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Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples

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Abstract The notion of quasi boundary triples and their Weyl functions is reviewed and applied to self-adjointness and spectral problems for a class of elliptic, formally symmetric, second order partial differential expressions with variable coefficients on bounded domains.

1.1 Introduction

Boundary triples and associated Weyl functions are a powerful and efficient tool to parameterize the self-adjoint extensions of a symmetric operator and to describe their spectral properties. There are numerous papers applying boundary triple techniques to spectral problems for various types of ordinary differential operators in Hilbert spaces; see, e.g. [Behrndt and Langer, 2010; Behrndt, Malamud and Neidhardt, 2008; Behrndt and Trunk, 2007; Brasche, Malamud and Neidhardt, 2002; Brüning, Geyler and Pankrashkin, 2008; Derkach, Hassi and de Snoo, 2003; Gorbachuk and Gorbachuk, 1991; Derkach and Malamud, 1995; Karabash, Kostenko and Malamud, 2009; Kostenko and Malamud, 2010; Posilicano, 2008] and the references therein.

The abstract notion of boundary triples and Weyl functions is strongly inspired by Sturm–Liouville operators on a half-line and their Titchmarsh–Weyl coefficients. To make this more precise, let us consider the ordinary differential expression $\ell = -D^2 + q$ on the half-line $\mathbb{R}^+ = (0,\infty)$, where D denotes the derivative, and suppose that q is a real-valued L^∞-function. The maximal operator associated with ℓ in $L^2(\mathbb{R}^+)$ is defined on the Sobolev space $H^2(\mathbb{R}^+)$ and turns out to be the adjoint of the minimal operator $Sf = \ell(f)$, $\operatorname{dom} S = H^2_0(\mathbb{R}^+)$, where $H^2_0(\mathbb{R}^+)$ is the subspace of $H^2(\mathbb{R}^+)$ consisting of functions f that satisfy the boundary conditions $f(0) = f'(0) = 0$. Here S is a densely defined closed symmetric operator in $L^2(\mathbb{R}^+)$ with deficiency numbers $(1,1)$. In this situation it is natural to define boundary mappings Γ_0 and Γ_1 on the domain $H^2(\mathbb{R}^+)$ of the maximal operator S^* (the adjoint of S) by

$$\Gamma_0, \Gamma_1 \colon \operatorname{dom} S^* \to \mathbb{C}, \qquad \Gamma_0 f := f(0) \quad \text{and} \quad \Gamma_1 f := f'(0).$$

The mapping $(\Gamma_0; \Gamma_1)^\top : \operatorname{dom} S^* \to \mathbb{C} \times \mathbb{C}$ is surjective, and the Lagrange identity reads as

$$(S^* f, g) - (f, S^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \qquad f, g \in \operatorname{dom} S^*,$$

where on the right-hand side the standard inner product of the boundary space $\mathbb{C}$ is used. The abstract Lagrange identity and the surjectivity of $(\Gamma_0; \Gamma_1)^\top$ are the defining relations for a boundary triple, in this case the triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$. In a general situation, one has a triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ where $\mathcal{G}$ is an auxiliary Hilbert space (the space of boundary values) and Γ_0, Γ_1 are linear mappings from $\operatorname{dom} S^*$ to $\mathcal{G}$; see Definition 1.1 below for details. The self-adjoint extensions of the symmetric Sturm–Liouville operator S can be parameterized in the form

$$A_\alpha f = \ell(f), \qquad \operatorname{dom} A_\alpha = \bigl\{ f \in \operatorname{dom} S^* \colon \Gamma_1 f = \alpha \Gamma_0 f \bigr\},$$

where $\alpha \in \mathbb{R} \cup \{\infty\}$. For an arbitrary closed symmetric operator with equal deficiency indices and a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for its adjoint, the domains of the self-adjoint extensions A_Θ are characterized formally in the same way, namely by the boundary conditions $\Gamma_1 f = \Theta \Gamma_0 f$, where Θ is a self-adjoint operator (or relation) in $\mathcal{G}$; cf. Proposition 1.2.

If S is an arbitrary closed symmetric operator with equal deficiency indices in some Hilbert space and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triple for the adjoint S^*, then the corresponding Weyl function M is defined as the map $\Gamma_0 f_\lambda \mapsto \Gamma_1 f_\lambda$, where f_λ belongs to $\ker(S^* - \lambda)$ and $\lambda \in \mathbb{C} \backslash \mathbb{R}$. In the Sturm–Liouville case the Weyl function corresponding to the boundary triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a scalar analytic function defined on $\mathbb{C} \backslash \mathbb{R}$, which maps Dirichlet boundary values $f_\lambda(0)$ of H^2-solutions f_λ of the differential equation $\ell(u) = \lambda u$ onto their Neumann boundary values $f'_\lambda(0)$. We note that the Weyl function M coincides with the Titchmarsh–Weyl coefficient associated with ℓ. In Sturm–Liouville theory it is well known that the complete spectral information of the self-adjoint realizations is encoded in the Titchmarsh–Weyl coefficient, that is, in the Weyl function of the boundary triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$. For example, the spectrum of the Dirichlet operator equals the set of real numbers to which M cannot be continued analytically; isolated eigenvalues coincide with poles of M.

Similar considerations can be made for a regular Sturm–Liouville expression $\ell = -D^2 + q$ on the interval $[0,1]$; here a usual choice of a boundary triple is

$$\mathcal{G} = \mathbb{C}^2, \qquad \Gamma_0 f = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f = \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix},$$

and the poles of the corresponding Weyl function (which is a 2×2-matrix function in this case) coincide with the eigenvalues of the Dirichlet operator. For more details, see Section 1.2.

Motivated by the above considerations for the case of second order ordinary differential operators it seems very desirable and natural to adapt the boundary triple concept in such a form that it can be applied to elliptic, formally symmetric, second order differential operators of the form

$$\mathcal{L} = -\sum_{j,k=1}^{n} \partial_j \, a_{jk} \, \partial_k + a$$

with variable coefficients a_{jk} and a on a bounded domain Ω by choosing the boundary mappings

$$\Gamma_0 f := f|_{\partial\Omega} \qquad \text{and} \qquad \Gamma_1 f := -\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = -\sum_{j,k=1}^{n} a_{jk} \mathfrak{n}_j \partial_k f\Big|_{\partial\Omega} \tag{1.1}$$

as the Dirichlet and (oblique) Neumann trace map, respectively. Here $\mathfrak{n}$ denotes the outward normal vector on $\partial\Omega$. One of the main motivations to choose the boundary maps in (1.1) is that in applications usually Dirichlet and Neumann data are used and that (formally) the corresponding Weyl function M coincides (up to a minus sign) with the Dirichlet-to-Neumann map. For $f, g \in H^2(\Omega)$, Green's identity takes the form

$$(\mathcal{L}f, g) - (f, \mathcal{L}g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g),$$

where the $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products appear on the left-hand and right-hand sides, respectively. However, $H^2(\Omega)$ is only a core for the maximal operator associated with $\mathcal{L}$, which is defined on the set

$$\mathfrak{D}_{\max} = \{ f \in L^2(\Omega) \colon \mathcal{L}(f) \in L^2(\Omega) \};$$

moreover, the mapping $(\Gamma_0; \Gamma_1)^\top \colon H^2(\Omega) \to L^2(\partial\Omega) \times L^2(\partial\Omega)$ is not surjective, but its range is only dense. Green's identity in the above form cannot be extended to functions f, g in $\mathfrak{D}_{\max}$. Therefore the triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ with Γ_0 and Γ_1 as in (1.1) is not a boundary triple in the usual sense and cannot be turned into one by enlarging the domain of the boundary mappings. These simple observations led to a generalization of the notion of boundary triples in [Behrndt and Langer, 2007]. There the concept of *quasi boundary triples* was introduced in an abstract setting and applied to second order elliptic differential operators on bounded domains. The $H^2(\Omega)$ setting, as well as the case of the larger domain

$$H^{3/2}_{\mathcal{L}}(\Omega) := \big\{ f \in H^{3/2}(\Omega)\colon \mathcal{L}(f) \in L^2(\Omega) \big\}$$

for the boundary mappings was discussed there in detail. In contrast to (ordinary) boundary triples there is no bijective correspondence of self-adjoint extensions A_Θ of the underlying symmetric operator S and self-adjoint parameters Θ in the boundary space via the formula

$$\Theta \mapsto A_\Theta = S^* \upharpoonright \{ f \in \operatorname{dom} S^* \colon \Gamma_1 f = \Theta \Gamma_0 f \}.$$

However, sufficient conditions for self-adjointness can be given with the help of a version of Krein's formula (see Theorems 1.17, 1.18 and 1.21). Many papers have been written on Krein's formula; see, e.g. [Derkach and Malamud, 1991; Gesztesy, Makarov and Tsekanovskii, 1998; Kreĭn, 1946; Langer and Textorius, 1977; Pankrashkin, 2006; Saakjan, 1965] for the general case and, e.g. [Gesztesy and Mitrea, 2008, 2009, 2011; Grubb, 2008; Posilicano and Raimondi, 2009] for applications to PDEs. Here we present a version of Krein's formula that is slightly more general than the one in [Behrndt and Langer, 2007] (but compare also [Behrndt, Langer and Lotoreichik, 2011]):

$$(A_\Theta - \lambda)^{-1} = (A_D - \lambda)^{-1} - \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^*,$$

which holds for any Θ such that $\rho(A_\Theta) \cap \rho(A_D) \neq \varnothing$ (see Theorem 1.22, and Theorem 1.16, and their corollaries for the abstract setting); here $\gamma(\lambda)$ is the so-called γ-field, which maps boundary values y onto solutions f of the equation $\mathcal{L}(f) = \lambda f$ with $\Gamma_0 f = y$.

Let us point out that a concept that is similar to quasi boundary triples was introduced and studied by V. Ryzhov independently in [Ryzhov, 2007, 2009]. In [Malinen and Staffans, 2007, Section 6.2] the idea to restrict boundary mappings in connection with colligations and boundary nodes is also used. A boundary triple concept for first order operators was introduced in [Post, 2007]. Other generalizations of boundary triples and abstract concepts of boundary mappings were studied, e.g. in [Arlinskiĭ, 2000; Derkach et al., 2006, 2009; Kopachevskiĭ and Kreĭn, 2004; Posilicano, 2004, 2008; Mogilevskiĭ, 2006, 2009],

The aim of the present paper is to give an introduction to and an overview of the properties of quasi boundary triples, associated γ-fields and Weyl functions, and to demonstrate how conveniently this technique can be applied to boundary value and spectral problems for elliptic operators. For simplicity, we restrict ourselves to the case that the underlying symmetric operator S is densely defined, so that the adjoint S^* is an operator and not a multi-valued linear relation. For the general case we refer the reader to [Behrndt and Langer, 2007]. In Section 2 we start by recalling the notion of boundary triples and Weyl functions and collect some well-known properties of these objects. Furthermore, we show in examples how boundary triples can be applied to ordinary, as well as elliptic differential operators. The boundary triple for an elliptic operator from Section 1.2 can already be found in a slightly different context in [Grubb, 1968] and was studied in slight variations in [Brown, Grubb and Wood, 2009; Brown et al., 2009, 2008; Grubb; Malamud, 2010; Posilicano, 2008; Posilicano and Raimondi, 2009]. We emphasize that there the boundary mapping Γ_1 is a regularized variant of the Neumann trace in (1.1) and that the corresponding Weyl function is not the Dirichlet-to-Neumann map.

The notion of quasi boundary triples, their γ-fields and Weyl functions is reviewed in Section 1.3. We also provide a full proof of Krein's formula, which is difficult to find in the literature in this form; cf. [Behrndt and Langer, 2007; Behrndt, Langer and Lotoreichik, 2011]. Furthermore, we give some sufficient criteria for self-adjointness of the extensions of the underlying symmetric operator. In Section 1.4 the quasi boundary triple concept is then applied to the elliptic differential expression $\mathcal{L}$. Here we have decided to work on the scale of spaces $H^s_{\mathcal{L}}(\Omega)$, $s \in [\frac{3}{2}, 2]$, the largest possible range of values of s for our purposes. We stress again that the essential idea here is to use the Dirichlet and Neumann boundary mappings Γ_0 and Γ_1 from (1.1) and to identify the corresponding Weyl function with the Dirichlet-to-Neumann map from the theory of elliptic differential equations. Furthermore, we compare and connect the quasi boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ and its Weyl function with the regularized (ordinary) boundary triple from Section 1.2 and the associated Weyl function.

Let us finish this introduction by fixing some notation. For Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, denote by $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$ the space of everywhere defined bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$; moreover, we set $\mathfrak{B}(\mathcal{H}_1) := \mathfrak{B}(\mathcal{H}_1, \mathcal{H}_1)$. For abstract boundary conditions we need linear relations in the boundary space; so let us recall a couple of definitions. A linear relation (or, in short, relation) T in a Hilbert space $\mathcal{H}$ is a subspace of the cartesian product $\mathcal{H} \times \mathcal{H}$; operators are identified as linear relations via their graphs. The elements in a linear relation T are denoted in the form $(f;g)^\top$ or as column vectors. For a linear relation T we define its domain, range, multi-valued part, inverse and adjoint as

$$\begin{aligned}
\operatorname{dom} T &:= \bigl\{ f : \exists\, g \text{ such that } (f;g)^\top \in T \bigr\}, \\
\operatorname{ran} T &:= \bigl\{ g : \exists\, f \text{ such that } (f;g)^\top \in T \bigr\}, \\
\ker T &:= \bigl\{ f : \text{ such that } (f;0)^\top \in T \bigr\}, \\
\operatorname{mul} T &:= \bigl\{ g : \text{ such that } (0;g)^\top \in T \bigr\}, \\
T^{-1} &:= \bigl\{ (g;f)^\top : (f;g)^\top \in T \bigr\}, \\
T^* &:= \bigl\{ (f;g)^\top : (v,f) = (u,g) \text{ for all } (u;v)^\top \in T \bigr\}.
\end{aligned}$$

A linear relation T in a Hilbert space $\mathcal{H}$ is called *symmetric* if $T \subset T^*$, and *self-adjoint* if $T = T^*$. The relation T is called *dissipative* if $\operatorname{Im}(g,f) \ge 0$ for all $(f;g)^\top \in T$ and *accumulative* if $\operatorname{Im}(g,f) \le 0$ for all $(f;g)^\top \in T$; T is called *maximal dissipative* (*maximal accumulative*) if T is dissipative (or accumulative, respectively) and $\operatorname{ran}(T - \lambda) = \mathcal{H}$ for $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$, respectively), where $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the upper and lower half-planes. The sum of two linear relations T_1, T_2 is defined as

$$T_1 + T_2 := \bigl\{ (f; g+h)^\top : (f;g)^\top \in T_1,\ (f;h)^\top \in T_2 \bigr\}.$$

The real and imaginary parts of a linear relation are defined as

$$\operatorname{Re} T = \frac{1}{2}\bigl(T + T^*\bigr), \qquad \operatorname{Im} T = \frac{1}{2\mathrm{i}}\bigl(T - T^*\bigr).$$

1.2 Boundary triples and Weyl functions for ordinary and partial differential operators

In this section we first review the concept of boundary triples and associated Weyl functions from abstract extension theory of symmetric operators in Hilbert spaces. As a standard example we discuss the case of a regular Sturm–Liouville operator. Furthermore, as a second example a class of second order elliptic differential operators is studied.

Boundary triples and Weyl functions Let $(\mathcal{H}, (\cdot,-))$ be a Hilbert space and let in the following S be a densely defined closed symmetric operator in $\mathcal{H}$. Everything what follows can be done also for non-densely defined operators S, in which case the adjoint S^* is a proper relation, but for simplicity we restrict ourselves to the case that S is densely defined. This is also sufficient for the applications we have in mind here. First we recall the notion of a boundary triple (originally also called boundary value space), which nowadays is very popular in extension theory of symmetric operators; cf. [Brown et al., 2008; Bruk, 1976; Brüning, Geyler and Pankrashkin, 2008; Derkach and Malamud, 1991, 1995; Gorbachuk and Gorbachuk, 1991; Kochubei, 1975; Malamud, 1992].

Definition 1.1 A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be a *boundary triple* for the adjoint operator S^* if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ are linear mappings such that

$$(S^* f, g) - (f, S^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) \tag{1.2}$$

holds for all $f, g \in \operatorname{dom} S^*$ and the map $\Gamma := (\Gamma_0; \Gamma_1)^\top : \operatorname{dom} S^* \to \mathcal{G} \times \mathcal{G}$ is surjective.

A boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* exists if and only if the deficiency numbers $n_\pm(S) = \dim \ker(S^* \pm i)$ are equal, that is, if and only if S admits self-adjoint extensions in $\mathcal{H}$. It follows that $\dim \mathcal{G} = n_\pm(S)$, and we point out that $\dim \mathcal{G}$ may be infinite. Moreover, if $S \neq S^*$ then a boundary triple for S^* (if it exists) is not unique.

In the following let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^*. Then the map $\Gamma = (\Gamma_0; \Gamma_1)^\top : \operatorname{dom} S^* \to \mathcal{G} \times \mathcal{G}$ is closed, continuous with respect to the graph norm of S^*, and

$$\operatorname{dom} S = \ker \Gamma = \ker \Gamma_0 \cap \ker \Gamma_1$$

holds. Furthermore, the restrictions of S^* to the dense subspaces $\ker \Gamma_0$ and $\ker \Gamma_1$,

$$A_0 := S^* \upharpoonright \ker \Gamma_0 \qquad \text{and} \qquad A_1 := S^* \upharpoonright \ker \Gamma_1,$$

are self-adjoint extensions of S in $\mathcal{H}$ which are transversal, that is, $\operatorname{dom} A_0 \cap \operatorname{dom} A_1 = \operatorname{dom} S$ and $\operatorname{dom} A_0 + \operatorname{dom} A_1 = \operatorname{dom} S^*$ hold. With the help of the boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ all closed extensions of S which are restrictions of S^* can be parameterized in a convenient way; see, e.g. [Derkach and Malamud, 1995, Proposition 1.4].

Proposition 1.2 *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^*. Then*

$$\Theta \mapsto A_\Theta := S^* \upharpoonright \ker(\Gamma_1 - \Theta \Gamma_0) \tag{1.3}$$

establishes a bijective correspondence between the closed linear relations

Θ in $\mathcal{G}$ and the closed extensions $A_\Theta \subset S^*$ of S. Furthermore, for every closed linear relation Θ in $\mathcal{G}$ the identity

$$(A_\Theta)^* = A_{\Theta^*}$$

holds, and A_Θ is a self-adjoint (symmetric, (maximal) dissipative, (maximal) accumulative) operator in $\mathcal{H}$ if and only if Θ is a self-adjoint (symmetric, (maximal) dissipative, (maximal) accumulative, respectively) relation in $\mathcal{G}$.

We mention that the dense subspace $\ker(\Gamma_1 - \Theta\Gamma_0)$ on the right-hand side of (1.3) coincides with

$$\big\{f \in \operatorname{dom} S^* \colon \Gamma f = (\Gamma_0 f; \Gamma_1 f)^\top \in \Theta\big\} = \Gamma^{-1}(\Theta) \tag{1.4}$$

and that the expression $\Gamma_1 - \Theta\Gamma_0$ has to be interpreted in the sense of linear relations if $\operatorname{mul}\Theta \neq \{0\}$. Observe that a linear relation Θ in $\mathcal{G}$ is self-adjoint if and only if there exists a pair $\{\Phi, \Psi\}$ of operators in $\mathcal{G}$ with the properties

$$\Phi, \Psi \in \mathfrak{B}(\mathcal{G}), \quad \Psi^*\Phi = \Phi^*\Psi \quad \text{and} \quad 0 \in \rho(\Psi \pm i\Phi) \tag{1.5}$$

such that

$$\Theta = \big\{(\Phi k; \Psi k)^\top : k \in \mathcal{G}\big\} = \big\{(h; h')^\top \in \mathcal{G} \times \mathcal{G} : \Psi^* h = \Phi^* h'\big\}. \tag{1.6}$$

With the help of this representation the condition (1.4) can also be written in the form

$$\big\{f \in \operatorname{dom} S^* : \Psi^*\Gamma_0 f = \Phi^*\Gamma_1 f\big\},$$

and hence the corresponding self-adjoint operator A_Θ in (1.3) is given by

$$A_\Theta = S^* \upharpoonright \ker(\Psi^*\Gamma_0 - \Phi^*\Gamma_1).$$

The following theorem from [Behrndt and Langer, 2010] is of a certain inverse nature and can be used to determine the adjoint of a given symmetric operator with the help of boundary mappings that satisfy (1.2) and a maximality condition. Very roughly speaking the problem of determining the adjoint is reduced to the much easier problem of checking self-adjointness. The method is inspired by the theory of isometric and unitary operators between indefinite inner product spaces; see, e.g. [Azizov and Iokhvidov, 1989; Derkach et al., 2006; Šmuljan, 1976].

Theorem 1.3 *Let T be a linear operator in $\mathcal{H}$ and let $\mathcal{G}$ be a Hilbert space. Assume that $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings which satisfy the following conditions:*

(i) *there exists a symmetric operator or relation Θ in $\mathcal{G}$ such that*

$$T \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$$

is the extension of a self-adjoint operator A in $\mathcal{H}$;

(ii) $\operatorname{ran}(\Gamma_0; \Gamma_1)^\top = \mathcal{G} \times \mathcal{G}$*;*

(iii) $(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)$ *for all* $f, g \in \operatorname{dom} T$*.*

Then the operator

$$S := T \upharpoonright \ker\Gamma_0 \cap \ker\Gamma_1$$

is a densely defined closed symmetric operator in $\mathcal{H}$ such that $S^* = T$ and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triple for S^*. Furthermore, Θ is a self-adjoint operator or relation in $\mathcal{G}$, and $A = S^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0) = A_\Theta$ holds.

Next the notion and essential properties of the γ-field and Weyl function corresponding to a boundary triple are recalled. Let again S be a densely defined closed symmetric operator in $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* with $A_0 = S^* \upharpoonright \ker\Gamma_0$. We first define $\mathcal{N}_\lambda(S^*) := \ker(S^* - \lambda)$ for $\lambda \in \mathbb{C}$. It follows from $A_0 = A_0^*$ that for all $\lambda \in \rho(A_0)$ the domain of S^* can be decomposed into a direct sum:

$$\operatorname{dom} S^* = \operatorname{dom} A_0 \dot{+} \mathcal{N}_\lambda(S^*) = \ker\Gamma_0 \dot{+} \mathcal{N}_\lambda(S^*). \tag{1.7}$$

In particular, the restriction of the map Γ_0 to $\mathcal{N}_\lambda(S^*)$, $\lambda \in \rho(A_0)$, is injective, and as a consequence of $\operatorname{ran}\Gamma_0 = \mathcal{G}$ it follows that $\Gamma_0 \upharpoonright \mathcal{N}_\lambda(S^*) \to \mathcal{G}$ is bijective.

Definition 1.4 The *γ-field* γ and *Weyl function* M corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* are defined by

$$\left.\begin{aligned} \gamma(\lambda) &:= \bigl(\Gamma_0 \upharpoonright \mathcal{N}_\lambda(S^*)\bigr)^{-1}, \\ M(\lambda) &:= \Gamma_1\gamma(\lambda) = \Gamma_1\bigl(\Gamma_0 \upharpoonright \mathcal{N}_\lambda(S^*)\bigr)^{-1}, \end{aligned}\right. \qquad \lambda \in \rho(A_0).$$

In the next two propositions we collect the basic properties of the γ-field and Weyl function of a boundary triple; see [Derkach and Malamud, 1991, Lemma 1 and Theorem 1] and Proposition 1.14 (iv) for the particular form of M in (1.9).

Proposition 1.5 *Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a boundary triple for S^* with $A_0 = S^* \restriction \ker\Gamma_0$. Then the corresponding γ-field $\lambda \mapsto \gamma(\lambda)$ is a holomorphic $\mathfrak{B}(\mathcal{G},\mathcal{H})$-valued function on $\rho(A_0)$, and the identities*

$$\gamma(\lambda) = \big(I + (\lambda-\mu)(A_0-\lambda)^{-1}\big)\gamma(\mu) \quad \textit{and} \quad \gamma(\bar\lambda)^* = \Gamma_1(A_0-\lambda)^{-1}$$

hold for all $\lambda,\mu \in \rho(A_0)$.

Proposition 1.6 *Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a boundary triple for S^* with $A_0 = S^* \restriction \ker\Gamma_0$. Then the corresponding Weyl function M is a holomorphic $\mathfrak{B}(\mathcal{G})$-valued function on $\rho(A_0)$, and the identities*

$$M(\lambda) - M(\mu)^* = (\lambda-\bar\mu)\gamma(\mu)^*\gamma(\lambda) \tag{1.8}$$

and

$$M(\lambda) = \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^*\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\bar\lambda_0)(A_0-\lambda)^{-1}\big)\gamma(\lambda_0) \tag{1.9}$$

hold for all $\lambda,\mu \in \rho(A_0)$ and any fixed $\lambda_0 \in \rho(A_0)$.

The identity (1.8) yields that the Weyl function M is a so-called *Nevanlinna* (or *Herglotz*) *function*, that is, M is holomorphic on $\mathbb{C}\backslash\mathbb{R}$, $M(\lambda) = M(\bar\lambda)^*$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $\operatorname{Im} M(\lambda)$ is a non-negative operator for all λ in the upper half-plane $\mathbb{C}^+$; see, e.g. [Gesztesy and Tsekanovskii, 2000; Kac and Kreĭn, 1974]. Moreover, it follows from (1.8) that $0 \in \rho(\operatorname{Im} M(\lambda))$ if $\lambda \in \mathbb{C}\backslash\mathbb{R}$, i.e. M is a *uniformly strict* Nevanlinna function; cf. [Derkach et al., 2006, p. 5354]. Conversely, every uniformly strict Nevanlinna function is the Weyl function of some boundary triple (where S may be non-densely defined); see [Derkach and Malamud, 1995, Section 5] and [Langer and Textorius, 1977]. We also mention, that a $\mathfrak{B}(\mathcal{G})$-valued function N is a Nevanlinna function if and only if there exist self-adjoint operators $\alpha,\beta \in \mathfrak{B}(\mathcal{G})$, $\beta \geq 0$, and a non-decreasing self-adjoint operator function $t \mapsto \Sigma(t) \in \mathfrak{B}(\mathcal{G})$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\Sigma(t) \in \mathfrak{B}(\mathcal{G})$ and

$$N(\lambda) = \alpha + \lambda\beta + \int_{-\infty}^{\infty} \Big(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\Big)\, d\Sigma(t), \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Let again $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a boundary triple for S^*. With the help of the corresponding Weyl function M the spectral properties of the closed extensions of S can be described. Roughly speaking the spectrum of A_Θ can be described by means of the singularities of the function $\lambda \mapsto (\Theta - M(\lambda))^{-1}$. The following theorem, see, e.g. [Derkach and Malamud, 1991, Propositions 1 and 2], illustrates this and provides a variant of Krein's formula for canonical extensions (which are not necessarily self-adjoint).

Theorem 1.7 *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* with γ-field γ and Weyl function M. Let $A_0 = S^* \upharpoonright \ker \Gamma_0$ and let A_Θ be a closed extension of S corresponding to some Θ via (1.3)–(1.4). Then the following statements hold for all $\lambda \in \rho(A_0)$:*

(i) $\lambda \in \rho(A_\Theta)$ *if and only if* $0 \in \rho(\Theta - M(\lambda))$*;*

(ii) $\lambda \in \sigma_i(A_\Theta)$ *if and only if* $0 \in \sigma_i(\Theta - M(\lambda))$, $i = \mathrm{p}, \mathrm{c}, \mathrm{r}$, *where* σ_p, σ_c, σ_r *denote the point, continuous and residual spectrum, respectively;*

(iii) *for all* $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$,

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^*.$$

We mention that Krein's formula in Theorem 1.7 (iii) above can also be written in the form

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\Phi\big(\Psi - M(\lambda)\Phi\big)^{-1}\gamma(\bar{\lambda})^*$$

if Θ is a self-adjoint relation in $\mathcal{G}$ which is represented by a pair $\{\Phi, \Psi\}$ as in (1.6).

We point out that the spectral characterization with the help of the Weyl function in Theorem 1.7 (i)–(ii) is only valid on $\rho(A_0)$; see [Brüning, Geyler and Pankrashkin, 2008, Section 4] for a certain extension to points which are isolated eigenvalues of A_0. We also want to point out that, if the symmetric operator S is simple, i.e. there exist no non-trivial reducing subspaces on which S is self-adjoint, it is well known that in the case $\Theta \in \mathfrak{B}(\mathcal{G})$ the function $\lambda \mapsto (\Theta - M(\lambda))^{-1}$ can be minimally represented by the extension A_Θ. In particular, this implies that the complete spectrum of A_Θ can be characterized with an analytic extension of the function $(\Theta - M(\cdot))^{-1}$. Moreover, again under the condition that S is simple, the spectrum of A_0 can be characterized with the singularities of the Weyl function M; cf. [Brasche, Malamud and Neidhardt, 2002].

Boundary triples for Sturm–Liouville operators Let (a, b) be a bounded interval and let p, q, w be real-valued functions on (a, b)

such that $p, w > 0$ almost everywhere and $1/p$, $q, w \in L^1(a,b)$. In the following we consider the regular Sturm–Liouville differential expression

$$\ell = \frac{1}{w}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right)$$

and differential operators associated with ℓ. For simplicity, only the regular case is discussed here, for singular problems in the limit circle case; see, e.g. [Allakhverdiev, 1991; Behrndt and Langer, 2010]. The limit point case is very well known and is also briefly discussed in the introduction of the present paper.

Let $L^2_w(a,b)$ denote the space of (equivalence classes) of complex-valued measurable functions on (a,b) such that $|f|^2 w \in L^1(a,b)$ and equip $L^2_w(a,b)$ with the inner product

$$(f,g) = \int_a^b f(x)\overline{g(x)}w(x)dx.$$

The differential operators associated with ℓ act in the Hilbert space $(L^2_w(a,b), (\cdot,\cdot))$ and are defined as follows: let

$$\begin{aligned}\mathfrak{D}_{\max} = \bigl\{f \in L^2_w(a,b) : f,\, pf' \text{ absolutely continuous on } (a,b)\\ \text{and } \ell(f) \in L^2_w(a,b)\bigr\},\\ \mathfrak{D}_{\min} = \bigl\{f \in \mathfrak{D}_{\max} : f(a) = (pf)'(a) = f(b) = (pf)'(b) = 0\bigr\},\end{aligned}$$

and let $Sf = \ell(f)$ with $\operatorname{dom} S = \mathfrak{D}_{\min}$ be the minimal operator associated with ℓ. Then S is a densely defined closed symmetric operator in $L^2_w(a,b)$ with deficiency numbers $n_\pm(S) = 2$. The maximal realization of ℓ coincides with the adjoint of the minimal operator:

$$S^* f = \ell(f), \qquad \operatorname{dom} S^* = \mathfrak{D}_{\max}.$$

As a basis for the two-dimensional space $\mathcal{N}_\lambda(S^*)$, $\lambda \in \mathbb{C}$, we choose the unique solutions φ_λ and ψ_λ of $\ell(f) = \lambda f$ fixed by the initial conditions

$$\begin{aligned}\varphi_\lambda(a) &= 1, \qquad & (p\varphi_\lambda')(a) &= 0,\\ \psi_\lambda(a) &= 0, \qquad & (p\psi_\lambda')(a) &= 1.\end{aligned}$$

The proof of the next proposition is straightforward and left to the reader.

Proposition 1.8 *The triple* $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, *where*

$$\Gamma_0 f := \begin{pmatrix} f(a) \\ f(b)\end{pmatrix} \qquad \textit{and} \qquad \Gamma_1 f := \begin{pmatrix} (pf')(a) \\ -(pf')(b)\end{pmatrix},$$

is a boundary triple for the maximal operator

$$S^* f = \ell(f), \quad \operatorname{dom} S^* = \mathfrak{D}_{\max}$$

such that $A_0 = S^* \upharpoonright \ker \Gamma_0$ *and* $A_1 = S^* \upharpoonright \ker \Gamma_1$ *are the Dirichlet realization and the Neumann realizations of* ℓ*. For* $\lambda \in \rho(A_0)$ *the corresponding* γ*-field and Weyl function are given by*

$$\gamma(\lambda)\eta = \eta_1 \left(\varphi_\lambda - \frac{\varphi_\lambda(b)}{\psi_\lambda(b)} \psi_\lambda \right) + \eta_2 \frac{1}{\psi_\lambda(b)} \psi_\lambda, \qquad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{C}^2,$$

and

$$M(\lambda) = \frac{1}{\psi_\lambda(b)} \begin{pmatrix} -\varphi_\lambda(b) & 1 \\ 1 & -(p\psi_\lambda')(b) \end{pmatrix}.$$

Note that the poles of M are exactly the eigenvalues of A_0, i.e. the Dirichlet eigenvalues.

Boundary triples for second order elliptic differential operators Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 1$, with C^∞-boundary $\partial\Omega$ and consider the second order differential expression

$$\mathcal{L} = -\sum_{j,k=1}^{n} \partial_j \, a_{jk} \, \partial_k + a \tag{1.10}$$

on Ω with real-valued coefficients $a_{jk} \in C^\infty(\overline{\Omega})$, $a \in L^\infty(\Omega)$ such that $a_{jk} = a_{kj}$ for all $j, k = 1, \dots, n$. In addition, it is assumed that the ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq C \sum_{k=1}^{n} \xi_k^2, \qquad \xi = (\xi_1, \dots, \xi_n)^\top \in \mathbb{R}^n, \ x \in \overline{\Omega},$$

holds for some constant $C > 0$.

The Sobolev space of kth order on Ω is denoted by $H^k(\Omega)$, $k \in \mathbb{N}$, and the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$ by $H_0^k(\Omega)$. Sobolev spaces on the boundary are denoted by $H^s(\partial\Omega)$, $s \in \mathbb{R}$. Let $(\cdot,\cdot)_{-1/2\times 1/2}$ and $(\cdot,\cdot)_{-3/2\times 3/2}$ be the extensions of the $L^2(\partial\Omega)$ inner product to $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ and $H^{-3/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$, respectively, and let

$$\iota_\pm \colon H^{\pm 1/2}(\partial\Omega) \to L^2(\partial\Omega)$$

be isomorphisms such that $(x, y)_{-1/2\times 1/2} = (\iota_- x, \iota_+ y)$ holds for every $x \in H^{-1/2}(\partial\Omega)$ and $y \in H^{1/2}(\partial\Omega)$.

Recall that the *Dirichlet operator*

$$A_D f = \mathcal{L}(f), \qquad \operatorname{dom} A_D = H^2(\Omega) \cap H^1_0(\Omega),$$

associated with the elliptic differential expression $\mathcal{L}$ in (1.10) is self-adjoint in $L^2(\Omega)$ and the resolvent of A_D is compact, cf. [Edmunds and Evans, 1987, Theorem VI.1.4] and, e.g. [Lions and Magenes, 1972; Wloka, 1987]. Furthermore, the *minimal operator*

$$Sf = \mathcal{L}(f), \qquad \operatorname{dom} S = H^2_0(\Omega),$$

is a densely defined closed symmetric operator in $L^2(\Omega)$ with equal infinite deficiency numbers, and the adjoint operator $S^*f = \mathcal{L}(f)$ is defined on the maximal domain

$$\operatorname{dom} S^* = \mathfrak{D}_{\max} = \bigl\{ f \in L^2(\Omega) \colon \mathcal{L}(f) \in L^2(\Omega) \bigr\}. \tag{1.11}$$

Let us fix some $\eta \in \mathbb{R} \cap \rho(A_D)$. Then for each function $f \in \mathfrak{D}_{\max}$ there is a unique decomposition $f = f_D + f_\eta$, where $f_D \in \operatorname{dom} A_D$ and $f_\eta \in \mathcal{N}_\eta(S^*) = \ker(S^* - \eta)$; cf. (1.7).

Next we recall the definition and some properties of the Dirichlet and (oblique) Neumann trace operators. Let $\mathfrak{n} = (\mathfrak{n}_1, \dots, \mathfrak{n}_n)^\top$ be the unit outward normal of Ω. It is well known that the map

$$C^\infty(\overline{\Omega}) \ni f \mapsto \left\{ f|_{\partial\Omega}, \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right\}, \text{ where } \frac{\partial f}{\partial \nu_{\mathcal{L}}} := \sum_{j,k=1}^n a_{jk} \mathfrak{n}_j \partial_k f, \tag{1.12}$$

can be extended to a linear operator from $\mathfrak{D}_{\max}$ into $H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)$ and that for $f \in \mathfrak{D}_{\max}$ and $g \in H^2(\Omega)$, Green's identity

$$(S^*f, g) - (f, S^*g) = \left(f|_{\partial\Omega}, \frac{\partial g}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right)_{-\frac{1}{2}\times\frac{1}{2}} - \left(\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, g|_{\partial\Omega} \right)_{-\frac{3}{2}\times\frac{3}{2}} \tag{1.13}$$

holds; see [Grubb, 1968; Lions and Magenes, 1972; Wloka, 1987].

The boundary triple in the following proposition can also be found in, e.g. [Brown, Grubb and Wood, 2009; Brown et al., 2008; Grubb, 2009; Malamud, 2010; Posilicano and Raimondi, 2009] and is already essentially contained in the classical paper [Grubb, 1968]. In [Posilicano, 2008] also a more abstract version of this construction was considered. For the convenience of the reader we repeat the short proof of the next proposition from [Behrndt, 2010] which is based on the general observations in [Grubb, 1968, 1971].

Proposition 1.9 *Let $\eta \in \mathbb{R} \cap \rho(A_D)$. The triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0 f := \iota_- f_\eta|_{\partial\Omega} = \iota_- f|_{\partial\Omega} \qquad \text{and} \qquad \Gamma_1 f := -\iota_+ \frac{\partial f_D}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}$$

$$\text{with } f = f_D + f_\eta \in \mathfrak{D}_{\max}, \quad f_D \in \operatorname{dom} A_D, \quad f_\eta \in \ker(S^* - \eta),$$

is a boundary triple for the maximal operator $S^* f = \mathcal{L}(f)$, $\operatorname{dom} S^* = \mathfrak{D}_{\max}$, such that $A_D = S^* \upharpoonright \ker \Gamma_0$. The corresponding γ-field and Weyl function are, for $\lambda \in \rho(A_D)$ and $y \in L^2(\partial\Omega)$, given by

$$\gamma(\lambda) y = \big(I + (\lambda - \eta)(A_D - \lambda)^{-1}\big) f_\eta(y),$$
$$M(\lambda) y = (\eta - \lambda)\iota_+ \frac{\partial (A_D - \lambda)^{-1} f_\eta(y)}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}$$

respectively, where $f_\eta(y)$ is the unique function in $\ker(S^* - \eta)$ satisfying $\iota_- f_\eta(y)|_{\partial\Omega} = y$.

Proof Let $f, g \in \mathfrak{D}_{\max}$ be decomposed in the form $f = f_D + f_\eta$ and $g = g_D + g_\eta$. Since A_D is self-adjoint and $\eta \in \mathbb{R}$, we find

$$(S^* f, g) - (f, S^* g) = (A_D f_D, g_\eta) - (f_D, S^* g_\eta) + (S^* f_\eta, g_D) - (f_\eta, A_D g_D).$$

Then $f_D|_{\partial\Omega} = g_D|_{\partial\Omega} = 0$ together with Green's identity (1.13) implies that

$$\begin{aligned}(S^* f, g) - (f, S^* g) &= -\left(\frac{\partial f_D}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, g_\eta|_{\partial\Omega}\right)_{\frac12 \times -\frac12} + \left(f_\eta|_{\partial\Omega}, \frac{\partial g_D}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}\right)_{-\frac12 \times \frac12} \\ &= (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g).\end{aligned}$$

Hence (1.2) in Definition 1.1 holds. By the classical trace theorem the map $H^2(\Omega) \cap H^1_0(\Omega) \ni f_D \mapsto \frac{\partial f_D}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ is onto and the same holds for the map $\ker(S^* - \eta) \ni f_\eta \mapsto f_\eta|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$, which is an isomorphism according to [Grubb, 1971, Theorem 2.1]. Hence $(\Gamma_0; \Gamma_1)^\top$ maps $\operatorname{dom} S^*$ onto $L^2(\partial\Omega) \times L^2(\partial\Omega)$ and therefore $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is a boundary triple for S^* with $A_D = \ker \Gamma_0$.

It remains to show that the corresponding γ-field and Weyl function have the asserted form. For this let $y \in L^2(\partial\Omega)$, choose the unique function $f_\eta(y)$ in $\ker(S^* - \eta)$ such that $y = \iota_- f_\eta(y)|_{\partial\Omega}$, and set

$$f_\lambda := (\lambda - \eta)(A_D - \lambda)^{-1} f_\eta(y) + f_\eta(y) \tag{1.14}$$

for $\lambda \in \rho(A_D)$. Then $(S^* - \lambda) f_\lambda = 0$ and since $(A_D - \lambda)^{-1} f_\eta(y) \in \operatorname{dom} A_D$ and $f_\eta(y) \in \ker(S^* - \eta)$, we obtain

$$\Gamma_0 f_\lambda = \iota_- f_\eta(y)|_{\partial\Omega} = y,$$

i.e. $\gamma(\lambda)y = f_\lambda = (I + (\lambda - \eta)(A_D - \lambda)^{-1})f_\eta(y)$. Finally, by the definition of the Weyl function and (1.14) we have

$$M(\lambda)y = \Gamma_1 f_\lambda = (\eta - \lambda)\,\iota_+ \frac{\partial (A_D - \lambda)^{-1} f_\eta(y)}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}. \qquad \square$$

Note that $A_1 = S^* \upharpoonright \ker \Gamma_1$ is not the Neumann operator but the restriction of S^* to the domain

$$\operatorname{dom} S \dotplus \ker(S^* - \eta) = H_0^2(\Omega) \dotplus \ker(S^* - \eta).$$

If $\eta = 0$, then the operator A_1 is the Krein–von Neumann or "soft" extension of S, which was studied, e.g. in [Ashbaugh et al., 2010; Behrndt and Langer, 2007; Everitt and Markus, 2003; Everitt, Markus and Plum, 2005; Grubb, 1968, 1983, 2006].

1.3 Quasi boundary triples and their Weyl functions

The notion of quasi boundary triples was introduced by the authors in [Behrndt and Langer, 2007] with a particular focus on the applicability to elliptic boundary value problems and elliptic differential operators. The concept is a natural generalization of the concept of boundary triples from the previous section and so-called generalized boundary triples from [Derkach and Malamud, 1995, Section 6] and [Derkach et al., 2006, Section 5.2]. In this section again $(\mathcal{H}, (\cdot,\cdot))$ is assumed to be a Hilbert space and S a densely defined closed symmetric operator in $\mathcal{H}$. The idea is that boundary mappings are defined not on the domain of S^* but only on a core of S^*, and that the abstract Green identity is supposed to be valid on this core. The restriction of S^* to this core is called T in the following.

Definition 1.10 A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be a *quasi boundary triple* for the operator S^* if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exists an operator T such that $\overline{T} = S^*$, and $\Gamma_0, \Gamma_1 \colon \operatorname{dom} T \to \mathcal{G}$ are linear mappings satisfying

$$(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) \tag{1.15}$$

for all $f, g \in \operatorname{dom} T$, the range of the map $\Gamma := (\Gamma_0; \Gamma_1)^\top \colon \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ is dense and $A_0 = T \upharpoonright \ker \Gamma_0$ is self-adjoint in $\mathcal{H}$.

A quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* exists if and only if the deficiency numbers of S are equal, and it follows that $\dim \mathcal{G} = n_\pm(S)$, just as for (ordinary) boundary triples. Clearly, every boundary triple is also a quasi boundary triple, and we point out that for the case of finite deficiency numbers also the converse holds. We also remark that a quasi boundary triple with the additional property $\operatorname{ran}\Gamma_0 = \mathcal{G}$ is a generalized boundary triple in the sense of [Derkach and Malamud, 1995, Definition 6.1] (cf. [Behrndt and Langer, 2007, Corollary 3.7]) and that quasi boundary triples are not necessarily boundary relations as studied in [Derkach et al., 2006, 2009].

In the following, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for S^*. Then the map $\Gamma = (\Gamma_0; \Gamma_1)^\top$ as a mapping from $\operatorname{dom} T$ endowed with the graph norm of T to $\mathcal{G} \times \mathcal{G}$ is closable (which follows from (1.15) and the assumption that $\operatorname{ran}\Gamma$ is dense), and

$$\operatorname{dom} S = \ker \Gamma = \ker \Gamma_0 \cap \ker \Gamma_1$$

holds; see [Behrndt and Langer, 2007, Proposition 2.2].

For a linear operator or relation Θ in $\mathcal{G}$ (not necessarily closed) we define the extension A_Θ of S in analogy to (1.3)–(1.4) by

$$A_\Theta := T \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0) = T \upharpoonright \{f \in \operatorname{dom} T \colon \Gamma f \in \Theta\}. \tag{1.16}$$

In contrast to boundary triples, relation (1.16) in general does not induce a bijective correspondence between the self-adjoint extensions of S and the self-adjoint operators and relations Θ in $\mathcal{G}$; cf. Proposition 1.2. However, if Θ is symmetric (dissipative, accumulative) in $\mathcal{G}$, then the corresponding extension A_Θ in (1.16) is also symmetric (dissipative, accumulative) in $\mathcal{H}$, but simple counterexamples show that self-adjointness of Θ does not even imply essential self-adjointness of A_Θ; see [Behrndt and Langer, 2007, Proposition 4.11].

The following result is a variant of Theorem 1.3 and will turn out to be useful when defining quasi boundary triples for elliptic operators in the next section. The advantage of this theorem is that one starts with some operator T and then constructs S and one does not have to show that $\operatorname{dom} T$ is a core of S^*; this follows from the theorem. Moreover, one only has to show that $T \upharpoonright \ker \Gamma_0$ is an extension of a self-adjoint operator and not that it is equal to one. For the proof see [Behrndt and Langer, 2007, Theorem 2.3].

Theorem 1.11 *Let T be a linear operator in $\mathcal{H}$ and let $\mathcal{G}$ be a Hilbert space. Assume that $\Gamma_0, \Gamma_1 \colon \operatorname{dom} T \to \mathcal{G}$ are linear mappings which satisfy the following conditions:*

(i) *$T \upharpoonright \ker \Gamma_0$ is the extension of a self-adjoint operator A in $\mathcal{H}$;*

(ii) $\operatorname{ran}(\Gamma_0;\Gamma_1)^\top$ *is dense in* $\mathcal{G}\times\mathcal{G}$ *and* $\ker\Gamma_0\cap\ker\Gamma_1$ *is dense in* $\mathcal{H}$*;*

(iii) $(Tf,g)-(f,Tg)=(\Gamma_1 f,\Gamma_0 g)-(\Gamma_0 f,\Gamma_1 g)$ *for all* $f,g\in\operatorname{dom}T$.

Then the operator

$$S := T\upharpoonright \ker\Gamma_0\cap\ker\Gamma_1$$

is a densely defined closed symmetric operator in $\mathcal{H}$ *such that* $S^*=\overline{T}$*, and* $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ *is a quasi boundary triple for* S^* *with* $A=T^*\upharpoonright\ker\Gamma_0 = A_0$.

Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a quasi boundary triple for $\overline{T}=S^*$. In the following we set $\mathcal{G}_0:=\operatorname{ran}\Gamma_0$ and $\mathcal{G}_1:=\operatorname{ran}\Gamma_1$. Because $\operatorname{ran}\Gamma$ is dense in $\mathcal{G}\times\mathcal{G}$, it follows that $\mathcal{G}_0$ and $\mathcal{G}_1$ are dense subspaces of $\mathcal{G}$. Since $A_0=T\upharpoonright\ker\Gamma_0$ is a self-adjoint extension of S in $\mathcal{H}$, the decomposition

$$\operatorname{dom}T=\operatorname{dom}A_0\dotplus\mathcal{N}_\lambda(T),\qquad \mathcal{N}_\lambda(T):=\ker(T-\lambda),$$

holds for all $\lambda\in\rho(A_0)$; cf. (1.7). The γ-field and Weyl function of a quasi boundary triple are defined in analogy to Definition 1.4.

Definition 1.12 Let S be a densely defined closed symmetric operator in $\mathcal{H}$ and let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a quasi boundary triple for $\overline{T}=S^*$ with $A_0=T\upharpoonright\ker\Gamma_0$. Then the functions γ and M defined by

$$\begin{aligned}\gamma(\lambda)&:=\bigl(\Gamma_0\upharpoonright\mathcal{N}_\lambda(T)\bigr)^{-1},\\ M(\lambda)&:=\Gamma_1\gamma(\lambda)=\Gamma_1\bigl(\Gamma_0\upharpoonright\mathcal{N}_\lambda(T)\bigr)^{-1},\end{aligned}\qquad \lambda\in\rho(A_0),\tag{1.17}$$

are called the *γ-field* and *Weyl function* corresponding to the quasi boundary triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$.

Note that $\gamma(\lambda)$ is a mapping from $\mathcal{G}_0$ to $\mathcal{H}$, and $M(\lambda)$ is a mapping from $\mathcal{G}_0$ to $\mathcal{G}_1\subset\mathcal{G}$ for $\lambda\in\rho(A_0)$. In the next propositions we collect some properties of the γ-field and the Weyl function of a quasi boundary triple, which are extensions of well-known properties of the γ-field and Weyl function of an ordinary boundary triple; cf. Propositions 1.5 and 1.6. For the convenience of the reader we repeat the proofs from [Behrndt and Langer, 2007, Proposition 2.6] which are similar to the ones for γ-fields and Weyl functions of ordinary boundary triples; cf. [Derkach and Malamud, 1991, 1995].

Proposition 1.13 *Let* $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ *be a quasi boundary triple with* $A_0=T\upharpoonright\ker\Gamma_0$ *and* γ*-field* γ*. Then the following assertions hold for all* $\lambda,\mu\in\rho(A_0)$*:*

(i) $\gamma(\lambda)$ *is a bounded operator from* $\mathcal{G}$ *to* $\mathcal{H}$ *with dense domain* $\operatorname{dom}\gamma(\lambda) = \mathcal{G}_0$ *and range* $\operatorname{ran}\gamma(\lambda) = \mathcal{N}_\lambda(T)$*, and hence* $\overline{\gamma(\lambda)} \in \mathfrak{B}(\mathcal{G},\mathcal{H})$*;*

(ii) *the function* $\lambda \mapsto \gamma(\lambda)g$ *is holomorphic on* $\rho(A_0)$ *for every* $g \in \mathcal{G}_0$*, and the relation*

$$\gamma(\lambda) = \big(I + (\lambda-\mu)(A_0-\lambda)^{-1}\big)\gamma(\mu) \tag{1.18}$$

holds;

(iii) $\gamma(\bar\lambda)^* \in \mathfrak{B}(\mathcal{H},\mathcal{G})$*,* $\operatorname{ran}\gamma(\bar\lambda)^* \subset \mathcal{G}_1$ *and for all* $h \in \mathcal{H}$ *we have*

$$\gamma(\bar\lambda)^* h = \Gamma_1(A_0-\lambda)^{-1}h. \tag{1.19}$$

Proof Let $\lambda \in \rho(A_0)$. Since Γ is closable from $\operatorname{dom} T$ (with the graph norm) to $\mathcal{G}\times\mathcal{G}$, it follows that $\Gamma(A_0-\lambda)^{-1}$ is closable and hence bounded from $\mathcal{H}$ to $\mathcal{G}\times\mathcal{G}$ by the closed graph theorem, which implies that the mapping $\Gamma_1(A_0-\lambda)^{-1}\colon \mathcal{H}\to\mathcal{G}$ is bounded. For $h\in\mathcal{H}$ and $x \in \operatorname{dom}\gamma(\bar\lambda) = \mathcal{G}_0$ we have (where we use (1.15), (1.17), the relation $T\gamma(\bar\lambda)x = \bar\lambda\gamma(\bar\lambda)x$ and the fact that Γ_0 vanishes on $\operatorname{dom} A_0$)

$$\begin{aligned}
(h,\gamma(\bar\lambda)x) &= \big((T-\lambda)(A_0-\lambda)^{-1}h,\gamma(\bar\lambda)x\big)\\
&= \big(T(A_0-\lambda)^{-1}h,\gamma(\bar\lambda)x\big) - \lambda\big((A_0-\lambda)^{-1}h,\gamma(\bar\lambda)x\big)\\
&= \big(T(A_0-\lambda)^{-1}h,\gamma(\bar\lambda)x\big) - \big((A_0-\lambda)^{-1}h,T\gamma(\bar\lambda)x\big)\\
&= \big(\Gamma_1(A_0-\lambda)^{-1}h,\Gamma_0\gamma(\bar\lambda)x\big) - \big(\Gamma_0(A_0-\lambda)^{-1}h,\Gamma_1\gamma(\bar\lambda)x\big)\\
&= \big(\Gamma_1(A_0-\lambda)^{-1}h,x\big),
\end{aligned}$$

which shows relation (1.19). The latter relation also yields $\operatorname{ran}\gamma(\bar\lambda)^* \subset \mathcal{G}_1$, and the boundedness of $\Gamma_1(A_0-\lambda)^{-1}$ implies (i). The resolvent identity and (1.19) show that the following equality is true for $\lambda,\mu \in \rho(A_0)$:

$$\gamma(\lambda)^* - \gamma(\mu)^* = (\bar\lambda-\bar\mu)\gamma(\mu)^*(A_0-\bar\lambda)^{-1}.$$

Taking the adjoint and rearranging we obtain (1.18), which also implies the analyticity of $\gamma(\cdot)g$, $g\in\mathcal{G}_0$. □

The first five items of the next proposition are taken from [Behrndt and Langer, 2007, Proposition 2.6]; for the last item see [Behrndt, Langer and Lotoreichik, 2011].

Proposition 1.14 *Let* $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ *be a quasi boundary triple with* $A_0 = T \restriction \ker\Gamma_0$ *and Weyl function* M*. Then the following assertions hold for all* $\lambda,\mu \in \rho(A_0)$*:*

(i) *$M(\lambda)$ maps $\mathcal{G}_0$ into $\mathcal{G}_1$. If also $A_1 := T \upharpoonright \ker \Gamma_1$ is a self-adjoint operator in $\mathcal{H}$ and $\lambda \in \rho(A_1)$, then $M(\lambda)$ maps $\mathcal{G}_0$ onto $\mathcal{G}_1$.*

(ii) *$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda$ for all $f_\lambda \in \mathcal{N}_\lambda(T)$.*

(iii) *$M(\lambda) \subset M(\bar{\lambda})^*$ and*

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda). \tag{1.20}$$

(iv) *The function $\lambda \mapsto M(\lambda)$ is holomorphic in the sense that it can be written as the sum of the possibly unbounded symmetric operator $\operatorname{Re} M(\mu)$, where μ is fixed, and a bounded holomorphic operator function:*

$$\begin{aligned} M(\lambda) &= \operatorname{Re} M(\mu) \\ &\quad + \gamma(\mu)^*\Big((\lambda - \operatorname{Re}\mu) + (\lambda - \mu)(\lambda - \bar{\mu})(A_0 - \lambda)^{-1}\Big)\gamma(\mu). \end{aligned} \tag{1.21}$$

(v) *$\operatorname{Im} M(\lambda)$ is a densely defined bounded operator in $\mathcal{G}$. For $\lambda \in \mathbb{C}^+ (\mathbb{C}^-)$ the operator $\operatorname{Im} M(\lambda)$ is positive (negative, respectively).*

(vi) *If $M(\lambda_0)$ is bounded for some $\lambda_0 \in \rho(A_0)$, then $M(\lambda)$ is bounded for all $\lambda \in \rho(A_0)$. In this case,*

$$\frac{1}{\operatorname{Im}\lambda} \operatorname{Im} \overline{M(\lambda)} > 0, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{1.22}$$

and, in particular, $\ker \overline{M(\lambda)} = \{0\}$ for $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Proof The first assertion in (i) and the statement in (ii) follow immediately from the definition of $M(\lambda)$. The second assertion in (i) follows from the relation $\operatorname{dom} T = \operatorname{dom} A_1 \dot{+} \mathcal{N}_\lambda(T)$ for $\lambda \in \rho(A_1)$.

(iii) Let $x, y \in \mathcal{G}_0$ and $\lambda, \mu \in \rho(A_0)$. Then

$$\begin{aligned} \big(M(\lambda)x, y\big) - \big(x, M(\mu)y\big) &= \big(\Gamma_1\gamma(\lambda)x, \Gamma_0\gamma(\mu)y\big) - \big(\Gamma_0\gamma(\lambda)x, \Gamma_1\gamma(\mu)y\big) \\ &= \big(T\gamma(\lambda)x, \gamma(\mu)y\big) - \big(\gamma(\lambda)x, T\gamma(\mu)y\big) \\ &= \big(\lambda\gamma(\lambda)x, \gamma(\mu)y\big) - \big(\gamma(\lambda)x, \mu\gamma(\mu)y\big) \\ &= (\lambda - \bar{\mu})\big(\gamma(\lambda)x, \gamma(\mu)y\big). \end{aligned} \tag{1.23}$$

For $\mu = \bar{\lambda}$ one obtains $\big(M(\lambda)x, y\big) = \big(x, M(\bar{\lambda})y\big)$, which shows that $\mathcal{G}_0 \subset \operatorname{dom} M(\bar{\lambda})^*$ and that $M(\lambda)$ is a restriction of $M(\bar{\lambda})^*$. Now it follows from (1.23) that

$$\big(M(\lambda)x, y\big) - \big(M(\mu)^*x, y\big) = (\lambda - \bar{\mu})\big(\gamma(\mu)^*\gamma(\lambda)x, y\big),$$

for all $x, y \in \mathcal{G}_0$, which yields (1.20) since $\mathcal{G}_0$ is dense in $\mathcal{G}$ and the operators on both sides of (1.20) are defined on $\mathcal{G}_0$.

(iv) Using (1.20) and (1.18) we obtain the following relations, which are valid on $\mathcal{G}_0$:

$$\begin{aligned} &M(\lambda) - \operatorname{Re} M(\mu) = M(\lambda) - \tfrac{1}{2}\big(M(\mu) + M(\mu)^*\big) \\ &= M(\lambda) - M(\mu)^* - \tfrac{1}{2}\big(M(\mu) - M(\mu)^*\big) \\ &= (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda) - \tfrac{1}{2}(\mu - \bar{\mu})\gamma(\mu)^*\gamma(\mu) \\ &= \gamma(\mu)^*\Big[(\lambda - \bar{\mu})\big(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\big)\gamma(\mu) - \tfrac{1}{2}(\mu - \bar{\mu})\gamma(\mu)\Big] \\ &= \gamma(\mu)^*\Big[\lambda - \bar{\mu} + (\lambda - \bar{\mu})(\lambda - \mu)(A_0 - \lambda)^{-1} - \tfrac{1}{2}(\mu - \bar{\mu})\Big]\gamma(\mu) \\ &= \gamma(\mu)^*\Big[\lambda - \operatorname{Re}\mu + (\lambda - \bar{\mu})(\lambda - \mu)(A_0 - \lambda)^{-1}\Big]\gamma(\mu). \end{aligned}$$

This shows (1.21) and the analyticity as claimed.

(v) Let $\lambda \in \mathbb{C}^+$; the case $\lambda \in \mathbb{C}^-$ is analogous. From (1.20) we obtain

$$\begin{aligned} \operatorname{Im} M(\lambda) &= \tfrac{1}{2\mathrm{i}}\big(M(\lambda) - M(\lambda)^*\big) = \tfrac{1}{2\mathrm{i}}(\lambda - \bar{\lambda})\gamma(\lambda)^*\gamma(\lambda) \\ &= (\operatorname{Im}\lambda)\gamma(\lambda)^*\gamma(\lambda), \end{aligned} \tag{1.24}$$

which is a bounded, positive operator since $\gamma(\lambda)$ is bounded and injective; it is defined on the dense subspace $\mathcal{G}_0$.

(vi) The first assertion follows immediately from (1.21). For the inequality (1.22), assume without loss of generality that $\operatorname{Im}\lambda > 0$. Observe that $\operatorname{Im}\overline{M(\lambda)} = \overline{\operatorname{Im} M(\lambda)}$ since $M(\lambda)$ is bounded. It follows from (v) that $\operatorname{Im} M(\lambda) > 0$. Hence it is sufficient to show that

$$\ker\big(\operatorname{Im}\overline{M(\lambda)}\big) = \{0\}.$$

Let $x \in \ker(\operatorname{Im}\overline{M(\lambda)}) = \ker(\overline{\operatorname{Im} M(\lambda)})$. Then there exist $x_n \in \operatorname{dom} M(\lambda)$ so that $x_n \to x$ and $(\operatorname{Im} M(\lambda))x_n \to 0$ when $n \to \infty$. By (1.24) we have

$$\big((\operatorname{Im} M(\lambda))x_n, x_n\big) = \big((\operatorname{Im}\lambda)\gamma(\lambda)^*\gamma(\lambda)x_n, x_n\big) = (\operatorname{Im}\lambda)\|\gamma(\lambda)x_n\|^2,$$

and since $\operatorname{Im}\lambda \neq 0$, this implies that $\gamma(\lambda)x_n \to 0$. The relation $\Gamma_0\gamma(\lambda)x_n = x_n \to x$ and the boundedness of $M(\lambda)$ imply that

$$\Gamma_1\gamma(\lambda)x_n = M(\lambda)\Gamma_0\gamma(\lambda)x_n = M(\lambda)x_n \to \overline{M(\lambda)}x,$$

i.e.

$$\Gamma\gamma(\lambda)x_n \to \begin{pmatrix} x \\ \overline{M(\lambda)}x \end{pmatrix}.$$

Since $\gamma(\lambda)x_n$ converges to 0 in the graph norm of T, the closability of Γ implies that $x = 0$, which shows (1.22).

If $x \neq 0$, then

$$\operatorname{Im}\big(\overline{M(\lambda)}x, x\big) = \big((\operatorname{Im}\overline{M(\lambda)})x, x\big) \neq 0,$$

which implies that $x \notin \ker \overline{M(\lambda)}$. Hence $\ker \overline{M(\lambda)} = \{0\}$. □

The next theorem gives a characterization of the class of Weyl functions corresponding to quasi boundary triples. It is a reformulation of [Alpay and Behrndt, 2009, Theorem 2.6] and can be regarded as a generalization of [Langer and Textorius, 1977, Theorem 2.2 and Theorem 2.4], [Derkach and Malamud, 1991, Corollary 2] and [Derkach and Malamud, 1995, Theorem 6.1]; see also [Derkach et al., 2006, Section 5].

Theorem 1.15 *Let $\mathcal{G}_0$ be a dense subspace of $\mathcal{G}$, $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, and let M be a function defined on $\mathbb{C}\backslash\mathbb{R}$ whose values $M(\lambda)$ are linear operators in $\mathcal{G}$ with* $\operatorname{dom} M(\lambda) = \mathcal{G}_0$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. *Then the following two statements are equivalent.*

(i) *There exists a separable Hilbert space $\mathcal{H}$, a densely defined closed symmetric operator S and a quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for $\overline{T} = S^*$ such that M is the corresponding Weyl function.*

(ii) *There exists a unique $\mathfrak{B}(\mathcal{G})$-valued Nevanlinna function N with the properties* (α), (β) *and* (γ):

(α) *the relations*

$$M(\lambda)h - \operatorname{Re} M(\lambda_0)h = N(\lambda)h,$$
$$M(\lambda)^*h - \operatorname{Re} M(\lambda_0)h = N(\lambda)^*h$$

hold for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$;

(β) $\operatorname{Im} N(\lambda)h = 0$ *for some $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ implies $h = 0$;*

(γ) *the conditions*

$$\lim_{\eta\to+\infty} \frac{1}{\eta}\big(N(i\eta)k, k\big) = 0 \quad \textit{and} \quad \lim_{\eta\to+\infty} \eta \operatorname{Im}\big(N(i\eta)k, k\big) = \infty$$

are valid for all $k \in \mathcal{G}$, $k \neq 0$.

The following theorem and corollary contain a variant of Krein's formula for the resolvents of canonical extensions parameterized by quasi boundary triples via (1.16). The theorem generalizes [Derkach and Malamud, 1991, Proposition 2] and can be found in a similar form in [Behrndt and Langer, 2007] and [Behrndt, Langer and Lotoreichik, 2011]. For completeness, the full proof is given after the corollaries below.

Theorem 1.16 *Let S be a densely defined closed symmetric operator in $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\overline{T} = S^*$ with $A_0 = T \upharpoonright \ker \Gamma_0$, γ-field γ and Weyl function M. Further, let Θ be a relation in $\mathcal{G}$ and assume that $\lambda \in \rho(A_0)$ is not an eigenvalue of A_Θ, or, equivalently, that* $\ker(\Theta - M(\lambda)) = \{0\}$. *Then the following assertions are true.*

(i) $g \in \operatorname{ran}(A_\Theta - \lambda)$ *if and only if* $\gamma(\bar{\lambda})^* g \in \operatorname{dom}(\Theta - M(\lambda))^{-1}$.

(ii) *For all* $g \in \operatorname{ran}(A_\Theta - \lambda)$ *we have*

$$(A_\Theta - \lambda)^{-1} g = (A_0 - \lambda)^{-1} g + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g. \quad (1.25)$$

If $\rho(A_\Theta) \cap \rho(A_0) \neq \varnothing$ or $\rho(\overline{A_\Theta}) \cap \rho(A_0) \neq \varnothing$, e.g. if A_Θ is self-adjoint or essentially self-adjoint, respectively, then for $\lambda \in \rho(\overline{A_\Theta}) \cap \rho(A_0)$, relation (1.25) is valid on $\mathcal{H}$ or a dense subset of $\mathcal{H}$, respectively. This, together with the fact that $\gamma(\bar{\lambda})^*$ is an everywhere defined bounded operator and

$$\gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* \subset \overline{\gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^*}$$

implies the following corollary.

Corollary *Let the assumptions be as in Theorem 1.16. Then the following assertions hold.*

(i) *If* $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$, *then*

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^*. \quad (1.26)$$

(ii) *If* $\lambda \in \rho(\overline{A_\Theta}) \cap \rho(A_0)$, *then*

$$(\overline{A_\Theta} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \overline{\gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^*}.$$

In particular, if A_Θ is self-adjoint, then Krein's formula (1.26) holds at least for all non-real λ.

If the relation Θ in Theorem 1.16 is self-adjoint, then Krein's formula can be rewritten as follows: let $\{\Phi, \Psi\}$ be a pair of bounded operators in $\mathcal{G}$ such that

$$\Theta = \big\{(\Phi k; \Psi k)^\top : k \in \mathcal{G}\big\} \quad (1.27)$$

holds; cf. (1.6) and note that (1.5) has to be satisfied. It follows that $\ker(\Theta - M(\lambda)) = \{0\}$ if and only if $\ker(\Psi - M(\lambda)\Phi) = \{0\}$; then

$$\big(\Theta - M(\lambda)\big)^{-1} = \big\{(\Psi k - M(\lambda)\Phi k; \Phi k)^\top : k \in \mathcal{G},\ \Phi k \in \operatorname{dom} M(\lambda)\big\}$$

together with Theorem 1.16 yield the following corollary.

Corollary *Let S be a densely defined closed symmetric operator in $\mathcal{H}$ and let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a quasi boundary triple for $\overline{T} = S^*$ with $A_0 = T \upharpoonright \ker\Gamma_0$, γ-field γ and Weyl function M. Further, let Θ be a self-adjoint relation in $\mathcal{G}$ represented with a pair $\{\Phi,\Psi\}$ in the form* (1.27) *and assume that $\lambda \in \rho(A_0)$ is not an eigenvalue of A_Θ, or, equivalently, that $\ker(\Psi - M(\lambda)\Phi) = \{0\}$. Then the following assertions are true.*

(i) *$g \in \operatorname{ran}(A_\Theta - \lambda)$ if and only if $\gamma(\bar\lambda)^* g \in \operatorname{dom}(\Psi - M(\lambda)\Phi)^{-1}$.*

(ii) *For all $g \in \operatorname{ran}(A_\Theta - \lambda)$ we have*

$$(A_\Theta - \lambda)^{-1} g = (A_0 - \lambda)^{-1} g + \gamma(\lambda)\Phi\big(\Psi - M(\lambda)\Phi\big)^{-1}\gamma(\bar\lambda)^* g.$$

Let us now turn to the proof of Theorem 1.16.

Proof of Theorem 1.16 Let us first show that $\lambda \in \rho(A_0)$ is not an eigenvalue of A_Θ if and only if $\ker(\Theta - M(\lambda)) = \{0\}$. Assume, e.g. that $f \in \ker(A_\Theta - \lambda)$ and $f \neq 0$. Then $f \in \mathcal{N}_\lambda(T)$ and as $\Gamma f \in \Theta$, we obtain

$$\begin{pmatrix} \Gamma_0 f \\ 0 \end{pmatrix} = \begin{pmatrix} \Gamma_0 f \\ \Gamma_1 f - M(\lambda)\Gamma_0 f \end{pmatrix} \in \Theta - M(\lambda).$$

Moreover, $\Gamma_0 f \neq 0$ because otherwise $f \in \operatorname{dom} A_0 \cap \mathcal{N}_\lambda(T)$, which would imply $f = 0$. Conversely, if $y \in \ker(\Theta - M(\lambda))$ and $y \neq 0$, then

$$\begin{pmatrix} y \\ M(\lambda) y \end{pmatrix} \in \Theta,$$

and for $f := \gamma(\lambda) y \in \mathcal{N}_\lambda(T)$ we obtain

$$\begin{pmatrix} \Gamma_0 f \\ \Gamma_1 f \end{pmatrix} = \begin{pmatrix} y \\ M(\lambda) y \end{pmatrix} \in \Theta.$$

Therefore $f \in \operatorname{dom} A_\Theta$, i.e. $\gamma(\lambda) y \in \ker(A_\Theta - \lambda)$. Thus $\lambda \in \rho(A_0)$ is not an eigenvalue of A_Θ if and only if $\ker(\Theta - M(\lambda)) = \{0\}$.

Now let us fix some point $\lambda \in \rho(A_0)$ which is not an eigenvalue of A_Θ. Then $(A_\Theta - \lambda)^{-1}$ and $(\Theta - M(\lambda))^{-1}$ are operators in $\mathcal{H}$ and $\mathcal{G}$, resp. Let $g \in \operatorname{ran}(A_\Theta - \lambda)$. We show that $\gamma(\bar\lambda)^* g \in \operatorname{dom}(\Theta - M(\lambda))^{-1}$ and that formula (1.25) holds. Set

$$f := (A_\Theta - \lambda)^{-1} g - (A_0 - \lambda)^{-1} g \quad \text{and} \quad h := (A_\Theta - \lambda)^{-1} g.$$

Then we have $f \in \mathcal{N}_\lambda(T)$ and $h \in \operatorname{dom} A_\Theta$. Moreover,

$$\Gamma_0 f = \Gamma_0 h - \Gamma_0 (A_0 - \lambda)^{-1} g = \Gamma_0 h$$

since $(A_0 - \lambda)^{-1} g \in \operatorname{dom} A_0 = \ker \Gamma_0$, and

$$\Gamma_1 f = \Gamma_1 h - \Gamma_1 (A_0 - \lambda)^{-1} g = \Gamma_1 h - \gamma(\bar{\lambda})^* g$$

by Proposition 1.13 (iii). These equalities together with Proposition 1.14 (ii) yield

$$\gamma(\bar{\lambda})^* g = \Gamma_1 h - \Gamma_1 f = \Gamma_1 h - M(\lambda)\Gamma_0 f = \Gamma_1 h - M(\lambda)\Gamma_0 h.$$

Since $h \in \operatorname{dom} A_\Theta$, we have $(\Gamma_0 h; \Gamma_1 h)^\top \in \Theta$ by (1.16) and hence

$$\begin{pmatrix} \Gamma_0 h \\ \gamma(\bar{\lambda})^* g \end{pmatrix} = \begin{pmatrix} \Gamma_0 h \\ \Gamma_1 h - M(\lambda)\Gamma_0 h \end{pmatrix} \in \Theta - M(\lambda), \tag{1.28}$$

which implies $\gamma(\bar{\lambda})^* g \in \operatorname{dom}(\Theta - M(\lambda))^{-1}$, i.e. $\Rightarrow$ in (i) is proved. Furthermore, it follows from (1.28) that $\Gamma_0 h = (\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^* g$ since $(\Theta - M(\lambda))^{-1}$ is an operator. Therefore

$$\begin{aligned} \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g &= \gamma(\lambda)\Gamma_0 h = \gamma(\lambda)\Gamma_0 f \\ &= f = (A_\Theta - \lambda)^{-1} g - (A_0 - \lambda)^{-1} g, \end{aligned}$$

which shows relation (1.25).

For $\Leftarrow$ in (i) assume that $\Theta - M(\lambda)$ is injective and let $\gamma(\bar{\lambda})^* g \in \operatorname{ran}(\Theta - M(\lambda))$ for some $g \in \mathcal{H}$. Then $(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^* g$ belongs to $\operatorname{dom}(\Theta - M(\lambda)) \subset \mathcal{G}_0$, and we claim that

$$f := \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g + (A_0 - \lambda)^{-1} g \in \operatorname{dom} A_\Theta.$$

Clearly, $f \in \operatorname{dom} T$. Moreover, the relations

$$\begin{aligned} \Gamma_0 f &= \big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g, \\ \Gamma_1 f &= M(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g + \gamma(\bar{\lambda})^* g \end{aligned}$$

and

$$\begin{pmatrix} \big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g \\ \gamma(\bar{\lambda})^* g \end{pmatrix} \in \Theta - M(\lambda),$$

imply that

$$\begin{aligned} \begin{pmatrix} \Gamma_0 f \\ \Gamma_1 f \end{pmatrix} &= \begin{pmatrix} \big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g \\ M(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g + \gamma(\bar{\lambda})^* g \end{pmatrix} \\ &\in M(\lambda) + \big(\Theta - M(\lambda)\big) \subset \Theta, \end{aligned}$$

that is, $f \in \operatorname{dom} A_\Theta$. Since $\gamma(\lambda)$ maps into $\ker(T - \lambda)$, we have

$$(A_\Theta - \lambda) f = (T - \lambda)\gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g + (T - \lambda)(A_0 - \lambda)^{-1} g = g,$$

which shows that $g \in \operatorname{ran}(A_\Theta - \lambda)$. $\square$

If Θ is a self-adjoint (maximal dissipative, maximal accumulative) relation in $\mathcal{G}$, then we can decompose Θ as follows. Let $\mathcal{G}_\infty := \operatorname{mul}\Theta$, $\mathcal{G}_{\mathrm{op}} := \mathcal{G}_\infty^\perp$ and denote by P_{op}, P_∞ the orthogonal projections onto $\mathcal{G}_{\mathrm{op}}$ and $\mathcal{G}_\infty$, respectively. Then the relation Θ can be written as

$$\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty,$$

where Θ_{op} is a self-adjoint (maximal dissipative, maximal accumulative, resp.) operator in $\mathcal{G}_{\mathrm{op}}$ and $\Theta_\infty = \{(0;y)^\top : y \in \mathcal{G}_\infty\}$. In the next corollary Krein's formula is rewritten in terms of this decomposition. The canonical embedding of $\mathcal{G}_{\mathrm{op}}$ in $\mathcal{G}$ is denoted by ι_{op}.

Corollary *Let S, T, $\{\mathcal{G},\Gamma_0,\Gamma_1\}$, γ and M be as in Theorem 1.16. Further, let Θ be a self-adjoint, maximal dissipative or maximal accumulative relation in $\mathcal{G}$ and assume that $\lambda \in \rho(A_0)$ is not an eigenvalue of A_Θ. Then*

$$\Theta_{\mathrm{op}} - P_{\mathrm{op}} M(\lambda)|_{\mathcal{G}_{\mathrm{op}}}$$

defined on $\operatorname{dom}\Theta_{\mathrm{op}} \cap \operatorname{dom} M(\lambda)$ *is an injective operator in* $\mathcal{G}_{\mathrm{op}}$ *and*

$$\begin{aligned}(A_\Theta - \lambda)^{-1} g \\ = (A_0-\lambda)^{-1} g + \gamma(\lambda)\,\iota_{\mathrm{op}}\big(\Theta_{\mathrm{op}} - P_{\mathrm{op}} M(\lambda)|_{\mathcal{G}_{\mathrm{op}}}\big)^{-1} P_{\mathrm{op}}\gamma(\bar\lambda)^* g \end{aligned} \tag{1.29}$$

holds for all $g \in \operatorname{ran}(A_\Theta - \lambda)$.

Proof First we show that $\Theta_{\mathrm{op}} - P_{\mathrm{op}} M(\lambda)|_{\mathcal{G}_{\mathrm{op}}}$ is an injective operator in $\mathcal{G}_{\mathrm{op}}$. Let $x \in \operatorname{dom}\Theta_{\mathrm{op}} \cap \operatorname{dom} M(\lambda)$ be such that $(\Theta_{\mathrm{op}} - P_{\mathrm{op}}M(\lambda))x = 0$. Then we have

$$\big(x; (\Theta_{\mathrm{op}} - P_{\mathrm{op}}M(\lambda))x \oplus (y - (I-P_{\mathrm{op}})M(\lambda)x)\big) \in \Theta - M(\lambda), \quad y \in \mathcal{G}_\infty,$$

and according to Theorem 1.16, $\Theta - M(\lambda)$ is injective; thus $x = 0$. It remains to show the equality

$$\big(\Theta - M(\lambda)\big)^{-1} = \iota_{\mathrm{op}}\big(\Theta_{\mathrm{op}} - P_{\mathrm{op}} M(\lambda)|_{\mathcal{G}_{\mathrm{op}}}\big)^{-1} P_{\mathrm{op}}, \tag{1.30}$$

which was proved in [Langer and Textorius, 1977, (1.3)] for the case when $M(\lambda) \in \mathfrak{B}(\mathcal{G})$. We have the following chain of equivalences (note that $\operatorname{dom}\Theta = \operatorname{dom}\Theta_{\mathrm{op}} \subset \mathcal{G}$):

$$\begin{aligned}
&(x;y)^\top \in \big(\Theta - M(\lambda)\big)^{-1} \\
&\iff y \in \operatorname{dom}\Theta \cap \operatorname{dom} M(\lambda),\ \exists\, u \in \mathcal{G}\colon (y;u)^\top \in \Theta,\ x = u - M(\lambda)y \\
&\iff y \in \operatorname{dom}\Theta \cap \operatorname{dom} M(\lambda),\ \exists\, u \in \mathcal{G}\colon P_{\mathrm{op}}u = \Theta_{\mathrm{op}}y,\ x = u - M(\lambda)y \\
&\iff y \in \operatorname{dom}\Theta \cap \operatorname{dom} M(\lambda),\ P_{\mathrm{op}}x = \big(\Theta_{\mathrm{op}} - P_{\mathrm{op}}M(\lambda)\big)y \\
&\iff (P_{\mathrm{op}}x;y)^\top \in \big(\Theta_{\mathrm{op}} - P_{\mathrm{op}}M(\lambda)|_{\mathcal{G}_{\mathrm{op}}}\big)^{-1},
\end{aligned}$$

which shows (1.30). Now formula (1.29) follows from Theorem 1.16. □

With the help of Krein's formula one can show the following theorem, which provides a sufficient condition for self-adjointness of the extension A_Θ and which was proved in [Behrndt, Langer and Lotoreichik, 2011]. We make use of the notation

$$\Theta^{-1}(X) := \left\{ x \in \mathcal{G}\colon \exists\, y \in X \text{ so that } \begin{pmatrix} x \\ y \end{pmatrix} \in \Theta \right\}$$

for a linear relation Θ in $\mathcal{G}$ and a subspace $X \subset \mathcal{G}$.

Theorem 1.17 *Let S be a densely defined closed symmetric operator in $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\overline{T} = S^*$ with $A_i = T \upharpoonright \ker\Gamma_i$, $i = 0, 1$, and Weyl function M. Assume that A_1 is self-adjoint and that $\overline{M(\lambda_0)}$ is a compact operator in $\mathcal{G}$ for some $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$. If Θ is a self-adjoint relation in $\mathcal{G}$ such that*

$$0 \notin \sigma_{\mathrm{ess}}(\Theta) \qquad \text{and} \qquad \Theta^{-1}\big(\operatorname{ran}\overline{M(\lambda_\pm)}\big) \subset \mathcal{G}_0 \tag{1.31}$$

hold for some $\lambda_+ \in \mathbb{C}^+$ and some $\lambda_- \in \mathbb{C}^-$, then A_Θ as defined in (1.16) *is a self-adjoint operator in $\mathcal{H}$. In particular, the second condition in* (1.31) *is satisfied if* $\operatorname{dom}\Theta \subset \mathcal{G}_0$.

Remark We also mention that if in the above theorem Θ is assumed to be maximal dissipative (maximal accumulative) and the second condition in (1.31) is replaced by the condition

$$\Theta^{-1}\big(\operatorname{ran}\overline{M(\lambda)}\big) \subset \mathcal{G}_0$$

for some $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$, respectively), then the operator A_Θ in (1.16) is maximal dissipative (maximal accumulative) in $\mathcal{H}$.

We formulate another variant of Theorem 1.17 below, which will be used later on. Observe that if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $\overline{T} = S^*$ with corresponding Weyl function M and if, in addition,

$A_1 = T \upharpoonright \ker \Gamma_1$ is self-adjoint, then $\{\mathcal{G}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where $\widetilde{\Gamma}_0 := -\Gamma_1$ and $\widetilde{\Gamma}_1 := \Gamma_0$ is also a quasi boundary triple for $\overline{T} = S^*$ with corresponding Weyl function $\widetilde{M} = -M^{-1}$ and self-adjoint operator $\widetilde{A}_1 = T \upharpoonright \ker \widetilde{\Gamma}_1 = A_0$. Moreover,

$$A_\Theta = T \upharpoonright \big\{f \in \operatorname{dom} T \colon \Gamma f \in \Theta\big\} = T \upharpoonright \big\{f \in \operatorname{dom} T \colon \widetilde{\Gamma} f \in \widetilde{\Theta}\big\} = \widetilde{A}_{\widetilde{\Theta}}$$

holds with $\widetilde{\Theta} = -\Theta^{-1}$. This transformation of quasi boundary triples leads to the following theorem.

Theorem 1.18 *Let S be a densely defined closed symmetric operator in $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\overline{T} = S^*$ with $A_i = T \upharpoonright \ker \Gamma_i$, $i = 0, 1$, and Weyl function M. Assume that A_1 is self-adjoint and that $\overline{M(\lambda_0)^{-1}}$ is a compact operator in $\mathcal{G}$ for some $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$. If Θ is a bounded self-adjoint operator in $\mathcal{G}$ such that*

$$\Theta\big(\operatorname{dom} \overline{M(\lambda_\pm)}\big) \subset \mathcal{G}_1 \tag{1.32}$$

holds for some $\lambda_+ \in \mathbb{C}^+$ and some $\lambda_- \in \mathbb{C}^-$, then

$$A_\Theta = T \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0) \tag{1.33}$$

is a self-adjoint operator in $\mathcal{H}$. In particular, the condition (1.32) *is satisfied if* $\operatorname{ran} \Theta \subset \mathcal{G}_1$.

Remark For completeness we also remark that for a dissipative (accumulative) bounded operator Θ with the property

$$\Theta\big(\operatorname{dom} \overline{M(\lambda)}\big) \subset \mathcal{G}_1$$

for some $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$, respectively) it follows that the operator A_Θ in (1.33) is maximal dissipative (maximal accumulative) in $\mathcal{H}$.

1.4 Quasi boundary triples for elliptic operators and Dirichlet-to-Neumann maps

In this section we consider the same type of elliptic operators as in Section 1.2 and we define and study a family of quasi boundary triples for the maximal operator. As boundary mappings we choose the Dirichlet and (oblique) Neumann trace so that the associated Weyl function turns out to be the Dirichlet-to-Neumann map. Only the case of a bounded domain Ω is treated here, although the considerations for unbounded domains with compact boundaries (so-called exterior domains) are very similar; cf. [Behrndt, Langer and Lotoreichik, 2011].

Let again Ω be a bounded domain in $\mathbb{R}^n$, $n > 1$, with C^∞-boundary $\partial\Omega$ and consider the expression

$$\mathcal{L} = -\sum_{j,k=1}^{n} \partial_j \, a_{jk} \, \partial_k + a$$

on Ω with real-valued coefficients $a_{jk} \in C^\infty(\overline{\Omega})$, $a \in L^\infty(\Omega)$ such that $a_{jk} = a_{kj}$ for all $j, k = 1, \dots, n$. In addition, $\mathcal{L}$ is assumed to be elliptic, that is,

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq C\sum_{k=1}^{n} \xi_k^2, \qquad \xi = (\xi_1, \dots, \xi_n)^\top \in \mathbb{R}^n, \; x \in \overline{\Omega},$$

holds for some constant $C > 0$. In the following, the spaces

$$H^s_{\mathcal{L}}(\Omega) := \bigl\{ f \in H^s(\Omega) \colon \mathcal{L}(f) \in L^2(\Omega) \bigr\}, \qquad s \in \bigl[\tfrac{3}{2}, 2\bigr],$$

are used as domains for the boundary mappings; the cases $s = 2$ and $s = \frac{3}{2}$ were already studied in [Behrndt and Langer, 2007]. The spaces $H^s_{\mathcal{L}}(\Omega)$ are frequently used in the theory of elliptic operators; see, e.g. [Grubb, 1968, 1971; Lions and Magenes, 1972] and are usually defined for all $s \in [0, \infty)$. Then, in particular, $H^0_{\mathcal{L}}(\Omega)$ coincides with the maximal domain $\mathfrak{D}_{\max}$ and $H^s_{\mathcal{L}}(\Omega) = H^s(\Omega)$ for $s \geq 2$. In the following we deal with the family of differential operators T_s, $s \in [\frac{3}{2}, 2]$, defined by

$$T_s f = \mathcal{L}(f), \qquad \operatorname{dom} T_s = H^s_{\mathcal{L}}(\Omega).$$

Recall that the minimal operator associated with $\mathcal{L}$ in $L^2(\Omega)$ is the densely defined closed symmetric operator $Sf = \mathcal{L}(f)$, $\operatorname{dom} S = H^2_0(\Omega)$, that S has equal and infinite deficiency indices, and that the adjoint S^* of S coincides with the maximal realization of $\mathcal{L}$ in $L^2(\Omega)$ defined on $\mathfrak{D}_{\max}$; see (1.11). The self-adjoint realizations of $\mathcal{L}$ in $L^2(\Omega)$ with Dirichlet or Neumann boundary conditions are denoted by A_D and A_N, respectively, i.e.

$$\begin{aligned} A_D f &= \mathcal{L}(f), \qquad \operatorname{dom} A_D = \bigl\{ f \in H^2(\Omega) \colon f|_{\partial\Omega} = 0 \bigr\}, \\ A_N f &= \mathcal{L}(f), \qquad \operatorname{dom} A_N = \left\{ f \in H^2(\Omega) \colon \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = 0 \right\}, \end{aligned}$$

where $\frac{\partial f}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega}$ is defined as in (1.12).

The proof of the next proposition consists in principle of applying Theorem 1.11. However, we provide a short proof here for the convenience of the reader; cf. [Behrndt and Langer, 2007, Proposition 4.6] for a similar consideration.

Proposition 1.19 *For each $s \in [\frac{3}{2}, 2]$ the triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0 f := f|_{\partial\Omega} \qquad \textit{and} \qquad \Gamma_1 f := -\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \quad f \in H^s_{\mathcal{L}}(\Omega), \tag{1.34}$$

is a quasi boundary triple for the maximal operator $\overline{T}_s = S^*$ such that

$$A_D = T_s \upharpoonright \ker \Gamma_0 \qquad \textit{and} \qquad A_N = T_s \upharpoonright \ker \Gamma_1.$$

Proof We apply Theorem 1.11. Since $H^2(\Omega) \subset H^s_{\mathcal{L}}(\Omega)$ for all $s \in [\frac{3}{2}, 2]$, the restriction of T_s to

$$\ker \Gamma_0 = \bigl\{ f \in H^s_{\mathcal{L}}(\Omega) \colon f|_{\partial\Omega} = 0 \bigr\}$$

is an extension of the self-adjoint Dirichlet operator A_D, i.e. condition (i) of Theorem 1.11 is satisfied. In order to verify condition (ii), note first that for $s \in [\frac{3}{2}, 2]$ and $f \in H^s_{\mathcal{L}}(\Omega)$ we have $f|_{\partial\Omega} \in L^2(\partial\Omega)$, and for $s \in (\frac{3}{2}, 2]$ and $f \in H^s_{\mathcal{L}}(\Omega)$ we have $\frac{\partial f}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} \in L^2(\partial\Omega)$. According to [Grubb, 1968, Theorem I.3.3] and [Lions and Magenes, 1972], $\frac{\partial f}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} \in L^2(\partial\Omega)$ holds also for $s = \frac{3}{2}$ and $f \in H^s_{\mathcal{L}}(\Omega)$. Hence Γ_0, Γ_1 are well defined. Since the map

$$H^2(\Omega) \ni f \mapsto \left\{ f|_{\partial\Omega}, \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right\} \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

is surjective onto the dense subset $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ of the space $L^2(\partial\Omega) \times L^2(\partial\Omega)$, see e.g. [Lions and Magenes, 1972, Theorem 1.8.3], and $H^2(\Omega) \subset H^s_{\mathcal{L}}(\Omega)$, $s \in [\frac{3}{2}, 2]$, it follows that $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top$ is dense in $L^2(\partial\Omega) \times L^2(\partial\Omega)$, i.e. the first condition in (ii) of Theorem 1.11 is satisfied; the second condition follows from $C_0^\infty(\Omega) \subset \ker \Gamma_0 \cap \ker \Gamma_1$. With Γ_0 and Γ_1 from (1.34), Green's identity reads as

$$\begin{aligned}(T_s f, g) - (f, T_s g) &= \left(f|_{\partial\Omega}, \frac{\partial g}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right) - \left(\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, g|_{\partial\Omega} \right) \\ &= (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)\end{aligned}$$

with the inner product $(\cdot, -)$ in $L^2(\partial\Omega)$ on the right-hand side. Hence also condition (iii) in Theorem 1.11 is fulfilled. Therefore the operator

$$T \upharpoonright \ker \Gamma_0 \cap \ker \Gamma_1 \tag{1.35}$$

is a densely defined closed symmetric operator in $L^2(\Omega)$, $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for its adjoint and $T \upharpoonright \ker \Gamma_0$ is equal to the

Dirichlet operator A_D. Since $\ker\Gamma_0 = \operatorname{dom} A_D \subset H^2(\Omega)$, we have

$$\begin{aligned}\ker\Gamma_0 \cap \ker\Gamma_1 &= \bigl\{f \in H^s_{\mathcal{L}}(\Omega) \colon \Gamma_0 f = 0,\ \Gamma_1 f = 0\bigr\} \\ &= \bigl\{f \in H^2(\Omega) \colon \Gamma_0 f = 0,\ \Gamma_1 f = 0\bigr\} = H^2_0(\Omega),\end{aligned}$$

which shows that the operator in (1.35) is the minimal operator S associated with $\mathcal{L}$. □

Remark We point out that for the statements in the above proposition the scale $[\frac{3}{2}, 2]$ cannot be enlarged. The upper bound 2 is necessary in order to ensure that the self-adjoint Dirichlet operator is contained in (and hence equal to) $T \upharpoonright \ker\Gamma_0$, whereas the lower bound $\frac{3}{2}$ is necessary to ensure Green's identity with boundary terms in $L^2(\partial\Omega)$. However, Green's identity could also be considered, e.g. for functions $f, g \in H^1_{\mathcal{L}}(\Omega)$ so that on the right-hand side the extension of the $L^2(\partial\Omega)$ inner product to $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ appears, which makes it necessary to modify the boundary mappings by isomorphisms $\iota_\pm$ as in Section **??**. In this case the corresponding Weyl function is not the Dirichlet-to-Neumann map in $L^2(\partial\Omega)$. For completeness we also mention that for $s = \frac{3}{2}$ the quasi boundary triple $\{L^2(\partial\Omega), -\Gamma_1, \Gamma_0\}$ is a generalized boundary triple in the sense of [Derkach and Malamud, 1995]; see also [Behrndt and Langer, 2007, Section 4.2].

In the next proposition the γ-field and Weyl function corresponding to the quasi boundary triples $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ from Proposition 1.19 are specified.

Proposition 1.20 *Let $s \in [\frac{3}{2}, 2]$ and let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple for the maximal operator $\overline{T}_s = S^*$ from Proposition 1.19 with $A_D = T_s \upharpoonright \ker\Gamma_0$. Then the following statements are true for all $\lambda \in \rho(A_D)$:*

(i) *For $y \in H^{s-1/2}(\partial\Omega)$ there exists a unique function $f_\lambda(y)$ in $H^s_{\mathcal{L}}(\Omega)$ that solves the boundary value problem*

$$\mathcal{L}(u) = \lambda u, \qquad u|_{\partial\Omega} = y. \tag{1.36}$$

(ii) *The γ-field of $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is given by*

$$\gamma_s(\lambda) \colon L^2(\partial\Omega) \to L^2(\Omega), \qquad y \mapsto f_\lambda(y),$$

with $\operatorname{dom}\gamma_s(\lambda) = H^{s-1/2}(\partial\Omega)$.

(iii) *The Weyl function of $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is given by*

$$M_s(\lambda) \colon L^2(\partial\Omega) \to L^2(\partial\Omega), \qquad y \mapsto -\frac{\partial f_\lambda(y)}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega},$$

with $\operatorname{dom} M_s(\lambda) = H^{s-1/2}(\partial\Omega)$ *and* $\operatorname{ran} M_s(\lambda) \subset H^{s-3/2}(\partial\Omega)$. *If, in addition,* $\lambda \in \rho(A_N)$, *then* $\operatorname{ran} M_s(\lambda) = H^{s-3/2}(\partial\Omega)$ *and* $\overline{M_s(\lambda)^{-1}}$ *is a compact operator in* $L^2(\partial\Omega)$ *with values in* $H^1(\partial\Omega)$.

Proof (i) Problem (1.36) is equivalent to

$$u \in \ker(T_s - \lambda), \qquad \Gamma_0 u = y.$$

Proposition 1.13 (i) shows that this boundary value problem has a unique solution, namely $u = \gamma_s(\lambda)y$.

(ii) is a consequence of (i). Observe that the domain of $\gamma_s(\lambda)$ is equal to $\operatorname{ran}\Gamma_0 = H^{s-1/2}(\partial\Omega)$.

(iii) The asserted form of the Weyl function M_s follows immediately from the definition. That $\operatorname{ran} M_s(\lambda) = H^{s-3/2}(\partial\Omega)$ for $\lambda \in \rho(A_N)$ is clear from Proposition 1.14 (i). Using duality and interpolation arguments one can show that $M_s(\lambda)^{-1}$ can be extended to a bounded operator from $H^r(\partial\Omega)$ to $H^{r+1}(\partial\Omega)$ for $r \in [-\frac{3}{2}, \frac{1}{2}]$ and $\lambda \in \rho(A_D)$; for details see, e.g. [Behrndt, Langer and Lotoreichik, 2011; Lions and Magenes, 1972; Seeley, 1969]. In particular, $\overline{M_s(\lambda)^{-1}}$ is a bounded mapping from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$. Since $H^1(\partial\Omega)$ is compactly embedded in $L^2(\partial\Omega)$ (see, e.g. [Wloka, 1987, Theorem 7.10]), this shows the compactness of $\overline{M_s(\lambda)^{-1}}$. □

The operator $\gamma_s(\lambda)$ in the previous proposition is often called Poisson operator. The Weyl function M_s is (up to a minus sign) the Dirichlet-to-Neumann operator connected with $\mathcal{L}$. It has been used, e.g. to solve inverse problems, see [Astala and Päivärinta, 2006; Nachman, 1988, 1996; Nachman, Sylvester and Uhlmann, 1988; Sylvester and Uhlmann, 1987], to detect spurious eigenvalues in numerical calculations, see [Brown and Marletta, 2004; Marletta, 2004, 2010] and to prove inequalities between Dirichlet and Neumann eigenvalues, see [Filonov, 2004; Friedlander, 1991; Safarov, 2008]. See also [Amrein and Pearson, 2004] where a Weyl function for elliptic operators was constructed. Moreover, Dirichlet-to-Neumann maps on rough domains were defined and studied in [Arendt and ter Elst, 2011].

As a consequence of Theorem 1.18, the remark below that theorem and Proposition 1.20 (iii), we obtain the following sufficient condition for a self-adjoint, maximal dissipative or maximal accumulative parameter Θ in $L^2(\partial\Omega)$ to determine a self-adjoint, maximal dissipative or maximal accumulative realization A_Θ of $\mathcal{L}$ in $L^2(\Omega)$.

Theorem 1.21 *Let* $s \in [\frac{3}{2}, 2]$ *and let* $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ *be the quasi* *boundary triple for* $\overline{T}_s = S^*$ *from Proposition 1.19. For every bounded self-adjoint* (*dissipative, accumulative*) *operator* Θ *in* $L^2(\partial\Omega)$ *such that*

$$\Theta\big(H^1(\partial\Omega)\big) \subset H^{s-3/2}(\partial\Omega) \tag{1.37}$$

is satisfied, the differential operator

$$A_\Theta f = \mathcal{L}(f),\ \operatorname{dom} A_\Theta = \left\{ f \in H^s_{\mathcal{L}}(\Omega) \colon \Theta f|_{\partial\Omega} = -\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right\} \tag{1.38}$$

is a self-adjoint (*maximal dissipative, maximal accumulative, respectively*) *realization of* $\mathcal{L}$ *in* $L^2(\Omega)$.

Since for $s = \frac{3}{2}$ the condition (1.37) in the above theorem reduces to $\Theta(H^1(\partial\Omega)) \subset L^2(\partial\Omega)$, which is trivially satisfied, we obtain the following corollary; cf. [Behrndt and Langer, 2007, Theorem 4.8].

Corollary *Let* $s = \frac{3}{2}$ *and let* $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ *be the quasi boundary triple for* $\overline{T}_{3/2} = S^*$ *from Proposition 1.19. For every bounded self-adjoint* (*dissipative, accumulative*) *operator* Θ *in* $L^2(\partial\Omega)$*, the differential operator* A_Θ *in* (1.38) *is a self-adjoint* (*maximal dissipative, maximal accumulative, respectively*) *realization of* $\mathcal{L}$ *in* $L^2(\Omega)$.

As a consequence of Theorem 1.16 and its corollaries, we obtain a variant of Krein's formula for the self-adjoint (maximal dissipative, maximal accumulative) realizations of $\mathcal{L}$ in Theorem 1.21 and the above corollary.

Theorem 1.22 *Let* $s \in [\frac{3}{2}, 2]$ *and let* $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ *be the quasi boundary triple for* $\overline{T}_s = S^*$ *from Proposition 1.19 with* γ*-field* γ_s *and Weyl function* M_s *from Proposition 1.20. Let* Θ *be a self-adjoint* (*dissipative, accumulative*) *operator in* $L^2(\partial\Omega)$ *such that* (1.37) *is satisfied. Then*

$$(A_\Theta - \lambda)^{-1} = (A_D - \lambda)^{-1} + \gamma_s(\lambda)\big(\Theta - M_s(\lambda)\big)^{-1}\gamma_s(\bar{\lambda})^* \tag{1.39}$$

holds for all $\lambda \in \rho(A_\Theta) \cap \rho(A_D)$.

Krein's formula (1.39) in the above theorem allows the following interpretation: since A_Θ and A_D act formally in the same way (as both operators are realizations of the same differential expression $\mathcal{L}$), only their domains are different, and since $\operatorname{dom} A_\Theta$ and $\operatorname{dom} A_D$ are specified by boundary conditions on functions from $H^s_{\mathcal{L}}(\Omega)$, the resolvent difference $(A_\Theta - \lambda)^{-1} - (A_D - \lambda)^{-1}$ can be "localized" on the boundary $\partial\Omega$, that is, as the perturbation term $(\Theta - M_s(\lambda))^{-1}$ on the right-hand side of (1.39).

If the Dirichlet and Neumann boundary mappings are swapped and the quasi boundary triple $\{L^2(\partial\Omega), -\Gamma_1, \Gamma_0\}$ is considered, one obtains a variant of Krein's formula where the operators A_Θ as above and A_N, the self-adjoint Neumann realization, are compared. The next result on spectral estimates for singular values of resolvent differences from [Behrndt et al., 2010, Theorem 3.5 and Remark 3.7] is essentially based on this idea.

Theorem 1.23 *Let $s = \frac{3}{2}$ and let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple for $\overline{T}_{3/2} = S^*$ from Proposition 1.19. Let Θ be a bounded self-adjoint (dissipative, accumulative) operator in $L^2(\partial\Omega)$ and let A_Θ be the corresponding self-adjoint (maximal dissipative, maximal accumulative, respectively) realization of $\mathcal{L}$ in $L^2(\Omega)$. Then for all $\lambda \in \rho(A_\Theta) \cap \rho(A_N)$ the singular values s_k of the resolvent difference*

$$(A_\Theta - \lambda)^{-1} - (A_N - \lambda)^{-1} \tag{1.40}$$

satisfy $s_k = O(k^{-\frac{3}{n-1}})$, $k \to \infty$, and hence the expression in (1.40) belongs to the Schatten–von Neumann ideal $\mathfrak{S}_p(L^2(\Omega))$ for all $p > \frac{n-1}{3}$.

Proof First we express the resolvent difference (1.40) in a similar form as in Theorem 1.22. Observe that the γ-field and Weyl function corresponding to the quasi boundary triple $\{L^2(\partial\Omega), -\Gamma_1, \Gamma_0\}$ are given by

$$-\gamma_{3/2}(\lambda) M_{3/2}(\lambda)^{-1} \quad \text{and} \quad -M_{3/2}(\lambda)^{-1}, \quad \lambda \in \rho(A_D) \cap \rho(A_N), \tag{1.41}$$

and that the operator A_Θ is self-adjoint (maximal dissipative, maximal accumulative, respectively) by the corollary below Theorem 1.22. Observe that the boundary parameter Θ with respect to the triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ has to be replaced by the self-adjoint (maximal dissipative, maximal accumulative) relation $\widetilde{\Theta} := -\Theta^{-1}$ when one expresses the boundary condition in terms of the quasi boundary triple $\{L^2(\partial\Omega), -\Gamma_1, \Gamma_0\}$. Now it follows from Theorem 1.16 and its corollaries applied to the quasi boundary triple $\{L^2(\partial\Omega), -\Gamma_1, \Gamma_0\}$ and (1.41) that for $\lambda \in \rho(A_\Theta) \cap \rho(A_N) \cap \rho(A_D)$ the resolvent difference in (1.40) coincides with

$$\overline{\gamma_{3/2}(\lambda)} M_{3/2}(\lambda)^{-1} \big(\widetilde{\Theta}_{\mathrm{op}} + P_{\mathrm{op}} M_{3/2}(\lambda)^{-1}|_{\mathcal{G}_{\mathrm{op}}}\big)^{-1} P_{\mathrm{op}} M_{3/2}(\lambda)^{-1} \gamma_{3/2}(\bar{\lambda})^*, \tag{1.42}$$

where $\widetilde{\Theta}$ is decomposed into $\widetilde{\Theta}_{\mathrm{op}} \oplus \widetilde{\Theta}_\infty$ with P_{op} being the projection onto $\mathcal{G}_{\mathrm{op}}$. Note that $M_{3/2}(\lambda)^{-1}$ is defined on the whole space $\mathcal{G}$ and that in fact in (1.42) $\overline{\gamma_{3/2}(\lambda)}$ can be replaced by $\gamma_{3/2}(\lambda)$ (as an operator from $L^2(\partial\Omega)$ to $L^2(\Omega)$). Since $M_{3/2}(\lambda)^{-1}$ is a compact operator in $L^2(\partial\Omega)$

(see Proposition 1.20 (iii)), a Fredholm argument for $\widetilde{\Theta}_{\rm op}$ shows that

$$\big(\widetilde{\Theta}_{\rm op} + P_{\rm op} M_{3/2}(\lambda)^{-1}|_{\mathcal{G}_{\rm op}}\big)^{-1}$$

is a bounded and everywhere defined operator in $\mathcal{G}_{\rm op}$; cf. the proof of [Behrndt and Langer, 2007, Theorem 4.8]. Moreover, by Proposition 1.13 (iii) the operator $\gamma_{3/2}(\bar{\lambda})^*$ is bounded from $L^2(\Omega)$ into $L^2(\partial\Omega)$ with range in $H^{1/2}(\partial\Omega)$. Hence it is closed from $L^2(\Omega)$ to $H^{1/2}(\partial\Omega)$ and therefore bounded by the closed graph theorem. It follows from [Behrndt et al., 2010, Lemma 3.4] (and its proof) that the singular values of $\gamma_{3/2}(\lambda)^*$ satisfy $O(k^{-\frac{1}{2(n-1)}})$, $k \to \infty$. A similar argument using [Behrndt et al., 2010, Lemma 3.4] shows that the singular values of $M_{3/2}(\lambda)^{-1}$ satisfy $O(k^{-\frac{1}{n-1}})$, $k \to \infty$, and hence the singular values of the operators

$$M_{3/2}(\lambda)^{-1}\gamma_{3/2}(\bar{\lambda})^* \quad \text{and} \quad \overline{\gamma_{3/2}(\lambda)} M_{3/2}(\lambda)^{-1}$$

in (1.42) both satisfy $O(k^{-\frac{3}{2(n-1)}})$, $k \to \infty$. This implies the spectral estimates in Theorem 1.23. □

The asymptotics of singular values of resolvent differences of the Dirichlet, Neumann and Robin realizations of $\mathcal{L}$ have been studied already in [Birman, 1962] and later among others in [Grubb, 1974; Birman and Solomjak, 1980; Grubb, 1984a,b; Grubb; Malamud, 2010; Grubb, 2011]. For usual Robin boundary conditions, Theorem 1.23 reads as follows.

Corollary *Assume that the values of $\beta \in L^\infty(\partial\Omega)$ are real (or have positive, negative imaginary parts) and let A_β be the self-adjoint (maximal dissipative, maximal accumulative, respectively) realization of $\mathcal{L}$ defined on*

$$\operatorname{dom} A_\beta = \left\{ f \in H^{3/2}_{\mathcal{L}}(\Omega) \colon \beta f|_{\partial\Omega} = -\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right\}.$$

Then for all $\lambda \in \rho(A_\beta) \cap \rho(A_N)$ the singular values s_k of the resolvent difference

$$(A_\beta - \lambda)^{-1} - (A_N - \lambda)^{-1} \tag{1.43}$$

satisfy $s_k = O(k^{-\frac{3}{n-1}})$, $k \to \infty$, and hence the expression in (1.43) belongs to the Schatten–von Neumann ideal $\mathfrak{S}_p(L^2(\Omega))$ for all $p > \frac{n-1}{3}$.

Finally, we want to relate the quasi boundary triple from this section to the boundary triple from Section 1.2. Denote the boundary mappings and the Weyl function for the boundary triple for the maximal operator S^* from Section 1.2 by $\{L^2(\partial\Omega), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where

$$\widetilde{\Gamma}_0 f = \iota_- f|_{\partial\Omega} \qquad \text{and} \qquad \widetilde{\Gamma}_1 f = -\iota_+ \frac{\partial f_D}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \tag{1.44}$$

with $f = f_D + f_\eta$, $f_D \in \operatorname{dom} A_D$, $f_\eta \in \ker(S^* - \eta)$, and η is some fixed point in $\mathbb{R} \cap \rho(A_D)$. The corresponding Weyl function is denoted by $\widetilde{M}$.

Proposition 1.24 *Let $s \in [\frac{3}{2}, 2]$ and let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple for $\overline{T}_s = S^*$ from Proposition 1.19 with Weyl function M_s. The following relations hold for all $f = f_D + f_\eta \in \operatorname{dom} T_s$ with $f_D \in \operatorname{dom} A_D$ and $f_\eta \in \ker(T_s - \eta)$:*

(i) $\widetilde{\Gamma}_0 f = \iota_- \Gamma_0 f$ *and* $\widetilde{\Gamma}_1 f = \iota_+\big(\Gamma_1 f - M_s(\eta)\Gamma_0 f\big)$;

(ii) $\widetilde{\Gamma}_1 f = \Theta \widetilde{\Gamma}_0 f$ *if and only if* $\Gamma_1 f = \big(\iota_+^{-1}\Theta\iota_- + M_s(\eta)\big)\Gamma_0 f$;

(iii) $\widetilde{M}(\lambda)\iota_-\Gamma_0 f = \iota_+\big(M_s(\lambda) - M_s(\eta)\big)\Gamma_0 f$.

In particular, the Weyl functions $\widetilde{M}$ and M_s are connected via

$$\iota_+^{-1}\widetilde{M}(\lambda)\iota_- = M_s(\lambda) - M_s(\eta), \qquad \lambda \in \rho(A_D). \tag{1.45}$$

Proof (i) The first relation is immediate from (1.44) and (1.34) in Proposition 1.19. The second statement follows with $f = f_D + f_\eta$ from

$$\widetilde{\Gamma}_1 f = -\iota_+ \frac{\partial f_D}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = \iota_+\Gamma_1 f_D = \iota_+(\Gamma_1 f - \Gamma_1 f_\eta) = \iota_+\big(\Gamma_1 f - M_s(\eta)\Gamma_0 f\big).$$

(ii) is a simple consequence of (i).

(iii) Let $f \in \ker(T_s - \lambda)$ and $\lambda \in \rho(A_D)$. Then

$$\begin{aligned}\widetilde{M}(\lambda)\iota_-\Gamma_0 f = \widetilde{M}(\lambda)\widetilde{\Gamma}_0 f = \widetilde{\Gamma}_1 f &= \iota_+\big(\Gamma_1 f - M_s(\eta)\Gamma_0 f\big)\\ &= \iota_+\big(M_s(\lambda)\Gamma_0 f - M_s(\eta)\Gamma_0 f\big)\\ &= \iota_+\big(M_s(\lambda) - M_s(\eta)\big)\Gamma_0 f\end{aligned}$$

implies the third assertion.

Relation (1.45) is an immediate consequence of (iii). □

Note that the operators on both sides of (1.45) are bounded operators in $L^2(\partial\Omega)$. For the right-hand side this follows from Proposition 1.14 (iv). We remark that relation (1.45) also shows that the operator on the right-hand side can be extended to a bounded operator from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. Let us also mention that relation (1.45) implies that the Weyl function of the boundary triple from Section 1.2 is a regularization of the

Dirichlet-to-Neumann map. Such regularizations were also considered in, e.g. [Grubb, 1968; Malamud, 2010; Ryzhov, 2007; Višik, 1952].

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