$ from (3.5), the trace maps $\mathcal{T}_\pm^D$ on $H(\sigma, \Omega_\pm)$ in Lemma 3.1, and that the operator $A_{\eta,\tau}$ in (1.2) is defined by

$$\begin{aligned} A_{\eta,\tau} f &= (-\mathrm{i}\sigma \cdot \nabla + m\sigma_3) f_+ \oplus (-\mathrm{i}\sigma \cdot \nabla + m\sigma_3) f_-, \\ \operatorname{dom} A_{\eta,\tau} &= \Big\{ f = f_+ \oplus f_- \in H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-) : \\ &\qquad - \mathrm{i}(\sigma \cdot \nu)\big(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-\big) = \frac{1}{2}(\eta\sigma_0 + \tau\sigma_3)\big(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\big) \Big\}. \end{aligned} \tag{4.1}$$

Before analyzing the properties of $A_{\eta,\tau}$ we would like to mention that for special values of the interaction strengths $A_{\eta,\tau}$ decouples to Dirac operators in $L^2(\Omega_+;\mathbb{C}^2)$ and $L^2(\Omega_-;\mathbb{C}^2)$ subject to certain boundary conditions. Similar effects are known in dimension three, see [19, Section V], [4, Section 5], and [7, Lemma 3.1]. The result reads as follows:

Lemma 4.1. *Let $\eta, \tau \in \mathbb{R}$. Then the following holds:*

(i) *If $\eta^2 - \tau^2 \neq -4$, then there is an invertible matrix M, which is explicitly given below in (4.4), such that $f = f_+ \oplus f_- \in \operatorname{dom} A_{\eta,\tau}$ if and only if $\mathcal{T}_+^D f_+ = M \mathcal{T}_-^D f_-$.*
(ii) *If $\eta^2 - \tau^2 = -4$, then $A_{\eta,\tau} = A_+ \oplus A_-$, where $A_\pm$ is a Dirac operator in $L^2(\Omega_\pm;\mathbb{C}^2)$ and $f_\pm \in \operatorname{dom} A_\pm$ if and only if*

$$\mathcal{T}_\pm^D f_\pm = \pm \frac{\mathrm{i}}{2} (\sigma \cdot \nu)\, (\eta\sigma_0 + \tau\sigma_3)\, \mathcal{T}_\pm^D f_\pm. \tag{4.2}$$

Remark 4.2. Assume that $\eta^2 - \tau^2 = -4$, which is equivalent to $\frac{\eta^2}{\tau^2} + \frac{4}{\tau^2} = 1$. Thus, there exists $\vartheta \in [0, 2\pi] \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ such that $\eta/\tau = -\sin\vartheta$ and $2/\tau = \cos\vartheta$. Using (1.4) we see that (4.2) for f_+ is equivalent to

$$0 = \frac{2i}{\tau}\sigma_3(\sigma\cdot\nu)\left(\sigma_0 - \frac{\mathrm{i}}{2}(\sigma\cdot\nu)\,(\eta\sigma_0 + \tau\sigma_3)\right)\mathcal{T}_+^D f_+ = \big(\sigma_0 + \mathrm{i}\sigma_3(\sigma\cdot\nu)\cos\vartheta - \sin\vartheta\sigma_3\big)\mathcal{T}_+^D f_+,$$

i.e. the operators A_+ in the bounded domain Ω_+ are exactly those investigated in [11]. The case $\vartheta = 0$ corresponds to the well-known infinite mass boundary condition, which is the two dimensional analog of the MIT bag boundary condition, studied in [2,26,36]. We would like to point out that our results on $A_{\eta,\tau}$ obtained later in Section 4.2 can be used for a deeper understanding for $A_\pm$.

Proof of Lemma 4.1. The transmission condition in the definition of $A_{\eta,\tau}$ takes the form

$$\left(\mathrm{i}(\sigma\cdot\nu) + \frac{1}{2}(\eta\sigma_0 + \tau\sigma_3)\right)\mathcal{T}_+^D f_+ = \left(\mathrm{i}(\sigma\cdot\nu) - \frac{1}{2}(\eta\sigma_0 + \tau\sigma_3)\right)\mathcal{T}_-^D f_-.$$

Multiplying this equation with $-\mathrm{i}\,(\sigma\cdot\nu)$ we obtain the equivalent form

$$(\sigma_0 - R)\,\mathcal{T}_+^D f_+ = (\sigma_0 + R)\,\mathcal{T}_-^D f_-, \quad R := \frac{\mathrm{i}}{2}\,(\sigma\cdot\nu)(\eta\sigma_0 + \tau\sigma_3) = \frac{\mathrm{i}}{2}\,(\eta\sigma_0 - \tau\sigma_3)(\sigma\cdot\nu), \tag{4.3}$$

where (1.4) was used. One computes

$$R^2 = \frac{\mathrm{i}}{2}\,(\eta\sigma_0 - \tau\sigma_3)(\sigma\cdot\nu)\,\frac{\mathrm{i}}{2}\,(\sigma\cdot\nu)(\eta\sigma_0 + \tau\sigma_3) = -\frac{\eta^2 - \tau^2}{4}\,\sigma_0,$$

which implies $(\sigma_0 - R)(\sigma_0 + R) = \sigma_0 - R^2 = \sigma_0 + \frac{\eta^2 - \tau^2}{4}\,\sigma_0$. Assume now $\eta^2 - \tau^2 \neq -4$. Then both $\sigma_0 \pm R$ are invertible with $(\sigma_0 \pm R)^{-1} = \frac{4}{4 + \eta^2 - \tau^2}\,(\sigma_0 \mp R)$. Therefore, the transmission condition can be equivalently rewritten as

$$\mathcal{T}_+^D f_+ = (\sigma_0 - R)^{-1}(\sigma_0 + R)\,\mathcal{T}_-^D f_- \quad \text{or} \quad \mathcal{T}_-^D f_- = (\sigma_0 + R)^{-1}(\sigma_0 - R)\,\mathcal{T}_+^D f_+, \tag{4.4}$$

which shows assertion (i). On the other hand, for $\eta^2 - \tau^2 = -4$ one has $R^2 = \sigma_0$ and multiplying (4.3) by $\sigma_0 - R$ or $\sigma_0 + R$ leads to the two conditions $\mathcal{T}_\pm^D f_\pm = \pm R\mathcal{T}_\pm^D f_\pm$. It follows that the operator $A_{\eta,\tau}$ decouples in an orthogonal sum of operators $A_\pm$ acting in $\Omega_\pm$ and hence, also statement (ii) has been shown. □

Let us represent $A_{\eta,\tau}$ using the boundary triple $\{L^2(\Sigma;\mathbb{C}^2), \Gamma_0, \Gamma_1\}$ constructed in Proposition 3.5. Note that the definition of Γ_0 and Γ_1 can be rewritten as

$$\mathrm{i}(\sigma\cdot\nu)\left(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-\right) = \Lambda\Gamma_0 f, \tag{4.5}$$

$$\frac{1}{2}\left(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\right) = \Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f. \tag{4.6}$$

Proposition 4.3. *Let $\eta, \tau \in \mathbb{R}$. Then the following holds:*

(i) *Assume $|\eta| \neq |\tau|$. Let Θ be the linear operator in $L^2(\Sigma;\mathbb{C}^2)$ obtained as the maximal realization of the periodic pseudodifferential operator $\theta \in \Psi^1_\Sigma$ given by*

$$\theta = -\Lambda\left[\frac{1}{\eta^2-\tau^2}(\eta\sigma_0 - \tau\sigma_3) + \frac{1}{2}\left(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\right)\right]\Lambda, \tag{4.7}$$

i.e. $\operatorname{dom}\Theta = \{\varphi \in L^2(\Sigma;\mathbb{C}^2) : \theta\varphi \in L^2(\Sigma;\mathbb{C}^2)\}$ *and $\Theta\varphi = \theta\varphi$. Then*

$$\operatorname{dom} A_{\eta,\tau} = \{f \in \operatorname{dom} S^* : \Gamma_0 f \in \operatorname{dom}\Theta,\ \Gamma_1 f = \Theta\Gamma_0 f\}. \tag{4.8}$$

(ii) *Assume $\eta = \tau \neq 0$, let $\Pi_+ : L^2(\Sigma;\mathbb{C}^2) \ni (\varphi_1,\varphi_2) \mapsto \varphi_1 \in L^2(\Sigma)$ and let Θ_+ be the linear operator in $L^2(\Sigma)$ obtained as the maximal realization of the periodic pseudodifferential operator $\theta_+ \in \Psi^1_\Sigma$ given by*

$$\theta_+ = -\Lambda\left(\frac{1}{2\eta} + \frac{1}{2}\Pi_+(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Pi_+^*\right)\Lambda, \tag{4.9}$$

i.e. $\operatorname{dom}\Theta_+ = \{\varphi \in L^2(\Sigma) : \theta_+\varphi \in L^2(\Sigma)\}$ *and $\Theta_+\varphi = \theta_+\varphi$. Then*

$$\operatorname{dom} A_{\eta,\tau} = \Big\{f \in \operatorname{dom} S^* : \Pi_+\Gamma_1 f = \Theta_+\Pi_+\Gamma_0 f,\ (\sigma_0 - \Pi_+^*\Pi_+)\Gamma_0 f = 0\Big\}. \tag{4.10}$$

(iii) *Assume $\eta = -\tau \neq 0$, let $\Pi_- : L^2(\Sigma;\mathbb{C}^2) \ni (\varphi_1,\varphi_2) \mapsto \varphi_2 \in L^2(\Sigma)$ and let Θ_- be the linear operator in $L^2(\Sigma)$ obtained as the maximal realization of the periodic pseudodifferential operator $\theta_- \in \Psi^1_\Sigma$ given by*

$$\theta_- = -\Lambda\left(\frac{1}{2\eta} + \frac{1}{2}\Pi_-(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Pi_-^*\right)\Lambda, \tag{4.11}$$

i.e. $\operatorname{dom}\Theta_- = \{\varphi \in L^2(\Sigma) : \theta_-\varphi \in L^2(\Sigma)\}$ *and $\Theta_-\varphi = \theta_-\varphi$. Then*

$$\operatorname{dom} A_{\eta,\tau} = \Big\{f \in \operatorname{dom} S^* : \Pi_-\Gamma_1 f = \Theta_-\Pi_-\Gamma_0 f,\ (\sigma_0 - \Pi_-^*\Pi_-)\Gamma_0 f = 0\Big\}. \tag{4.12}$$

Note that the case $\eta = \tau = 0$ is not discussed in the previous statement because $A_{\eta,\tau}$ simply becomes the free Dirac operator A_0 introduced in (3.1).

Remark 4.4.

(i) The operators Θ and $\Theta_\pm$ in Proposition 4.3 are well-defined due to the fact that θ and $\theta_\pm$ are periodic pseudodifferential operators of order 1. For example $\theta\varphi$ makes sense as an element of $H^{-1}(\Sigma;\mathbb{C}^2)$ for any $\varphi \in L^2(\Sigma;\mathbb{C}^2)$, and $H^1(\Sigma;\mathbb{C}^2) \subset \operatorname{dom}\Theta$.

(ii) In items (ii) and (iii) of Proposition 4.3 we decomposed $\mathcal{G} = L^2(\Sigma;\mathbb{C}^2) = \mathcal{G}_{\Pi_+} \oplus \mathcal{G}_{\Pi_-}$,

$$\mathcal{G}_{\Pi_+} := \big\{\varphi = (\varphi_1, \varphi_2) \in L^2(\Sigma;\mathbb{C}^2) : \varphi_2 = 0\big\} \simeq L^2(\Sigma),$$
$$\mathcal{G}_{\Pi_-} := \big\{\varphi = (\varphi_1, \varphi_2) \in L^2(\Sigma;\mathbb{C}^2) : \varphi_1 = 0\big\} \simeq L^2(\Sigma).$$

Proof. With the help of (4.5) and (4.6) the transmission condition in (4.1) is

$$-\Lambda\Gamma_0 f = (\eta\sigma_0 + \tau\sigma_3)\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Lambda\Gamma_0 f\Big). \tag{4.13}$$

Now let us distinguish between several cases.

(i) For $|\eta| \neq |\tau|$ the matrix $\eta\sigma_0 + \tau\sigma_3$ is invertible with

$$(\eta\sigma_0 + \tau\sigma_3)^{-1} = \frac{1}{\eta^2 - \tau^2}(\eta\sigma_0 - \tau\sigma_3).$$

Hence, we can rewrite the equality (4.13) as

$$\Gamma_1 f = -\Lambda\left[\frac{1}{\eta^2 - \tau^2}(\eta\sigma_0 - \tau\sigma_3) + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\right]\Lambda\Gamma_0 f = \Theta\Gamma_0 f,$$

which gives the claimed representation in (4.8)

The cases (ii) are and (iii) are almost identical, so we only give a proof for (ii). By (4.13) we have that $f \in \operatorname{dom} A_{\eta,\tau}$ if and only if

$$\begin{aligned}
-\Lambda\Gamma_0 f &= (\eta\sigma_0 + \tau\sigma_3)\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Lambda\Gamma_0 f\Big)\\
&= \begin{pmatrix} 2\eta & 0 \\ 0 & 0 \end{pmatrix}\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Lambda\Gamma_0 f\Big)\\
&= 2\eta\Pi_+^*\Pi_+\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Lambda\Gamma_0 f\Big).
\end{aligned}$$

Writing this equation in components it follows that this boundary condition is equivalent to the conditions $(\sigma_0 - \Pi_+^*\Pi_+)\Gamma_0 f = 0$ and

$$\begin{aligned}
\Pi_+\Gamma_1 f &= -\Lambda\Big(\frac{1}{2\eta} + \frac{1}{2}\,\Pi_+(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Big)\Lambda\Gamma_0 f\\
&= -\Lambda\Big(\frac{1}{2\eta} + \frac{1}{2}\,\Pi_+(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Pi_+^*\Big)\Lambda\Pi_+\Gamma_0 f = \Theta_+\Pi_+\Gamma_0 f.
\end{aligned}$$

Hence, we find that (4.10) is true. □

In view of the general theory of boundary triples, see Subsection 2.2, many properties of $A_{\eta,\tau}$ can be deduced from the respective properties of the operators Θ and $\Theta_\pm$ from Proposition 4.3. We prefer to consider separately the non-critical case $\eta^2 - \tau^2 \neq 4$ and the critical case $\eta^2 - \tau^2 = 4$, where the latter one is more involved.

4.2. Non-critical case

Throughout this subsection we assume that

$$\eta^2 - \tau^2 \neq 4.$$

In order to show the self-adjointness of $A_{\eta,\tau}$ we use Theorem 2.10. For that it is necessary to investigate the operators Θ and $\Theta_\pm$ in Proposition 4.3.

Lemma 4.5. *Let $\eta, \tau \in \mathbb{R}$ with $\eta^2 - \tau^2 \neq 4$. Then the following holds:*

(i) *If $\eta^2 - \tau^2 \neq 0$, then $\operatorname{dom} \Theta = H^1(\Sigma; \mathbb{C}^2)$ and Θ is self-adjoint in $L^2(\Sigma; \mathbb{C}^2)$.*
(ii) *If $\eta = \pm\tau$, then $\operatorname{dom} \Theta_\pm = H^1(\Sigma)$ and $\Theta_\pm$ is self-adjoint in $L^2(\Sigma)$.*

Proof. (i) Let us consider the restriction $\Theta_1 := \Theta \upharpoonright H^1(\Sigma; \mathbb{C}^2)$. Since $\theta \in \Psi^1_\Sigma$, the operator Θ_1 is well-defined as an operator in $L^2(\Sigma; \mathbb{C}^2)$. We show $\Theta = \Theta_1$ and that Θ_1 is self-adjoint in $L^2(\Sigma; \mathbb{C}^2)$.

First, it follows from Proposition 3.3 that $(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})^* = \mathcal{C}_{\bar{\zeta}} + \mathcal{C}_\zeta$ and hence Θ_1 is a symmetric operator in $L^2(\Sigma; \mathbb{C}^2)$. Moreover, since Θ_1 is a symmetric extension of the symmetric operator $\Theta_\infty := \Theta \upharpoonright C^\infty(\Sigma; \mathbb{C}^2)$, Lemma 2.4 implies $\Theta_1^* \subset \Theta_\infty^* = \Theta$. Hence, $\Theta = \Theta_1$ and $\Theta_1 = \Theta_1^*$ follows if we show $\Theta \subset \Theta_1$, for which it suffices to check the inclusion

$$\operatorname{dom} \Theta \subset \operatorname{dom} \Theta_1 = H^1(\Sigma; \mathbb{C}^2). \tag{4.14}$$

To see (4.14) fix some $\varphi \in \operatorname{dom} \Theta$. Then $\theta\varphi \in L^2(\Sigma; \mathbb{C}^2)$. Using Proposition 3.3 we find

$$\theta\varphi = -\frac{1}{2}\Lambda P \Lambda \varphi + \widehat{\Psi}\varphi, \quad \text{where } P = \begin{pmatrix} \frac{2}{\eta+\tau} & C_\Sigma \overline{T} \\ T C'_\Sigma & \frac{2}{\eta-\tau} \end{pmatrix} \text{ and } \widehat{\Psi} \in \Psi^0_\Sigma.$$

Hence, $\Lambda P \Lambda \varphi \in L^2(\Sigma; \mathbb{C}^2)$ and as $\Lambda : H^{\frac{1}{2}}(\Sigma; \mathbb{C}^2) \to L^2(\Sigma; \mathbb{C}^2)$ is bijective, this amounts to $P\Lambda\varphi \in H^{\frac{1}{2}}(\Sigma; \mathbb{C}^2)$. Since $C_\Sigma, C'_\Sigma \in \Psi^0_\Sigma$ by Proposition 2.8, these operators give rise to bounded operators in $H^{\frac{1}{2}}(\Sigma; \mathbb{C}^2)$, which implies that

$$\begin{aligned} &\begin{pmatrix} \frac{2}{\eta-\tau} & -C_\Sigma \overline{T} \\ -T C'_\Sigma & \frac{2}{\eta+\tau} \end{pmatrix} \begin{pmatrix} \frac{2}{\eta+\tau} & C_\Sigma \overline{T} \\ T C'_\Sigma & \frac{2}{\eta-\tau} \end{pmatrix} \Lambda\varphi \\ &\qquad = \begin{pmatrix} \frac{4}{\eta^2-\tau^2} - C_\Sigma \overline{T} T C'_\Sigma & 0 \\ 0 & \frac{4}{\eta^2-\tau^2} - T C'_\Sigma C_\Sigma \overline{T} \end{pmatrix} \Lambda\varphi \in H^{\frac{1}{2}}(\Sigma; \mathbb{C}^2). \end{aligned}$$

Now we use that $\overline{T}T = T\overline{T}$ is the multiplication operator with the constant function 1 and that $C_\Sigma C'_\Sigma - \mathbb{1}, C'_\Sigma C_\Sigma - \mathbb{1} \in \Psi^{-\infty}_\Sigma$ by Proposition 2.8. We then obtain from the last line that $\frac{4-\eta^2+\tau^2}{\eta^2-\tau^2}\Lambda\varphi + \widetilde{\Psi}\varphi \in H^{\frac{1}{2}}(\Sigma; \mathbb{C}^2)$ with some $\widetilde{\Psi} \in \Psi^{-\infty}_\Sigma$ and hence we get

$\frac{4-\eta^2+\tau^2}{\eta^2-\tau^2}\Lambda\varphi \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. Since $\eta^2-\tau^2 \neq 4$ by assumption, this implies $\Lambda\varphi \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ and thus, $\varphi \in H^1(\Sigma;\mathbb{C}^2)$. We have shown (4.14). This completes the proof of (i).

(ii) We consider the case $\eta = \tau$, the other one being similar. Recall that Θ_+ is the maximal operator in $L^2(\Sigma)$ associated to the periodic pseudodifferential operator

$$\theta_+ = -\frac{1}{2}\Lambda\left(\frac{1}{\eta} + \Pi_+(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Pi_+^*\right)\Lambda.$$

Using Proposition 3.3 we find for $\varphi \in \operatorname{dom}\Theta_+$ that

$$\Theta_+\varphi = -\frac{1}{2\eta}\Lambda^2\varphi - \frac{1}{2}\Pi_+\begin{pmatrix} 0 & \Lambda C_\Sigma \overline{T}\Lambda \\ \Lambda T C_\Sigma' \Lambda & 0 \end{pmatrix}\Pi_+^*\varphi + \widehat{\Psi}\varphi = -\frac{1}{2\eta}\Lambda^2\varphi + \widehat{\Psi}\varphi$$

with some symmetric operator $\widehat{\Psi} \in \Psi_\Sigma^0$. This implies $\operatorname{dom}\Theta_+ = \operatorname{dom}\Lambda^2 = H^1(\Sigma;\mathbb{C})$ and since Λ^2 is self-adjoint we conclude that also Θ_+ is self-adjoint in $L^2(\Sigma)$. □

After the preparatory considerations of Lemma 4.5 we are now ready to show the self-adjointness of $A_{\eta,\tau}$ for non-critical interaction strengths. To formulate the result we recall the definitions of the free Dirac operator A_0 from (3.1), of Φ_z and Φ_z' from (3.7) and (3.6), and of $\mathcal{C}_z$ in (3.10), respectively.

Theorem 4.6. *Assume that $\eta, \tau \in \mathbb{R}$ with $\eta^2 - \tau^2 \neq 4$ and $(\eta,\tau) \neq (0,0)$. Then the operator $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with $\operatorname{dom} A_{\eta,\tau} \subset H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$. Moreover, for all $z \in \operatorname{res} A_{\eta,\tau} \cap \operatorname{res} A_0$ the operator $\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z$ is bounded and boundedly invertible in $H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ and*

$$(A_{\eta,\tau} - z)^{-1} = (A_0 - z)^{-1} - \Phi_z\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Phi_{\bar{z}}' \tag{4.15}$$

holds.

Proof. First, by Theorem 2.10 the self-adjointness of Θ and $\Theta_\pm$ in $L^2(\Sigma;\mathbb{C}^2)$ and $L^2(\Sigma)$, respectively, implies the self-adjointness of $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. In addition, since $\operatorname{dom}\Theta = H^1(\Sigma;\mathbb{C}^2)$ and $\operatorname{dom}\Theta_\pm = H^1(\Sigma)$, Lemma 3.6 yields $\operatorname{dom} A_{\eta,\tau} \subset H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$.

It remains to show the Krein type resolvent formula in (4.15). First, for $|\eta| \neq |\tau|$ we have by Theorem 2.10 that $\Theta - M_z$, $z \in \operatorname{res} A_{\eta,\tau} \cap \operatorname{res} A_0$, is boundedly invertible in $L^2(\Sigma;\mathbb{C}^2)$ and

$$(A_{\eta,\tau} - z)^{-1} = (A_0 - z)^{-1} + G_z\big(\Theta - M_z\big)^{-1}G_{\bar{z}}^*.$$

Taking the special form of Θ and $M_z = \Lambda\big(\mathcal{C}_z - \frac{1}{2}\big(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}}\big)\big)\Lambda$ into account and using $\frac{1}{\eta^2-\tau^2}(\eta\sigma_0 - \tau\sigma_3) = (\eta\sigma_0 + \tau\sigma_3)^{-1}$, we find

$$\begin{aligned}\Theta - M_z &= -\Lambda\left[\frac{1}{\eta^2-\tau^2}(\eta\sigma_0 - \tau\sigma_3) + \frac{1}{2}\left(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\right)\right]\Lambda - \Lambda\left(\mathcal{C}_z - \frac{1}{2}\left(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\right)\right)\Lambda \\ &= -\Lambda\left[\frac{1}{\eta^2-\tau^2}(\eta\sigma_0 - \tau\sigma_3) + \mathcal{C}_z\right]\Lambda \\ &= -\Lambda(\eta\sigma_0+\tau\sigma_3)^{-1}\big(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)\Lambda.\end{aligned} \tag{4.16}$$

As $\Theta - M_z$ is a bijective operator in $L^2(\Sigma;\mathbb{C}^2)$ defined on $\operatorname{dom}\Theta = H^1(\Sigma;\mathbb{C}^2)$ this implies that $\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z$ is bijective in $H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. In particular, the inverse $(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z)^{-1}$ is well-defined and bounded in $H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. Using $G_z = \Phi_z\Lambda$ and $G_{\bar z}^* = \Lambda\Phi_{\bar z}'$ we get

$$\begin{aligned}G_z\big(\Theta - M_z\big)^{-1}G_{\bar z}^* &= -\Phi_z\Lambda\Lambda^{-1}\big(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Lambda^{-1}\Lambda\Phi_{\bar z}' \\ &= -\Phi_z\big(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Phi_{\bar z}',\end{aligned} \tag{4.17}$$

which leads to (4.15).

The proof of (4.15) for $|\eta| = |\tau| \neq 0$ is similar as above. First, one notes in the same way as in (4.16) that

$$\Theta_\pm - \Pi_\pm M_z \Pi_\pm^* = -\Lambda\left(\frac{1}{2\eta} + \Pi_\pm\mathcal{C}_z\Pi_\pm^*\right)\Lambda = -\frac{1}{2\eta}\Pi_\pm\Lambda\big(\sigma_0 + 2\eta\Pi_\pm^*\Pi_\pm\mathcal{C}_z\big)\Lambda\Pi_\pm^*, \tag{4.18}$$

which implies with $2\eta\Pi_\pm^*\Pi_\pm = \eta\sigma_0 + \tau\sigma_3$

$$\begin{aligned}\Pi_\pm^*(\Theta_\pm - \Pi_\pm M_z\Pi_\pm^*)^{-1}\Pi_\pm &= \Lambda^{-1}\Pi_\pm^*\big(\Pi_\pm(\sigma_0 + 2\eta\Pi_\pm^*\Pi_+\mathcal{C}_z)\Pi_\pm^*\big)^{-1}2\eta\Pi_\pm\Lambda^{-1} \\ &= \Lambda^{-1}\big(\Pi_\pm^*\Pi_\pm(\sigma_0 + 2\eta\Pi_\pm^*\Pi_+\mathcal{C}_z)\big)^{-1}2\eta\Pi_\pm^*\Pi_\pm\Lambda^{-1} \\ &= \Lambda^{-1}\big(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Lambda^{-1}.\end{aligned}$$

With this observation and the same ideas as above one shows (4.15) also in the case $|\eta| = |\tau|$. This finishes the proof of this theorem. □

In the following proposition we discuss the basic spectral properties of $A_{\eta,\tau}$:

Theorem 4.7. *Let $\eta,\tau \in \mathbb{R}$ be such that $\eta^2 - \tau^2 \neq 4$. Then the following holds:*

(i) *We have* $\operatorname{spec}_{\mathrm{ess}} A_{\eta,\tau} = (-\infty, -|m|] \cup [|m|, +\infty)$. *In particular, for $m = 0$ we have* $\operatorname{spec} A_{\eta,\tau} = \operatorname{spec}_{\mathrm{ess}} A_{\eta,\tau} = \mathbb{R}$.
(ii) *Assume $m \neq 0$. Then $z \in (-|m|, |m|)$ is a discrete eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\varphi \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ such that $(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z)\varphi = 0$.*
(iii) *If $m \neq 0$, then $A_{\eta,\tau}$ has at most finitely many eigenvalues in $(-|m|, |m|)$.*

Proof. By Proposition 3.7, the set $(-\infty, -|m|] \cup [|m|, +\infty)$ is contained in the essential spectrum of $A_{\eta,\tau}$. Moreover, by Theorem 4.6 we have $\operatorname{dom} A_{\eta,\tau} \subset H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$, which implies by Proposition 3.8 that the spectrum of $A_{\eta,\tau}$ in $(-|m|,|m|)$ is discrete and finite. This proves the items (i) and (iii).

It remains to prove (ii). Assume first that $|\eta| \neq |\tau|$. By Theorem 2.10 a number $z \in \operatorname{res} A_0$ is an eigenvalue of $A_{\eta,\tau}$ if and only if zero is an eigenvalue of $\Theta - M_z$. Using (4.16) this means that $z \in \operatorname{res} A_0$ is an eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\psi \in \operatorname{dom}\Theta = H^1(\Sigma;\mathbb{C}^2)$ such that

$$-\Lambda(\eta\sigma_0+\tau\sigma_3)^{-1}\big(\sigma_0+(\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)\Lambda\psi = 0,$$

i.e. if and only if $\varphi := \Lambda\psi \in H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ satisfies $(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z)\varphi = 0$. The proof of (ii) for $|\eta| = |\tau|$ is similar, one just has to use (4.18) instead of (4.16). $\square$

Finally, we provide some symmetry relations for the point spectrum of $A_{\eta,\tau}$, which can be seen as consequences of commutator relations of $A_{\eta,\tau}$. The following results are the two-dimensional analogues of [7, Proposition 4.2].

Proposition 4.8. *Let $\eta,\tau \in \mathbb{R}$ and assume that $\eta^2-\tau^2 \neq 4$. Then the following holds:*

(i) *If $|\eta| \neq |\tau|$, then $z \in \operatorname{spec_p} A_{\eta,\tau}$ if and only if $z \in \operatorname{spec_p} A_{-\frac{4\eta}{\eta^2-\tau^2},-\frac{4\tau}{\eta^2-\tau^2}}$.*
(ii) *$z \in \operatorname{spec_p} A_{\eta,\tau}$ if and only if $-z \in \operatorname{spec_p} A_{-\eta,\tau}$.*

Proof. (i) Consider the unitary and self-adjoint operator

$$U : L^2(\Omega_+;\mathbb{C}^2)\oplus L^2(\Omega_-;\mathbb{C}^2) \to L^2(\Omega_+;\mathbb{C}^2)\oplus L^2(\Omega_-;\mathbb{C}^2), \quad U(f_+\oplus f_-) = f_+\oplus(-f_-).$$

We claim that

$$A_{\eta,\tau} = UA_{-\frac{4\eta}{\eta^2-\tau^2},-\frac{4\tau}{\eta^2-\tau^2}}U. \tag{4.19}$$

For this purpose we note first that $f = f_+\oplus f_- \in H^1(\Omega_+;\mathbb{C}^2)\oplus H^1(\Omega_-;\mathbb{C}^2)$ belongs to $\operatorname{dom} A_{\eta,\tau}$, if and only if

$$-\mathrm{i}(\sigma\cdot\nu)\big(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-\big) = \frac{1}{2}(\eta\sigma_0+\tau\sigma_3)\big(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\big), \tag{4.20}$$

which is equivalent to

$$-\mathrm{i}(\sigma\cdot\nu)\big(\mathcal{T}_+^D (Uf)_+ + \mathcal{T}_-^D (Uf)_-\big) = \frac{1}{2}(\eta\sigma_0+\tau\sigma_3)\big(\mathcal{T}_+^D (Uf)_+ - \mathcal{T}_-^D (Uf)_-\big).$$

By multiplying the last equation with $(\eta\sigma_0+\tau\sigma_3)^{-1} = \frac{1}{\eta^2-\tau^2}(\eta\sigma_0-\tau\sigma_3)$ and using (1.4) we find that $f \in \operatorname{dom} A_{\eta,\tau}$ if and only if

$$-\mathrm{i}(\sigma\cdot\nu)\frac{1}{\eta^2-\tau^2}(\eta\sigma_0+\tau\sigma_3)\big(\mathcal{T}_+^D (Uf)_+ + \mathcal{T}_-^D (Uf)_-\big) = \frac{1}{2}\big(\mathcal{T}_+^D (Uf)_+ - \mathcal{T}_-^D (Uf)_-\big),$$

which is equivalent to

$$-\frac{4}{\eta^2-\tau^2}(\eta\sigma_0+\tau\sigma_3)\frac{1}{2}\big(\mathcal{T}_+^D(Uf)_+ + \mathcal{T}_-^D(Uf)_-\big) = -\mathrm{i}(\sigma\cdot\nu)\big(\mathcal{T}_+^D(Uf)_+ - \mathcal{T}_-^D(Uf)_-\big)$$

i.e. to $Uf \in \operatorname{dom} A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)}$. Hence, we have shown the domain equality $\operatorname{dom} A_{\eta,\tau} = \operatorname{dom} A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)}U$. Moreover, a straightforward calculation shows $UA_{\eta,\tau}f = A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)}Uf$ for any $f \in \operatorname{dom} A_{\eta,\tau}$. This gives (4.19), which yields (i).

(ii) Define the antilinear charge conjugation operator $Cf = \sigma_1\overline{f}$, $f \in L^2(\mathbb{R}^2;\mathbb{C}^2)$. Then we see immediately $C^2 f = f$ for all $f \in L^2(\mathbb{R}^2;\mathbb{C}^2)$. We claim that

$$CA_{\eta,\tau} = -A_{-\eta,\tau}C, \tag{4.21}$$

which yields then the claim of statement (ii). To prove (4.21), we note first by taking the complex conjugate of equation (4.20) that $f \in \operatorname{dom} A_{\eta,\tau}$ if and only if

$$\mathrm{i}(\overline{\sigma}\cdot\nu)\big(\mathcal{T}_+^D\overline{f_+} - \mathcal{T}_-^D\overline{f_-}\big) = \frac{1}{2}(\eta\sigma_0+\tau\sigma_3)\big(\mathcal{T}_+^D\overline{f_+} + \mathcal{T}_-^D\overline{f_-}\big), \tag{4.22}$$

where $\overline{\sigma} = (\overline{\sigma_1},\overline{\sigma_2})$ and $\overline{\sigma_j}$ is the matrix with the complex conjugate entries of σ_j. By multiplying this equation with σ_1 and using (1.4), $\overline{\sigma_1} = \sigma_1$, and $\overline{\sigma_2} = -\sigma_2$ we find that (4.22) is equivalent to

$$\mathrm{i}(\sigma\cdot\nu)\big(\mathcal{T}_+^D(\sigma_1\overline{f_+}) - \mathcal{T}_-^D(\sigma_1\overline{f_-})\big) = \frac{1}{2}(\eta\sigma_0-\tau\sigma_3)\big(\mathcal{T}_+^D(\sigma_1\overline{f_+}) + \mathcal{T}_-^D(\sigma_1\overline{f_-})\big),$$

i.e. $Cf \in \operatorname{dom} A_{-\eta,\tau}$. Moreover, using again (1.4) and $\overline{\sigma_2} = -\sigma_2$ we get

$$\begin{aligned}(-\mathrm{i}\sigma\cdot\nabla + m\sigma_3)Cf &= (-\mathrm{i}\sigma\cdot\nabla + m\sigma_3)\sigma_1\overline{f} = \sigma_1(-\mathrm{i}\overline{\sigma}\cdot\nabla - m\sigma_3)\overline{f}\\ &= -\sigma_1\overline{(-\mathrm{i}\sigma\cdot\nabla + m\sigma_3)f} = -C\big(-\mathrm{i}\sigma\cdot\nabla + m\sigma_3)f\big),\end{aligned}$$

which implies (4.21). □

4.3. Critical case

In this subsection we study the self-adjointness and the spectral properties of $A_{\eta,\tau}$ for the critical interaction strengths, i.e. when $\eta^2-\tau^2 = 4$. To show the self-adjointness of $A_{\eta,\tau}$ we prove that the corresponding operator Θ in Proposition 4.3 is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$.

Lemma 4.9. *Let $\eta,\tau \in \mathbb{R}$ be such that $\eta^2-\tau^2 = 4$. Then the operator Θ is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$ and the restriction of Θ onto $H^1(\Sigma;\mathbb{C}^2)$ is essentially self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$.*

Remark 4.10. According to Lemma 4.9 the operator Θ is essentially self-adjoint on $H^1(\Sigma;\mathbb{C}^2)$. It will turn out later in the proof of Proposition 4.12 that $\mathrm{spec}_{\mathrm{ess}}\,\Theta$ is non-empty. Hence, one has $\mathrm{dom}\,\Theta \not\subset H^s(\Sigma;\mathbb{C}^2)$ for all $s>0$.

Proof of Lemma 4.9. As in the proof of Lemma 4.5 we consider $\Theta_1 := \Theta \upharpoonright H^1(\Sigma;\mathbb{C}^2)$. It follows in the same way as in the proof of Lemma 4.5 that Θ_1 is a symmetric operator in $L^2(\Sigma;\mathbb{C}^2)$ and together with Lemma 2.4 we see $\overline{\Theta_1} \subset \Theta_1^* \subset \Theta$. To see $\Theta \subset \overline{\Theta_1}$, which then implies the claims, we will show (the slightly stronger fact) that

$$\mathrm{dom}\,\Theta = \mathrm{dom}\,\overline{\Theta_1}. \tag{4.23}$$

For this we consider the associated periodic pseudodifferential operator θ defined in (4.7) and recall that with the aid of Proposition 3.3 we have

$$\theta = -\frac{1}{2}\,\upsilon + \Psi, \quad \text{where } \upsilon = \begin{pmatrix} \frac{2}{\eta+\tau}\Lambda^2 & \Lambda C_\Sigma \overline{T}\Lambda \\ \Lambda T C_\Sigma' \Lambda & \frac{2}{\eta-\tau}\Lambda^2 \end{pmatrix},$$

with some operator $\Psi \in \Psi_\Sigma^0$, which is symmetric and hence self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$. In the following we denote by Υ the maximal realization of υ in $L^2(\Sigma;\mathbb{C}^2)$, that is

$$\Upsilon\varphi = \upsilon\varphi, \quad \mathrm{dom}\,\Upsilon = \big\{\varphi \in L^2(\Sigma;\mathbb{C}^2) : \upsilon\varphi \in L^2(\Sigma;\mathbb{C}^2)\big\} = \mathrm{dom}\,\Theta,$$

and $\Upsilon_1 = \Upsilon \upharpoonright H^1(\Sigma;\mathbb{C}^2)$. Note that $\mathrm{dom}\,\overline{\Upsilon_1} = \mathrm{dom}\,\overline{\Theta_1}$. Since Λ and hence also Λ^2 are invertible, we get (as operators on distributions)

$$\upsilon = \begin{pmatrix} \mathbb{1} & 0 \\ \frac{\eta+\tau}{2}\Lambda T C_\Sigma' \Lambda^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \frac{2}{\eta+\tau}\Lambda^2 & 0 \\ 0 & \mathcal{S}(\upsilon) \end{pmatrix} \begin{pmatrix} \mathbb{1} & \frac{\eta+\tau}{2}\Lambda^{-1} C_\Sigma \overline{T}\Lambda \\ 0 & \mathbb{1} \end{pmatrix}, \tag{4.24}$$

where the Schur complement $\mathcal{S}(\upsilon)$ has the form

$$\mathcal{S}(\upsilon) = \frac{2}{\eta-\tau}\Lambda^2 - \frac{\eta+\tau}{2}\Lambda T C_\Sigma' \Lambda (\Lambda^2)^{-1} \Lambda C_\Sigma \overline{T}\Lambda = \frac{2}{\eta-\tau}\Lambda^2 - \frac{\eta+\tau}{2}\Lambda T C_\Sigma' C_\Sigma \overline{T}\Lambda. \tag{4.25}$$

Using that $C_\Sigma' C_\Sigma = \mathbb{1} + R$ with $R \in \Psi_\Sigma^{-\infty}$, see Proposition 2.8, we can rewrite this expression as

$$\mathcal{S}(\upsilon) = \frac{2}{\eta-\tau}\Lambda^2 - \frac{\eta+\tau}{2}\Lambda T\overline{T}\Lambda - \frac{\eta+\tau}{2}\Lambda T R \overline{T}\Lambda = -\frac{\eta+\tau}{2}\Lambda T R \overline{T}\Lambda \in \Psi_\Sigma^{-\infty}, \tag{4.26}$$

where we used in the last step that $T\overline{T}$ is the identity operator and $\eta^2 - \tau^2 = 4$. From this, (4.24), and $\mathrm{dom}\,\Lambda^2 = H^1(\Sigma)$ we obtain now

$$\mathrm{dom}\,\Theta = \mathrm{dom}\,\Upsilon = \Big\{(\varphi_1,\varphi_2) \in L^2(\Sigma;\mathbb{C}^2) : \varphi_1 + \frac{\eta+\tau}{2}\Lambda^{-1} C_\Sigma \overline{T}\Lambda\varphi_2 \in H^1(\Sigma)\Big\}.$$

Let us now consider the operator realizations Θ_1, Υ_1 of θ, υ and their closures $\overline{\Theta_1}, \overline{\Upsilon_1}$ in $L^2(\Sigma; \mathbb{C}^2)$. In the following we view Λ^2 as an operator defined on $H^1(\Sigma)$ and note that Λ^2 is self-adjoint and $0 \in \operatorname{res}(\Lambda^2)$. Moreover, since $\frac{\eta+\tau}{2}\Lambda^{-1}C_\Sigma\overline{T}\Lambda \in \Psi^0_\Sigma$, we get that the operator

$$\mathcal{A}_1\varphi = \frac{\eta+\tau}{2}\Lambda^{-2}\Lambda C_\Sigma\overline{T}\Lambda\varphi, \quad \operatorname{dom}\mathcal{A}_1 = H^1(\Sigma),$$

which is the product of the inverse of the upper left corner and the upper right corner of Υ_1, is bounded in $L^2(\Sigma)$ and has a bounded and everywhere defined closure. Since the Schur complement $\mathcal{S}_1(\upsilon)$ of Υ_1, which is the expression from (4.25) defined on $H^1(\Sigma)$, has a bounded closure in $L^2(\Sigma)$ by (4.26), we conclude from [38, Theorem 2.2.14] applied for $\mu = 0$ that

$$\begin{aligned}\operatorname{dom}\overline{\Theta_1} = \operatorname{dom}\overline{\Upsilon_1} &= \big\{(\varphi_1,\varphi_2) \in L^2(\Sigma;\mathbb{C}^2) : \varphi_1 + \overline{\mathcal{A}_1}\varphi_2 \in \operatorname{dom}\Lambda^2, \varphi_2 \in \operatorname{dom}\overline{\mathcal{S}_1(\upsilon)}\big\}\\ &= \Big\{(\varphi_1,\varphi_2) \in L^2(\Sigma;\mathbb{C}^2) : \varphi_1 + \frac{\eta+\tau}{2}\Lambda^{-1}C_\Sigma\overline{T}\Lambda\varphi_2 \in H^1(\Sigma)\Big\} = \operatorname{dom}\Theta.\end{aligned}$$

Hence, we have shown (4.23), which finishes the proof of this proposition. □

With Lemma 4.9 we are now ready to show the self-adjointness of $A_{\eta,\tau}$ for critical interaction strengths. To formulate the result we recall the definitions of the free Dirac operator A_0 from (3.1), of Φ_z and Φ'_z from (3.7) and (3.6), and of $\mathcal{C}_z$ in (3.10), respectively.

Theorem 4.11. *Let $\eta, \tau \in \mathbb{R}$ with $\eta^2 - \tau^2 = 4$. Then the operator $A_{\eta,\tau}$ is self-adjoint and its restriction to $\operatorname{dom} A_{\eta,\tau} \cap H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ is essentially self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Moreover, for all $z \in \operatorname{res} A_{\eta,\tau} \cap \operatorname{res} A_0$ the operator $\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z$ admits a bounded inverse from $H^{\frac12}(\Sigma; \mathbb{C}^2)$ to $H^{-\frac12}(\Sigma; \mathbb{C}^2)$, and*

$$(A_{\eta,\tau} - z)^{-1} = (A_0 - z)^{-1} - \Phi_z\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Phi'_{\bar z}. \tag{4.27}$$

Proof. First, according to Theorem 2.10 the self-adjointness of Θ in $L^2(\Sigma; \mathbb{C}^2)$ implies the self-adjointness of $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$, and the essential self-adjointness of the restriction $\Theta_1 = \Theta \upharpoonright H^1(\Sigma; \mathbb{C}^2)$ in $L^2(\Sigma; \mathbb{C}^2)$ implies the essential self-adjointness of $A_{\eta,\tau}$ restricted to $\operatorname{dom} A_{\eta,\tau} \cap H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. For the latter observation we have also used that by Lemma 3.6

$$S^* \upharpoonright \ker(\Gamma_1 - \Theta_1\Gamma_0) = A_{\eta,\tau} \upharpoonright \big(\operatorname{dom} A_{\eta,\tau} \cap H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)\big).$$

It remains to verify the Krein type resolvent formula in (4.27). By Theorem 2.10 we have that $\Theta - M_z$ is boundedly invertible in $L^2(\Sigma; \mathbb{C}^2)$ and

$$(A_{\eta,\tau} - z)^{-1} = (A_0 - z)^{-1} + G_z\big(\Theta - M_z\big)^{-1}G^*_{\bar z}.$$

Taking the special form of Θ and $M_z = \Lambda\big(\mathcal{C}_z - \frac{1}{2}\big(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\big)\big)\Lambda$ into account we find with a similar calculation as in (4.16)-(4.17) that

$$(\Theta - M_z)^{-1} = -\Lambda^{-1}\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Lambda^{-1}.$$

As $(\Theta - M_z)^{-1}$ is bounded in $L^2(\Sigma;\mathbb{C}^2)$ we deduce that $(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z)^{-1}$ is bounded from $H^{\frac{1}{2}}(\Sigma;\mathbb{C}^2)$ to $H^{-\frac{1}{2}}(\Sigma;\mathbb{C}^2)$. Using $G_z = \Phi_z\Lambda$ and $G_{\bar z}^* = \Lambda\Phi_{\bar z}'$ we get

$$\begin{aligned} G_z\big(\Theta - M_z\big)^{-1}G_{\bar z}^* &= -\Phi_z\Lambda\Lambda^{-1}\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Lambda^{-1}\Lambda\Phi_{\bar z}' \\ &= -\Phi_z\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Phi_{\bar z}', \end{aligned}$$

and thus (4.27). □

In the next proposition we analyze the essential spectrum of the self-adjoint operator Θ. Note that our assumption $\eta^2 - \tau^2 = 4$ implies $|\tau| < |\eta|$, and hence $-\frac{\tau}{\eta}m \in (-|m|, |m|)$.

Proposition 4.12. *Let $\eta, \tau \in \mathbb{R}$ be such that $\eta^2 - \tau^2 = 4$ and let $m \neq 0$. Then for $z \in (-|m|, |m|)$ one has $0 \in \mathrm{spec}_{\mathrm{ess}}(M_z - \Theta)$ if and only if $z = -\frac{\tau}{\eta}m$.*

Proof. Throughout the proof we assume that $z \in (-|m|, |m|)$. In particular, M_z is a bounded self-adjoint operator in $L^2(\Sigma;\mathbb{C}^2)$. Recall that

$$M_z - \Theta = \Lambda\frac{1}{\eta^2 - \tau^2}(\eta\sigma_0 - \tau\sigma_3)\Lambda + \Lambda\mathcal{C}_z\Lambda,$$

and using Proposition 3.3 we decompose $M_z - \Theta = \Xi_1 + \Xi_2$, where

$$\Xi_1 := \begin{pmatrix} \frac{1}{\eta+\tau}\Lambda^2 + \frac{\ell}{4\pi}(z+m)\mathbb{1} & \frac{1}{2}\Lambda C_\Sigma\overline{T}\Lambda \\ \frac{1}{2}\Lambda T C_\Sigma'\Lambda & \frac{1}{\eta-\tau}\Lambda^2 + \frac{\ell}{4\pi}(z-m)\mathbb{1} \end{pmatrix}$$

and $\Xi_2 \in \Psi_\Sigma^{-1}$ is a compact self-adjoint operator in $L^2(\Sigma;\mathbb{C}^2)$. We note that Ξ_1 defined on $\mathrm{dom}(M_z - \Theta) = \mathrm{dom}\,\Theta$ is a self-adjoint operator in $L^2(\Sigma;\mathbb{C}^2)$. Therefore, it follows that $\mathrm{spec}_{\mathrm{ess}}(M_z - \Theta) = \mathrm{spec}_{\mathrm{ess}}\,\Xi_1$ and, in particular,

$$0 \in \mathrm{spec}_{\mathrm{ess}}(M_z - \Theta) \text{ if and only if } 0 \in \mathrm{spec}_{\mathrm{ess}}\,\Xi_1.$$

In the following we will show that $0 \in \mathrm{spec}_{\mathrm{ess}}\,\Xi_1$ if and only if $z = -\frac{\tau}{\eta}m$. For this, the Schur complement of Ξ_1 and [38, Theorem 2.4.6] (applied for $\mu = 0$) will be used. To proceed, let $\Xi_{1,1} := \Xi_1 \upharpoonright H^1(\Sigma;\mathbb{C}^2)$. Then, by Lemma 4.9 we have $\Xi_1 = \overline{\Xi_{1,1}}$. We shall use the operator $\Lambda \in \Psi_\Sigma^{\frac{1}{2}}$ from (2.7) (see also (2.6)). Recall also that $\Lambda^2 \geq c_0^2$ for $c_0 > 0$. Now we choose c_0 such that $c_0^2 > \frac{|m|\ell}{2\pi}|\eta + \tau|$. Then the upper left corner of $\Xi_{1,1}$,

$$\mathcal{A} := \frac{1}{\eta + \tau}\Lambda^2 + \frac{\ell}{4\pi}(z+m)\mathbb{1},$$

is self-adjoint in $L^2(\Sigma)$ with $0 \in \operatorname{res} \mathcal{A}$. Hence, the Schur complement $\mathcal{S} := \mathcal{S}(\Xi_{1,1})$, that is defined on $\operatorname{dom} \mathcal{S} = H^1(\Sigma)$ and given by

$$\mathcal{S} = \frac{1}{\eta - \tau} \Lambda^2 + \frac{\ell(z-m)}{4\pi} \mathbb{1} - \frac{\eta + \tau}{4} \Lambda T C_\Sigma' \Lambda \left(\Lambda^2 + \frac{\ell(z+m)(\eta+\tau)}{4\pi} \mathbb{1} \right)^{-1} \Lambda C_\Sigma \overline{T} \Lambda,$$

is well-defined. It is easy to see that $\mathcal{S}$ is symmetric and hence closable. We leave it to the reader to check that the other assumptions in [38, Theorem 2.4.6] are also satisfied for the block operator matrix $\Xi_{1,1}$. Thus, it follows from [38, Theorem 2.4.6] that $0 \in \operatorname{spec}_{\mathrm{ess}} \Xi_1$ if and only if $0 \in \operatorname{spec}_{\mathrm{ess}} \overline{\mathcal{S}}$. We are going to prove that $\mathcal{S}$ is bounded in $L^2(\Sigma)$ and that $0 \in \operatorname{spec}_{\mathrm{ess}} \overline{\mathcal{S}}$ if and only if $z = -\frac{\tau}{\eta} m$.

To simplify the last summand in the above expression of $\mathcal{S}$ we use the identity

$$(\Lambda^2 + a\mathbb{1})^{-1} = \Lambda^{-2} - a\Lambda^{-1}(\Lambda^2 + a\mathbb{1})^{-1}\Lambda^{-1} = \Lambda^{-2} - a\Lambda^{-2}(\Lambda^2 + a\mathbb{1})^{-1} \tag{4.28}$$

and rewrite $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ with

$$\mathcal{S}_1 = \frac{1}{\eta - \tau} \Lambda^2 + \frac{\ell(z-m)}{4\pi} \mathbb{1} - \frac{\eta + \tau}{4} \Lambda T C_\Sigma' C_\Sigma \overline{T} \Lambda,$$

$$\mathcal{S}_2 = \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi} \Lambda T C_\Sigma' \left(\Lambda^2 + \frac{\ell(z+m)(\eta+\tau)}{4\pi} \mathbb{1} \right)^{-1} C_\Sigma \overline{T} \Lambda.$$

By Proposition 2.8 one has $C_\Sigma' C_\Sigma = \mathbb{1} + K_1$ with $K_1 \in \Psi_\Sigma^{-\infty}$, so

$$\frac{\eta + \tau}{4} \Lambda T C_\Sigma' C_\Sigma \overline{T} \Lambda = \frac{\eta + \tau}{4} \Lambda^2 + K_2$$

with $K_2 \in \Psi_\Sigma^{-\infty}$. Because of $\eta^2 - \tau^2 = 4$ one arrives at

$$\mathcal{S}_1 = \frac{1}{\eta - \tau} \Lambda^2 + \frac{\ell(z-m)}{4\pi} \mathbb{1} - \frac{\eta + \tau}{4} \Lambda^2 - K_2 = \frac{\ell(z-m)}{4\pi} \mathbb{1} - K_2.$$

In order to deal with $\mathcal{S}_2$ we use again the identity (4.28), which gives

$$\frac{4}{(\eta+\tau)^2} \cdot \frac{4\pi}{\ell(z+m)} \mathcal{S}_2 = \Lambda T C_\Sigma' \left(\Lambda^2 + \frac{\ell(z+m)(\eta+\tau)}{4\pi} \mathbb{1} \right)^{-1} C_\Sigma \overline{T} \Lambda = K_3 + K_4,$$

where $K_3 = \Lambda T C_\Sigma' \Lambda^{-2} C_\Sigma \overline{T} \Lambda$ and

$$K_4 = -\frac{\ell(z+m)(\eta+\tau)}{4\pi} \Lambda T C_\Sigma' \Lambda^{-2} \left(\Lambda^2 + \frac{\ell(z+m)(\eta+\tau)}{4\pi} \mathbb{1} \right)^{-1} C_\Sigma \overline{T} \Lambda.$$

Using Proposition 2.2 one finds that $K_4 \in \Psi_\Sigma^{-1}$ and hence this operator is compact in $L^2(\Sigma; \mathbb{C}^2)$. In order to simplify K_3 we note first that

$$K_5 := TC'_\Sigma \Lambda^{-2} - \Lambda^{-2} TC'_\Sigma \in \Psi_\Sigma^{-2}$$

by Proposition 2.2 (ii). Hence,

$$K_3 = \Lambda\Lambda^{-2}TC'_\Sigma C_\Sigma \overline{T}\Lambda + \Lambda K_5 C_\Sigma \overline{T}\Lambda =: \Lambda\Lambda^{-2}TC'_\Sigma C_\Sigma \overline{T}\Lambda + K_6$$

with $K_6 \in \Psi_\Sigma^{-1}$. Using again $C'_\Sigma C_\Sigma - \mathbb{1} \in \Psi^{-\infty}$, see Proposition 2.8, we arrive at $K_3 = \mathbb{1} + K_7$ with $K_7 \in \Psi_\Sigma^{-1}$. With this we find

$$\mathcal{S}_2 = \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi}(K_3 + K_4) = \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi}\mathbb{1} + K_8$$

with $K_8 \in \Psi_\Sigma^{-1}$. Using this in the expression of the Schur complement $\mathcal{S}$ we conclude, with some $K_9 \in \Psi_\Sigma^{-1}$, that

$$\begin{aligned} \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 &= \left(\frac{\ell(z-m)}{4\pi} + \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi} \right) \mathbb{1} + K_9 \\ &= \frac{\ell}{4\pi} \left[\left(\frac{(\eta+\tau)^2}{4} + 1 \right) z + \left(\frac{(\eta+\tau)^2}{4} - 1 \right) m \right] \mathbb{1} + K_9. \end{aligned}$$

From this we conclude that $\mathcal{S}$ is bounded and admits a bounded closure $\overline{\mathcal{S}}$. Moreover, as K_9 is compact and symmetric, it does not influence the essential spectrum, and we have

$$0 \in \operatorname{spec}_{\mathrm{ess}} \overline{\mathcal{S}} \text{ if and only if } z = -\frac{(\eta+\tau)^2 - 4}{(\eta+\tau)^2 + 4} m.$$

With $\eta^2 - \tau^2 = 4$ we can simplify the last expression to

$$\frac{(\eta+\tau)^2 - 4}{(\eta+\tau)^2 + 4} = \frac{\eta^2 + \tau^2 + 2\eta\tau - \eta^2 + \tau^2}{\eta^2 + \tau^2 + 2\eta\tau + \eta^2 - \tau^2} = \frac{2\tau^2 + 2\eta\tau}{2\eta^2 + 2\eta\tau} = \frac{2\tau(\eta+\tau)}{2\eta(\eta+\tau)} = \frac{\tau}{\eta}.$$

Hence, $0 \in \operatorname{spec}_{\mathrm{ess}} \overline{\mathcal{S}}$ if and only if $z = -\frac{\tau}{\eta} m$. This finishes the proof. □

We are now ready to describe the spectral properties of $A_{\eta,\tau}$ for critical interaction strengths. Compared to Proposition 4.7, the following theorem shows that the spectral properties of $A_{\eta,\tau}$ differ significantly from the non-critical case.

Theorem 4.13. *Let $\eta, \tau \in \mathbb{R}$ with $\eta^2 - \tau^2 = 4$. Then the following is true:*

(i) *There holds $\operatorname{spec}_{\mathrm{ess}} A_{\eta,\tau} = (-\infty, -|m|] \cup \{-\frac{\tau}{\eta} m\} \cup [|m|, +\infty)$. In particular, for $m = 0$ we have $\operatorname{spec} A_{\eta,\tau} = \operatorname{spec}_{\mathrm{ess}} A_{\eta,\tau} = \mathbb{R}$.*
(ii) *Assume $m \neq 0$. Then $z \notin \operatorname{spec}_{\mathrm{ess}} A_{\eta,\tau}$ is a discrete eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\varphi \in H^{-\frac{1}{2}}(\Sigma; \mathbb{C}^2)$ such that $(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z)\varphi = 0$.*
(iii) *For all $s > 0$ we have $\operatorname{dom} A_{\eta,\tau} \not\subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$.*

Remark 4.14. Item (ii) in the above theorem is slightly weaker as Proposition 4.7 (ii), since one has to search for eigenfunctions φ of $\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z$ in the larger space $H^{-\frac{1}{2}}(\Sigma; \mathbb{C}^2)$. However, as there is no Sobolev regularity in $\operatorname{dom} A_{\eta,\tau}$ the smoothness of the eigenfunctions of $\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z$ can not be improved.

Proof of Theorem 4.13. (i) The inclusion $(-\infty, -|m|] \cup [|m|, +\infty) \subset \operatorname{spec}_{\mathrm{ess}} A_{\eta,\tau}$ holds by Proposition 3.7. In addition, due to Theorem 2.10 and Proposition 4.12 one has $\operatorname{spec}_{\mathrm{ess}} A_{\eta,\tau} \cap (-|m|, |m|) = \{-\frac{\tau}{\eta} m\}$, which gives the claim.

To prove item (ii) we note first that by Theorem 2.10 a point $z \in \operatorname{res} A_0$ is an eigenvalue of $A_{\eta,\tau}$ if and only if zero is an eigenvalue of $\Theta - M_z$. Using a similar calculation as in (4.16) this shows that $z \in \operatorname{res} A_0$ is an eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\psi \in \operatorname{dom} \Theta \subset L^2(\Sigma; \mathbb{C}^2)$ such that $-\Lambda(\eta\sigma_0 + \tau\sigma_3)^{-1}(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z)\Lambda\psi = 0$, i.e. if and only if $\varphi := \Lambda\psi \in H^{-\frac{1}{2}}(\Sigma; \mathbb{C}^2)$ satisfies $\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)\varphi = 0$.

Eventually, since $\operatorname{dom} A_{\eta,\tau}$ is independent of m, it suffices to prove statement (iii) for $m \neq 0$. In this case the claim is a consequence of Proposition 3.8, as we have in this case $\operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) \cap (-|m|, |m|) \neq \emptyset$. □

Finally, we state several symmetry relations in the spectrum of $A_{\eta,\tau}$. The following proposition is the counterpart of Proposition 4.8 for critical interaction strengths.

Proposition 4.15. *Let $\eta, \tau \in \mathbb{R}$ with $\eta^2 - \tau^2 = 4$. Then the following holds:*

(i) *$z \in \operatorname{spec}_{\mathrm{p}} A_{\eta,\tau}$ if and only if $z \in \operatorname{spec}_{\mathrm{p}} A_{-\eta,-\tau}$.*
(ii) *$z \in \operatorname{spec}_{\mathrm{p}} A_{\eta,\tau}$ if and only if $-z \in \operatorname{spec}_{\mathrm{p}} A_{-\eta,\tau}$.*

Proof. In the following set $A^1_{\eta,\tau} := A_{\eta,\tau} \upharpoonright (\operatorname{dom} A_{\eta,\tau} \cap H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2))$. Then by Theorem 4.11 the operator $A^1_{\eta,\tau}$ is essentially self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ and, in particular, $\overline{A^1_{\eta,\tau}} = A_{\eta,\tau}$.

(i) Consider the unitary and self-adjoint mapping

$$U : L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2) \to L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2), \quad U(f_+ \oplus f_-) = f_+ \oplus (-f_-).$$

As in the proof of Proposition 4.8 (i) one verifies $A^1_{\eta,\tau} = U A^1_{-\eta,-\tau} U$. By taking closures we find $A_{\eta,\tau} = U A_{-\eta,-\tau} U$ and hence the claim follows.

(ii) Consider the nonlinear charge conjugation operator $Cf = \sigma_1 \overline{f}$, $f \in L^2(\mathbb{R}^2; \mathbb{C}^2)$. Then $C^2 f = f$ for $f \in L^2(\mathbb{R}^2; \mathbb{C}^2)$ and in the same way as in the proof of Proposition 4.8 (ii) one obtains $C A^1_{\eta,\tau} = -A^1_{-\eta,\tau} C$. Taking closures leads to $C A_{\eta,\tau} = -A_{-\eta,\tau} C$, which implies (ii). □

4.4. Case of several loops

To prove Theorem 1.3 we use similar constructions as in the case of one loop. We give some comments on necessary modifications in this subsection. Let $N \geq 1$ and let $\Sigma_j, j \in \{1, \dots, N\}$, be non-intersecting C^∞-smooth loops with normals ν_j. We set $\Sigma := \bigcup_{j=1}^N \Sigma_j$, and for $f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma)$ we denote its Dirichlet traces from Lemma 3.1 on the two sides of Σ_j by $\mathcal{T}^D_{\pm,j} f$, where $-$ corresponds to the side to which ν_j is directed. The Sobolev spaces on Σ are defined by $H^s(\Sigma) := \bigoplus_{j=1}^N H^s(\Sigma_j)$, and for $\varphi \in H^s(\Sigma)$ we denote by φ_j its restriction on Σ_j. Furthermore, if Λ_j denotes the isomorphism defined in (2.7) on Σ_j, then we set $\Lambda := \bigoplus_{j=1}^N \Lambda_j$. As in the case of one loop one starts with the symmetric operator $S := A_0 \upharpoonright H^1_0(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$. For $z \in \operatorname{res} A_0$ and $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ we introduce

$$\Phi_z \varphi(x) = \int_\Sigma \phi_z(x-y)\varphi(y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \Sigma.$$

As for the single loop one shows that the map Φ_z extends to a bounded linear operator $\Phi_z : H^{-\frac{1}{2}}(\Sigma; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2)$ with $\operatorname{ran} \Phi_z = \ker(S^* - z)$. The associated principal value operator $\mathcal{C}_z$,

$$\big(\mathcal{C}_z \varphi\big)(x) := \text{p.v.} \int_\Sigma \phi_z(x-y)\varphi(y)\, \mathrm{d}s(y), \quad \varphi \in C^\infty(\Sigma; \mathbb{C}^2),\ x \in \Sigma,$$

has a block structure of the form

$$(\mathcal{C}_z \varphi)_j(x) = \mathcal{C}^j_z \varphi_j(x) + \sum_{k \neq j} (\mathcal{K}^{j,k}_z \varphi_k)(x), \quad \varphi \in C^\infty(\Sigma; \mathbb{C}^2),\ x \in \Sigma_j, \tag{4.29}$$

$$(\mathcal{C}^j_z \varphi_j)(x) = \text{p.v.} \int_{\Sigma_j} \phi_z(x-y)\varphi_j(y)\, \mathrm{d}s(y), \quad x \in \Sigma_j, \tag{4.30}$$

$$(\mathcal{K}^{j,k}_z \varphi_k)(x) = \int_{\Sigma_k} \phi_z(x-y)\varphi_k(y)\, \mathrm{d}s(y), \quad x \in \Sigma_j. \tag{4.31}$$

The operators $\mathcal{C}^j_z$ are the same as in the one loop case, while the operators $\mathcal{K}^{j,k}_z$ have smooth integral kernels and are bounded from $H^s(\Sigma_k, \mathbb{C}^2)$ to $H^t(\Sigma_j, \mathbb{C}^2)$ for any $s, t \in \mathbb{R}$. Using Proposition 3.4, the trace equality $\mathcal{T}^D_{\pm,j} \Phi_z \varphi = \mp \frac{\mathrm{i}}{2} (\sigma \cdot \nu_j)\, \varphi_j + (\mathcal{C}_z \varphi)_j$ can be shown. The construction of the boundary triple takes then literally the same form as for a single loop. Let $\zeta \in \operatorname{res} A_0$ be fixed and set $(\mathcal{T}^D_\pm f) := (\mathcal{T}^D_{\pm,j} f)_{j=1}^N$. Then $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0, \Gamma_1\}$,

$$\Gamma_0 f = \mathrm{i}\Lambda^{-1}(\sigma \cdot \nu)\big(\mathcal{T}^D_+ f - \mathcal{T}^D_- f\big), \quad \Gamma_1 f = \frac{1}{2}\Lambda\Big((\mathcal{T}^D_+ f_+ + \mathcal{T}^D_- f_-) - (\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big),$$

is a boundary triple for S^*. The corresponding γ-field G and Weyl function M are $z \mapsto G_z = \Phi_z \Lambda$ and $z \mapsto M_z = \Lambda(\mathcal{C}_z - \frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}))\Lambda$.

Assume first that $|\eta_j| \neq |\tau_j|$ for all $j \in \{1, \dots, N\}$. Define an operator Θ in $L^2(\Sigma; \mathbb{C}^2)$ by $\Theta = -\Lambda\big[\Xi + \frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\big]\Lambda$, $(\Xi\varphi)_j := \frac{1}{\eta_j^2 - \tau_j^2}(\eta_j \sigma_0 - \tau_j \sigma_3)\, \varphi_j$, on its maximal domain, then $A_{\Sigma,\mathcal{P}}$ corresponds to the boundary condition $\Gamma_1 f = \Theta\Gamma_0 f$. Using (4.29) one sees that Θ can be written as $\Theta = \bigoplus_{j=1}^N \Theta_j + \widetilde{\Theta}$, where Θ_j acts in $L^2(\Sigma_j;\mathbb{C}^2)$ by

$$\Theta_j = -\Lambda_j \left[\frac{1}{\eta_j^2-\tau_j^2}(\eta_j\sigma_0-\tau_j\sigma_3)+\frac{1}{2}\left(\mathcal{C}_\zeta^j+\mathcal{C}_{\bar\zeta}^j\right)\right]\Lambda_j,$$

with maximal domain, while $\widetilde{\Theta}$ is a bounded operator from $H^s(\Sigma,\mathbb{C}^2)$ to $H^t(\Sigma,\mathbb{C}^2)$ for any $s,t\in\mathbb{R}$ and self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$. Hence, the self-adjointness of Θ is determined by the self-adjointness of $\bigoplus_{j=1}^N \Theta_j$, and each Θ_j is exactly of the form as in the single-loop case. Hence, Θ_j is self-adjoint by Lemma 4.5 and Lemma 4.9 and thus, also Θ is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$. This implies also the statements concerning the domain regularity.

In order to study the essential spectrum we decompose M_z to blocks as in (4.29) and remark that the terms $\mathcal{K}_z^{j,k}$ produce compact remainders, which do not influence the essential spectrum. Hence, the condition $0\in\mathrm{spec}_{\mathrm{ess}}(M_z-\Theta)$ is equivalent to

$$0\in\mathrm{spec}_{\mathrm{ess}}\left(\bigoplus_{j=1}^N\left(\Lambda_j\frac{1}{\eta_j^2-\tau_j^2}(\eta_j\sigma_0-\tau_j\sigma_3)\Lambda_j+\Lambda_j\mathcal{C}_z^j\Lambda_j\right)\right).$$

As each of the terms on the right-hand side is covered by the analysis of the single-loop case, the statement on the essential spectrum of $M_z-\Theta$ and thus, with the help of Theorem 2.10, also of $A_{\Sigma,\mathcal{P}}$, follows.

If for some j one has $|\eta_j|=|\tau_j|$, then one follows the same technical strategy as the one in Section 4.2 for $|\eta|=|\tau|$, i.e. one has to deal with additional orthogonal projectors, and all other constructions are easily adapted.

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