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NON-REAL EIGENVALUES OF SINGULAR INDEFINITE STURM-LIOUVILLE OPERATORS

JUSSI BEHRNDT, QUTAIBEH KATATBEH, AND CARSTEN TRUNK

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ABSTRACT. We study a Sturm-Liouville expression with indefinite weight of the form $\operatorname{sgn}(-d^2/dx^2+V)$ on $\mathbb{R}$ and the non-real eigenvalues of an associated selfadjoint operator in a Krein space. For real-valued potentials V with a certain behaviour at $\pm\infty$ we prove that there are no real eigenvalues and that the number of non-real eigenvalues (counting multiplicities) coincides with the number of negative eigenvalues of the selfadjoint operator associated to $-d^2/dx^2+V$ in $L^2(\mathbb{R})$. The general results are illustrated with examples.

1. INTRODUCTION

We consider a singular Sturm-Liouville differential expression of the form

$$\operatorname{sgn}(x)\big(-f''(x)+V(x)f(x)\big),\qquad x\in\mathbb{R}, \tag{1.1}$$

with the signum function as indefinite weight and a real-valued locally summable function V. Under the assumption that $-d^2/dx^2+V$ is in the limit point case at $+\infty$ and $-\infty$ the maximal operator A associated to (1.1) is selfadjoint in the Krein space $(L^2(\mathbb{R}),[\cdot,\cdot])$, where the indefinite inner product $[\cdot,\cdot]$ is defined by

$$[f,g]=\int_{\mathbb{R}} f(x)\overline{g(x)}\operatorname{sgn}(x)\,dx,\qquad f,g\in L^2(\mathbb{R}). \tag{1.2}$$

The spectral properties of indefinite Sturm-Liouville operators differ essentially from the spectral properties of selfadjoint Sturm-Liouville operators in the Hilbert space $L^2(\mathbb{R})$; e.g. the real spectrum of A necessarily accumulates to $+\infty$ and $-\infty$ and A may have non-real eigenvalues which possibly accumulate to the real axis (see [3, 4, 9, 14, 16, 20]). For further indefinite Sturm-Liouville problems, applications and references, see, e.g., [2, 6, 7, 11, 13, 15, 23, 26].

The main objective of this paper is to study the number of non-real eigenvalues of the operator A. For this it will be assumed that the negative spectrum of the selfadjoint definite Sturm-Liouville operator $Bf=-f''+Vf$ consists of $\kappa<\infty$ eigenvalues. Then the hermitian form $[A\cdot,\cdot]$ has exactly κ negative squares, and it follows from the considerations in [9] and [21] that the spectrum $\sigma(A)$ of A in the open upper half-plane $\mathbb{C}^+$ consists of at most κ eigenvalues (counting multiplicities). Inspired by results of I. Knowles from [18, 19], we give a sufficient condition on V such that $\sigma(A)\cap\mathbb{C}^+$ consists of exactly κ eigenvalues (counting multiplicities) and

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the continuous spectrum of A covers the whole real line; see Theorem 2.3 and Corollary 2.4 below. These results can be viewed as a partial answer to the open problem X in [26, p. 300]. We present two explicitly solvable examples illustrating our results. In the first example, potentials of hyperbolic secant type are considered and with the help of numerical methods we find κ different eigenvalues in $\mathbb{C}^+$. The second example shows that, in general, non-real eigenvalues of A may have non-trivial Jordan chains, and hence the number of distinct eigenvalues in $\mathbb{C}^+$ is less than κ.

2. Eigenvalues of indefinite Sturm-Liouville operators

In this section we consider the indefinite Sturm-Liouville differential expression on $\mathbb{R}$ given by (1.1), where $V : \mathbb{R} \to \mathbb{R}$ is a real function with $V \in L^1_{\mathrm{loc}}(\mathbb{R})$. We equip the Hilbert space $(L^2(\mathbb{R}), (\cdot,\cdot))$ with the indefinite inner product $[\cdot,\cdot]$ defined in (1.2) and denote the corresponding Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$ by $L^2_{\mathrm{sgn}}(\mathbb{R})$. As a corresponding fundamental symmetry we choose $J := \mathrm{sgn}(\cdot)$; hence we have $[\cdot,\cdot] = (J\cdot,\cdot)$ and $[J\cdot,\cdot] = (\cdot,\cdot)$. For the basic properties of indefinite inner product spaces and linear operators therein, we refer to [1] and [8].

Suppose that the definite Sturm-Liouville differential expression

$$-\frac{d^2}{dx^2} + V \tag{2.1}$$

is in the limit point case at $+\infty$ and $-\infty$; that is, for each $\lambda \in \mathbb{C}\backslash\mathbb{R}$ there exist (up to scalar multiples) unique solutions of the differential equation $-y'' + Vy = \lambda y$ which are square integrable in a neighbourhood of $+\infty$ and $-\infty$, respectively. A sufficient criterion for (2.1) to be in the limit point case at $\pm\infty$ is, e.g.

$$\liminf_{|x|\to\infty} \frac{V(x)}{x^2} > -\infty;$$

cf. [25, Satz 13.27] or [26, Example 7.4.1].[1] Denote by $\mathcal{D}_{\max}$ the linear space of all $f \in L^2(\mathbb{R})$ such that f and f' are absolutely continuous and $-f'' + Vf \in L^2(\mathbb{R})$ holds. Then it is well-known that the maximal operator

$$Bf := -f'' + Vf, \qquad \mathrm{dom}\, B = \mathcal{D}_{\max}, \tag{2.2}$$

associated to (2.1) is selfadjoint in the Hilbert space $L^2(\mathbb{R})$ and all eigenvalues are real and simple; i.e., $\dim\ker(B-\lambda) = 1$ for $\lambda \in \sigma_p(B)$. As a consequence we obtain the following statement for the operator $A = JB$.

Proposition 2.1. *Assume that* (2.1) *is in the limit point case at* $\pm\infty$*. Then the indefinite Sturm-Liouville operator defined by*

$$(Af)(x) = \mathrm{sgn}\,(x)\big(-f''(x) + V(x)f(x)\big), \quad x \in \mathbb{R}, \quad \mathrm{dom}\, A = \mathcal{D}_{\max}, \tag{2.3}$$

is selfadjoint in the Krein space $L^2_{\mathrm{sgn}}(\mathbb{R})$*, and the eigenspaces* $\ker(A-\lambda)$*,* $\lambda \in \sigma_p(A)$*, have dimension one.*

In the following it will be assumed that condition (I), stated below, holds.

(I) The set $\sigma(B) \cap (-\infty, 0)$ consists of $\kappa < \infty$ eigenvalues.

[1] In the formulation of [26, Example 7.4.1] a minus sign is missing.

Hence, the selfadjoint operator B in the Hilbert space $L^2(\mathbb{R})$ is semi-bounded from below and the eigenvalues do not accumulate to zero from the negative half-axis. A sufficient condition for (I) to hold is, e.g., $\int_{\mathbb{R}}(1+x^2)|V(x)|dx < \infty$ for continuous V; cf. [22].

We collect some properties of the non-real spectrum of the indefinite Sturm-Liouville operator A in the next proposition. Recall first that the spectrum of a selfadjoint operator in a Krein space is symmetric with respect to the real axis and denote by $\mathcal{L}_\lambda(A)$ the algebraic eigenspace of A corresponding to an eigenvalue λ.

Proposition 2.2. *The spectrum of the indefinite Sturm-Liouville operator A in the open upper half-plane $\mathbb{C}^+$ consists of at most finitely many eigenvalues with*

$$\sum_{\lambda\in\sigma_p(A)\cap\mathbb{C}^+} \dim \mathcal{L}_\lambda(A) \leq \kappa, \tag{2.4}$$

where κ is as in (I)*. In particular, for some $l \leq \kappa$ we have*

$$\mathbb{C}\setminus\mathbb{R} \subset \rho(A) \cup \{\lambda_1, \bar{\lambda}_1, \dots, \lambda_l, \bar{\lambda}_l\}. \tag{2.5}$$

If $V(x) = V(-x)$, $x \in \mathbb{R}$, then $\sigma_p(A)$ is symmetric with respect to the imaginary axis.

Proof. Assumption (I) and the relation $[Af, f] = (JAf, f) = (Bf, f)$, $f \in \mathcal{D}_{\max}$, imply that the hermitian form $[A\cdot, \cdot]$ has exactly κ negative squares; that is, there exists a κ-dimensional subspace $\mathcal{M}$ in $\mathcal{D}_{\max}$ such that $[Af, f] < 0$ if $f \in \mathcal{M}$, $f \neq 0$, but no $(\kappa + 1)$-dimensional subspace with this property. This, together with well-known properties of operators with κ negative squares (see, e.g., [21], [9] and [5, Theorem 3.1 and §4.2]) imply (2.4) and (2.5).

Moreover, if V is symmetric, then λ is an eigenvalue of A with corresponding eigenfunction $x \mapsto y(x)$ if and only if $-\lambda$ is an eigenvalue of A with corresponding eigenfunction $x \mapsto y(-x)$. Therefore, as $\sigma_p(A)$ is symmetric with respect to the real axis, $\sigma_p(A)$ is also symmetric with respect to the imaginary axis. □

Under some additional assumptions on V we prove in Theorem 2.3 the absence of eigenvalues on the real axis and, hence, improve the estimate in (2.4). We mention that in [15, Section 4] a similar result is proved if V satisfies $\int_{\mathbb{R}}(1+|x|)|V(x)|dx < \infty$. By $\sigma_c(A)$ we denote the continuous part of the spectrum of A, i.e. the set of all $\lambda \in \sigma(A) \setminus \sigma_p(A)$ such that the range of $A - \lambda$ is dense.

Theorem 2.3. *Assume that condition* (I) *holds and that there exist real functions q and r with $V = q + r$ such that $\lim_{|x|\to\infty} r(x) = \lim_{|x|\to\infty} q(x) = 0$, r is locally of bounded variation and*

$$\lim_{t\to\infty} \frac{1}{\log t} \int_{-t}^{t} |q(x)|dx = \lim_{t\to\infty} \frac{1}{\log t} \int_{-t}^{t} |dr(x)| = 0, \tag{2.6}$$

where dr denotes the measure induced by r. Then $\sigma_c(A)\setminus\{0\} = \mathbb{R}\setminus\{0\}$ and hence zero is the only possible real eigenvalue of the indefinite Sturm-Liouville operator A. If, in addition, $0 \notin \sigma_p(B)$, then we have $\sigma_c(A) = \mathbb{R}$ and

$$\sum_{\lambda\in\sigma_p(A)\cap\mathbb{C}^+} \dim \mathcal{L}_\lambda(A) = \kappa. \tag{2.7}$$

Proof. Let λ be an eigenvalue of A and let y be a corresponding eigenfunction. Then y satisfies the equations

$$-y''(x) + V(x)y(x) = \lambda y(x), \qquad x \in (0,\infty), \tag{2.8}$$

and

$$y''(x) - V(x)y(x) = \lambda y(x), \qquad x \in (-\infty,0). \tag{2.9}$$

Condition (2.6) implies

$$\lim_{t\to\infty} \frac{1}{\log t}\int_0^t |q(x)|dx = \lim_{t\to\infty} \frac{1}{\log t}\int_0^t |dr(x)| = 0.$$

This and [19, Theorem 3.2] applied to (2.8) yield $\lambda \notin (0,\infty)$. Similarly, (2.6) implies

$$\lim_{t\to\infty} \frac{1}{\log t}\int_{-t}^0 |q(x)|dx = \lim_{t\to\infty} \frac{1}{\log t}\int_{-t}^0 |dr(x)| = 0,$$

and, with [19, Theorem 3.2] applied to (2.9), we find $\lambda \notin (-\infty,0)$. Therefore, as a selfadjoint operator in a Krein space has no real points in the residual spectrum (see, e.g., [8, Corollary VI.6.2]), we obtain

$$\big(\sigma(A)\cap(\mathbb{R}\backslash\{0\})\big) \subset \sigma_c(A) \quad \text{and} \quad \sigma_p(A) \subset \{0\}\cup\mathbb{C}\backslash\mathbb{R}.$$

Moreover, from $A = JB$ we get $0 \in \sigma_p(A)$ if and only if $0 \in \sigma_p(B)$. Hence, if $0 \notin \sigma_p(B)$, we conclude $\sigma_p(A) \subset \mathbb{C}\setminus\mathbb{R}$. Since the operator A has exactly κ negative squares (cf. the proof of Proposition 2.2), it follows from, e.g. [5, Theorem 3.1], that A has κ eigenvalues (counted with multiplicities) in $\mathbb{C}^+$ and thus (2.7) holds.

It remains to show $\mathbb{R} \subset \sigma(A)$. For this, consider the differential expressions $\ell_+ = -\frac{d^2}{dx^2} + V$ on $\mathbb{R}^+$ and $\ell_- = \frac{d^2}{dx^2} - V$ on $\mathbb{R}^-$. By assumption ℓ_+ and ℓ_- are regular at zero and in the limit point case at ∞ and $-\infty$, respectively. Let A_+ and A_- be selfadjoint realizations of ℓ_+ and ℓ_- in the Hilbert spaces $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^-)$, respectively, e.g. corresponding to Dirichlet boundary conditions at zero. Under our assumptions

$$\lim_{x\to\infty} V(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} V(x) = 0,$$

and it is well known that $[0,\infty) \subset \sigma(A_+)$ and $(-\infty,0] \subset \sigma(A_-)$ hold. Since the rank of the operator

$$(A-\lambda)^{-1} - \big((A_+\times A_-)-\lambda\big)^{-1}, \qquad \lambda \in \rho(A)\cap\rho(A_+\times A_-),$$

is at most two and $\sigma(A_+\times A_-) = \mathbb{R}$, we conclude $\mathbb{R} \subset \sigma(A)$. □

A sufficient condition on V such that condition (I), (2.6) and $0 \notin \sigma_p(B)$ hold is given in the next corollary; cf. [24, Theorem 14.10], [26, §6.3] and [18, Remark after Corollary 3.3].

Corollary 2.4. *Assume that there exists $x_0 > 0$ with*

$$-\frac{1}{4x^2} \le V(x) \le \frac{3}{4x^2} \qquad \text{for all} \quad x \in \mathbb{R}\setminus(-x_0,x_0). \tag{2.10}$$

Then $\sigma_c(A) = \mathbb{R}$ and

$$\sum_{\lambda\in\sigma_p(A)\cap\mathbb{C}^+} \dim \mathcal{L}_\lambda(A) = \kappa.$$

Remark 2.5. We mention that (even under the condition (2.10)) for $\lambda \in \sigma_p(A)$, $\dim \mathcal{L}_\lambda(A) > 1$ may happen; i.e. there exists a Jordan chain of length greater than one, and the non-real spectrum does not consist of κ distinct eigenvalues. In Section 4 we give an example for an indefinite singular Sturm-Liouville operator with such a Jordan chain corresponding to a non-real eigenvalue.

3. A numerical example: Hyperbolic secant potentials

In this section we compute the non-real eigenvalues of singular indefinite Sturm-Liouville operators with potentials given by

$$V_\kappa(x) = -\kappa(\kappa+1)\operatorname{sech}^2(x), \qquad x \in \mathbb{R} \text{ and } \kappa \in \mathbb{N}, \tag{3.1}$$

with the help of the software package Mathematica (Wolfram Research).

It is well known (see, e.g., [12]) that the number of negative eigenvalues of the definite Sturm-Liouville operator $Bf = -f'' + V_\kappa f$ in (2.2) is exactly κ and condition (I) from Section 2 holds. Moreover, V_κ satisfies (2.10), and hence by Theorem 2.3 and Corollary 2.4 the continuous spectrum of the indefinite Sturm-Liouville operator

$$(Af)(x) = \operatorname{sgn}(x)\big(-f''(x) + V_\kappa(x)f(x)\big), \quad x \in \mathbb{R}, \quad \operatorname{dom} A = \mathcal{D}_{\max},$$

in the Krein space $L^2_{\operatorname{sgn}}(\mathbb{R})$ coincides with $\mathbb{R}$ and $\sum_{\lambda\in\sigma_p(A)\cap\mathbb{C}^+} \dim \mathcal{L}_\lambda(A) = \kappa$ holds. In order to determine the non-real eigenvalues of A, we divide the problem into two parts,

$$\begin{aligned} -y''(x;\lambda) + V_\kappa(x)y(x;\lambda) &= \lambda y(x;\lambda), \qquad x \in \mathbb{R}^+, \\ y''(x;\lambda) - V_\kappa(x)y(x;\lambda) &= \lambda y(x;\lambda), \qquad x \in \mathbb{R}^-. \end{aligned} \tag{3.2}$$

Since the potential V_κ in (3.1) satisfies $V_\kappa(x) = V_\kappa(-x)$ for $x \in \mathbb{R}$, it follows that a function $x \mapsto h(x;\lambda)$, $x \in \mathbb{R}^+$, is a solution of the first differential equation if and only if $x \mapsto h(-x;-\lambda)$, $x \in \mathbb{R}^-$, is a solution of the second differential equation in (3.2). Moreover, as both singular endpoints ∞ and $-\infty$ are in the limit point case, each of the equations in (3.2) has (up to scalar multiples) a unique square integrable solution. Since the functions in $\operatorname{dom} A$ and their derivatives are continuous at the point 0 it follows that $\lambda \in \mathbb{C}\setminus\mathbb{R}$ is an eigenvalue of A if and only if for the square integrable solution $x \mapsto h(x;\lambda)$, $x \in \mathbb{R}^+$, of the first equation in (3.2),

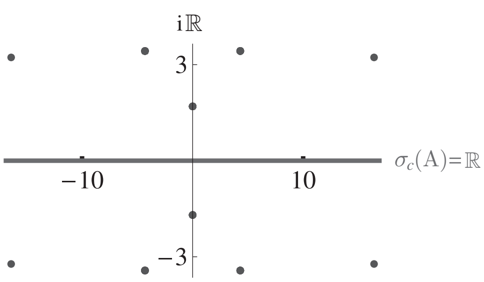

FIGURE 1. The case $\kappa = 5$.

$$h(0;\lambda) = \gamma h(0;-\lambda) \quad \text{and} \quad h'(0;\lambda) = -\gamma h'(0;-\lambda) \tag{3.3}$$

hold for some $\gamma \in \mathbb{C}$. For non-real λ we have $h(0;\lambda) \neq 0$ and $h(0;-\lambda) \neq 0$, and therefore (3.3) is satisfied if and only if the function

$$\mu \mapsto M(\mu) := \frac{h'(0;\mu)}{h(0;\mu)} + \frac{h'(0;-\mu)}{h(0;-\mu)}, \qquad \mu \in \mathbb{C}\setminus\mathbb{R}, \tag{3.4}$$

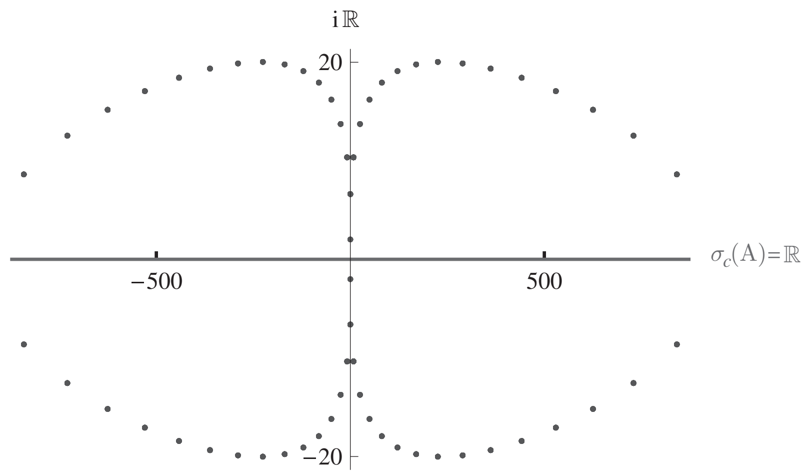

FIGURE 2. The operator $(Ay)(x) := \operatorname{sgn}(x)(-y''(x)+V_{30}(x)y(x))$, $x \in \mathbb{R}$, where $V_{30}(x) = -30 \cdot 31 \operatorname{sech}^2(x)$ has $\kappa = 30$ pairs of non-real eigenvalues.

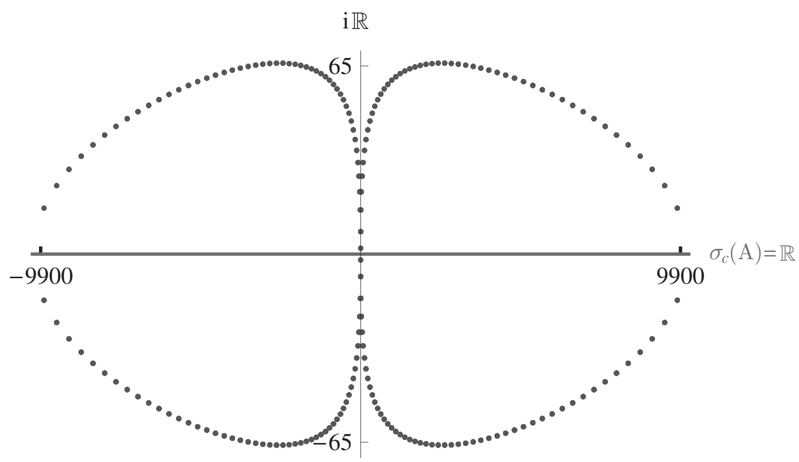

FIGURE 3. The operator $(Ay)(x) := \operatorname{sgn}(x)(-y''(x) + V_{100}(x)y(x))$, $x \in \mathbb{R}$, where $V_{100}(x) = -100 \cdot 101 \operatorname{sech}^2(x)$ has $\kappa = 100$ pairs of non-real eigenvalues.

has a zero at λ. As the equations in (3.2) are explicitly solvable in terms of Legendre functions we can determine numerically the zeros of M within the working default precision of the software package Mathematica.

Figures 1, 2 and 3 show the non-real eigenvalues of A for the cases $\kappa = 5$, $\kappa = 30$ and $\kappa = 100$. Here we find $5, 30$ and 100, respectively, distinct eigenvalues in $\mathbb{C}^+$, and hence $\dim \mathcal{L}_\lambda(A) = 1$ for each eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$; cf. Remark 2.5. Note also that by the symmetry of V_κ there is at least one pair of eigenvalues on the imaginary axis if κ is odd.

4. A counterexample: Jordan chains of singular indefinite Sturm-Liouville operators

In this section we show that the geometric eigenspaces of a singular indefinite Sturm-Liouville operator in $L^2_{\rm sgn}(\mathbb{R})$ in general do not coincide with the algebraic eigenspaces. In other words, there exist eigenvalues with non-trivial Jordan chains, and hence the number of non-real distinct eigenvalues is in general smaller than the dimension of the algebraic eigenspace corresponding to the non-real spectrum; cf. Remark 2.5. An explicit example of a non-trivial Jordan chain of a regular indefinite Sturm-Liouville operator can be found in [10].

We consider a family

$$(A_\eta f)(x) = \operatorname{sgn}(x)(-f''(x) + V_\eta(x)f(x)), \quad x \in \mathbb{R}, \quad \operatorname{dom} A_\eta = \mathcal{D}_{\max}, \quad \eta \geq 0,$$

of indefinite Sturm-Liouville operators in the Krein space $L^2_{\rm sgn}(\mathbb{R})$, where the potentials V_η, $\eta \geq 0$, are given by

$$V_\eta(x) = \begin{cases} 0 & |x| \geq 1, \\ -\eta & |x| < 1, \end{cases} \qquad \eta \geq 0.$$

The operators A_η, $\eta \geq 0$, are selfadjoint in $L^2_{\rm sgn}(\mathbb{R})$, and according to Theorem 2.3 and Corollary 2.4 there are no real eigenvalues and $\sigma_c(A_\eta)$ covers the whole real line. In the sequel we will show that the following statement holds.

Proposition 4.1. *There exist an $\eta_0 > 0$ and a $\lambda_0 \in \mathbb{C}^+$ such that*

$$2 = \dim\ker(A_{\eta_0} - \lambda_0)^2 > \dim\ker(A_{\eta_0} - \lambda_0) = 1.$$

In order to determine the eigenvalues of the operators A_η, we first consider the underlying differential equations (3.2) with V_κ replaced by V_η. The same reasoning as in Section 3 shows that the non-real eigenvalues of A_η are given by the zeros of the function

$$\lambda \mapsto M_\eta(\lambda) := \frac{h_\eta'(0;\lambda)}{h_\eta(0;\lambda)} + \frac{h_\eta'(0;-\lambda)}{h_\eta(0;-\lambda)}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{4.1}$$

where $h_\eta(\cdot;\lambda)$ is the square integrable solution of $-y''+V_\eta y = \lambda y$ on $\mathbb{R}^+$. Denote by $\sqrt{\cdot}$ the branch of the square root with cut along $[0,\infty)$ and $\sqrt{x} \geq 0$ for $x \in [0,\infty)$. Then it is easy to check that for $\lambda \notin [0,\infty)$ the function

$$h_\eta(x;\lambda) = \begin{cases} \exp(i\sqrt{\lambda}x) & x > 1, \\ \alpha_\eta(\lambda)\exp(i\sqrt{\lambda+\eta}x) + \beta_\eta(\lambda)\exp(-i\sqrt{\lambda+\eta}x) & x \in [0,1], \end{cases}$$

where

$$\alpha_\eta(\lambda) = \frac{1}{2}\left(1 + \sqrt{\lambda(\lambda+\eta)^{-1}}\right)\exp\big(i(\sqrt{\lambda} - \sqrt{\lambda+\eta})\big)$$

and

$$\beta_\eta(\lambda) = \frac{1}{2}\left(1 - \sqrt{\lambda(\lambda+\eta^{-1}}\right)\exp\big(i(\sqrt{\lambda} + \sqrt{\lambda+\eta})\big)$$

and its multiples are square integrable solutions of the first equation in (3.2) with V_κ replaced by V_η.

The function M_η in (4.1) can be expressed in terms of α_η and β_η in the following form:

$$M_\eta(\lambda) = i\sqrt{\lambda+\eta}\,\frac{\alpha_\eta(\lambda) - \beta_\eta(\lambda)}{\alpha_\eta(\lambda)+\beta_\eta(\lambda)} + i\sqrt{\eta-\lambda}\,\frac{\alpha_\eta(-\lambda)-\beta_\eta(-\lambda)}{\alpha_\eta(-\lambda)+\beta_\eta(-\lambda)}.$$

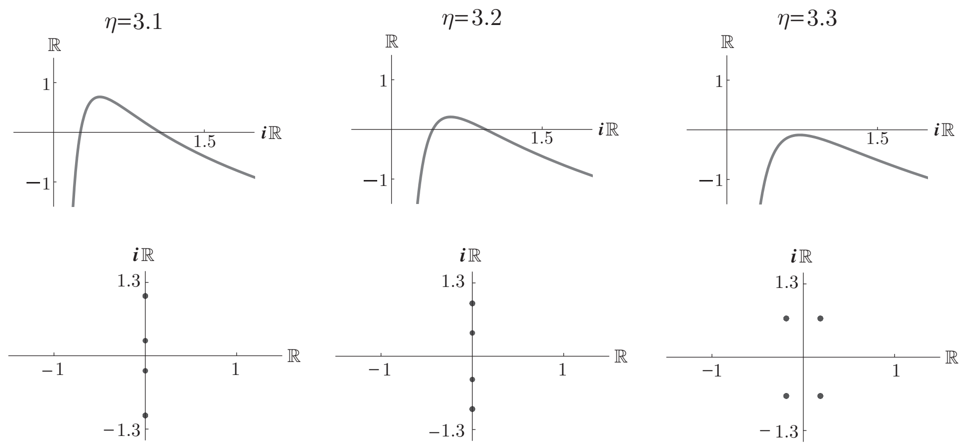

FIGURE 4. In the first row the function $\mu \mapsto M_\eta(i\mu)$ is plotted for $\mu > 0$ and $\eta = 3.1$, $\eta = 3.2$ and $\eta = 3.3$, respectively. In the second row the corresponding non-real eigenvalues of the operators $A_{3.1}$, $A_{3.2}$ and $A_{3.3}$ are shown.

We note that the values $M_\eta(i\mu)$, $\mu \in \mathbb{R}\backslash\{0\}$, are real since the solutions fulfill $h_\eta(x,\bar\lambda) = \overline{h_\eta(x,\lambda)}$ for $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Let us summarize some observations in the following lemma.

Lemma 4.2. *A non-real number λ is an eigenvalue of the indefinite Sturm-Liouville operator A_η if and only if $M_\eta(\lambda) = 0$. The restriction of M_η onto the imaginary axis is a real-valued function, and the non-real eigenvalues of A_η are symmetric with respect to the real and imaginary axes.*

One can check numerically that the selfadjoint operator $B_\eta = -\frac{d^2}{dx^2} + V_\eta$, $\operatorname{dom} B_\eta = \mathcal{D}_{\max}$, in the Hilbert space $L^2(\mathbb{R})$ has exactly two negative eigenvalues for $\eta = 3.1$, $\eta = 3.2$ and $\eta = 3.3$; cf. [12]. By Corollary 2.4 for these η the spectral subspace of A_η correponding to the eigenvalues in the upper half-plane $\mathbb{C}^+$ has dimension two.

The plots in the first row of Figure 4 show the function $\mu \mapsto M_\eta(i\mu)$, $\mu \in \mathbb{R}^+$, for $\eta = 3.1$, $\eta = 3.2$ and $\eta = 3.3$, respectively. For $\eta = 3.1$ and $\eta = 3.2$ the two zeros are the eigenvalues of $A_{3.1}$ and $A_{3.2}$ in the upper half-plane $\mathbb{C}^+$ which lie on the positive imaginary axis. These eigenvalues and their counterparts in $\mathbb{C}^-$ are plotted in the second row of Figure 4. For $\eta = 3.3$ the function $\mu \mapsto M_\eta(i\mu)$ has no zeros on the positive imaginary axis. Recall that a finite system of eigenvalues is continuous under perturbations small in norm; see [17, IV.3.5]. Hence the continuity and symmetry of the eigenvalues of A_η imply that the eigenvalues of $A_{3.3}$ are located as in the right lower plot in Figure 4. This can also be checked numerically by computing the non-real roots of $M_{3.3}$; see also Table 1. Again by continuity properties of the point spectrum there exists an $\eta_0 \in (3.2, 3.3)$ such that the spectrum of A_{η_0} in $\mathbb{C}^+$ (and hence also in $\mathbb{C}^-$) consists only of one eigenvalue λ_0 on the imaginary axis with corresponding algebraic eigenspace of dimension two. Recall that the dimension of the geometric eigenspaces of A_{η_0} is at most one since ∞ and $-\infty$ are in the limit point case. Hence there exists a Jordan chain of length

Table 1. In bold face is the (approximative) value of η where the eigenvalues $\lambda_{1,\eta}$ and $\lambda_{2,\eta}$ of A_η in $\mathbb{C}^+$ coincide and we have a Jordan chain of length two. With further increasing η, the eigenvalues $\lambda_{1,\eta}$ and $\lambda_{2,\eta}$ move away from the imaginary axis.

η	$\lambda_{1,\eta}$	$\lambda_{2,\eta}$
3.10000000000	0.26723799239 i	1.05923928894 i
3.26656565972	0.64287403712 i	0.72260288819 i
3.26796097363	0.67270918484 i	0.69312432044 i
3.26805876683	0.68293354062 i	0.68293354054 i
3.26805876685	**0.68292928856** i	**0.68292928856** i
3.26805890000	0.0003766+0.6829292 i	-0.0003766+0.6829292 i
3.27021280983	0.0479471+0.6832050 i	-0.0479471+0.6832050 i
3.28021280983	0.1143198+0.6844929 i	-0.1143198+0.6844929 i
3.30000000000	0.1866925 + 0.687078 i	-0.1866925 + 0.687078 i

two of A_{η_0} at the eigenvalue λ_0 (and $\bar{\lambda}_0$). We remark that for the function (4.1) we have $M_{\eta_0}(\lambda_0) = M'_{\eta_0}(\lambda_0) = 0$.

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Department of Mathematics MA 6–4, Technische Universität Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: behrndt@math.tu-berlin.de

Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan

E-mail address: qutaibeh@yahoo.com

Department of Mathematics, Technische Universität Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany

E-mail address: carsten.trunk@tu-ilmenau.de