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Spectral enclosures for non-self-adjoint extensions of symmetric operators

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The spectral properties of non-self-adjoint extensions $A_{[B]}$ of a symmetric operator in a Hilbert space are studied with the help of ordinary and quasi boundary triples and the corresponding Weyl functions. These extensions are given in terms of abstract boundary conditions involving an (in general non-symmetric) boundary operator B. In the abstract part of this paper, sufficient conditions for sectoriality and m-sectoriality as well as sufficient conditions for $A_{[B]}$ to have a non-empty resolvent set are provided in terms of the parameter B and the Weyl function. Special attention is paid to Weyl functions that decay along the negative real line or inside some sector in the complex plane, and spectral enclosures for $A_{[B]}$ are proved in this situation. The abstract results are applied to elliptic differential operators with local and non-local Robin boundary conditions on unbounded domains, to Schrödinger operators with δ-potentials of complex strengths supported on unbounded hypersurfaces or

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infinitely many points on the real line, and to quantum graphs with non-self-adjoint vertex couplings.

Contents

1. Introduction

Spectral problems for differential operators in Hilbert spaces and related boundary value problems have attracted a lot of attention in the last decades and have strongly influenced the development of modern functional analysis and operator theory. For example, the classical treatment of Sturm–Liouville operators and the corresponding Titchmarsh–Weyl theory in Hilbert spaces have led to the abstract concept of boundary triples and their Weyl functions (see [43,55,82,96]), which is an efficient and well-established tool to investigate closed extensions of symmetric operators and their spectral properties via abstract boundary maps and an analytic function; see, e.g. [1,5,40–42,44,53,56,115,117,125]. The more recent notion of quasi boundary triples and their Weyl functions are inspired by PDE analysis in a similar way. This abstract concept from [22,24] is tailor-made for spectral problems involving elliptic partial differential operators and the corresponding boundary value problems; the Weyl function of a quasi boundary triple is the abstract counterpart of the Dirichlet-to-Neumann map. For different abstract treatments of elliptic PDEs and Dirichlet-to-Neumann maps we refer to the classical works [84,128] and the more recent approaches [11–13,30,54,77–80,83,91,118,122,124].

To recall the notions of ordinary and quasi boundary triples in more detail, let S be a densely defined, closed, symmetric operator in a Hilbert space $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ and let S^* denote its adjoint; then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be an *ordinary boundary triple* for S^* if $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ are linear mappings from the domain of S^* into an auxiliary Hilbert space $(\mathcal{G}, (\cdot,\cdot)_{\mathcal{G}})$ that satisfy the abstract Lagrange or Green identity

$$(S^*f, g)_{\mathcal{H}} - (f, S^*g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}} \quad \text{for all } \ f, g \in \operatorname{dom} S^* \tag{1.1}$$

and a certain maximality condition. The corresponding *Weyl function* M is an operator-valued function in $\mathcal{G}$, which is defined by

$$M(\lambda)\Gamma_0 f = \Gamma_1 f, \qquad f \in \ker(S^* - \lambda),\ \lambda \in \rho(A_0), \tag{1.2}$$

where $A_0 = S^* \upharpoonright \ker \Gamma_0$ is a self-adjoint operator in $\mathcal{H}$. For a singular Sturm–Liouville expression $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ in $L^2(0,\infty)$ with a real-valued potential $V \in L^\infty(0,\infty)$ the operators S and S^* can be chosen as the minimal and maximal operators, respectively, together with $\mathcal{G} = \mathbb{C}$ and $\Gamma_0 f = f(0)$, $\Gamma_1 f = f'(0)$ for $f \in \operatorname{dom} S^*$; in this case the corresponding abstract Weyl function coincides with the classical Titchmarsh–Weyl m-function.

The notion of *quasi boundary triples* is a natural generalization of the concept above, inspired by, and developed for, the treatment of elliptic differential operators. The main difference is, that the boundary maps Γ_0 and Γ_1 are only defined on a subspace $\operatorname{dom} T$ of $\operatorname{dom} S^*$, where T is an operator in $\mathcal{H}$ which satisfies $\overline{T} = S^*$. The identities (1.1) and (1.2) are only required to hold for elements in $\operatorname{dom} T$; see Section 2 for precise definitions. For the Schrödinger operator $-\Delta + V$ in $L^2(\Omega)$ with a real-valued potential $V \in L^\infty(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ with a sufficiently regular boundary $\partial\Omega$, the operators S and S^* can again be taken as the minimal and maximal operator, respectively, and a convenient choice for the domain of $T = -\Delta + V$ is $H^2(\Omega)$. Then $\mathcal{G} = L^2(\partial\Omega)$ and $\Gamma_0 f = \partial_\nu f|_{\partial\Omega}$, $\Gamma_1 f = f|_{\partial\Omega}$ (where the latter denote the normal derivative and trace) form a quasi boundary triple, and the corresponding Weyl function is the energy-dependent Neumann-to-Dirichlet map.

The main focus of this paper is on non-self-adjoint extensions of S that are restrictions of S^* parameterized by an ordinary or quasi boundary triple and an (in general non-self-adjoint) boundary parameter, and to describe their spectral properties. For a quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and a linear operator B in $\mathcal{G}$ we consider the operator

$$A_{[B]} f = Tf, \qquad \operatorname{dom} A_{[B]} = \bigl\{ f \in \operatorname{dom} T : \Gamma_0 f = B\Gamma_1 f \bigr\} \tag{1.3}$$

in $\mathcal{H}$. The principal results of this paper include (a) a sufficient condition for $A_{[B]}$ to be m-sectorial and (b) enclosures for the numerical range and the spectrum of the operator $A_{[B]}$ in parabola-type regions. The latter make use of decay properties of the Weyl function M along the negative half-axis or inside sectors in the complex plane; in order to make these results easily applicable, we provide (c) an abstract sufficient condition for the Weyl function to decay appropriately. We point out that, to the best of our knowledge, these results are also new in the special case of ordinary boundary triples. While the operator $A_{[B]}$ can be regarded as a perturbation of the self-adjoint operator A_0 in the resolvent sense, let us mention that the spectra of additive non-self-adjoint perturbations of self-adjoint operators were studied recently in, e.g. [48–51,71]. In the second half of the present paper, we provide applications of these results to several classes of operators, namely to elliptic differential operators with local and non-local Robin boundary conditions on domains with possibly non-compact boundaries, to Schrödinger operators with δ-interactions of complex strength supported on hypersurfaces, to infinitely many point δ-interactions on the real line, and to quantum graphs with non-self-adjoint vertex couplings.

Let us explain in more detail the structure, methodology, and results of this paper. After the preliminary Section 2, our first main result is Theorem 3.1, where it is shown that, under certain assumptions on the Weyl function and the boundary parameter B, the operator $A_{[B]}$ in (1.3) is sectorial, and a sector containing the numerical range of $A_{[B]}$ is specified. However, in applications it is essential to ensure that a sectorial operator is m-sectorial; hence the next main objective is to prove that the resolvent set of the operator $A_{[B]}$ in (1.3) is non-empty, which is a non-trivial question particularly for quasi boundary triples. This problem is treated in Section 4. The principal result here is Theorem 4.1, in which we provide sufficient conditions for $\lambda_0 \in \rho(A_{[B]})$ in terms of the operator $M(\lambda_0)$ and the parameter B. In this context also a Krein-type resolvent formula is obtained, and the adjoint of $A_{[B]}$ is related to a dual parameter B'; cf. [27,29] for the special case of symmetric B. We list various corollaries of Theorem 4.1 for more specialized situations. We point out that an alternative description of sectorial and m-sectorial extensions of a symmetric operator can be found in [14,114]; see also the review article [15] and [16–18]. Section 4 is complemented by two propositions on Schatten–von Neumann properties for the resolvent difference of $A_{[B]}$ and A_0; cf. [27,55] for related abstract results and, e.g. [21,26,33,85,112,115] for applications to differential operators. Such estimates can be used, for instance, to get bounds on the discrete spectrum of $A_{[B]}$; cf. [51]. In Section 5 we consider the situation when the Weyl function M converges to 0 in norm along the negative half-axis or in some sector in the complex plane. The most important result in this section is Theorem 5.6 where, under the assumption that $\|M(\lambda)\|$ decays like a power of $\frac{1}{|\lambda|}$, the numerical range and the spectrum of $A_{[B]}$ are contained in a parabola-type region. Spectral enclosures of this type with more restrictive assumptions on B were obtained for elliptic partial differential operators in [19, 20,73]; similar enclosures for Schrödinger operators with complex-valued regular potentials can be found in [2,71,108]. They also appear in the abstract settings of so-called p-subordinate perturbations [131]. Finally, as the last topic within the abstract part of this paper, we prove in Theorem 6.1 that the Weyl function decays along the negative real line or in suitable complex sectors with a certain rate if the map $\Gamma_1|A_0 - \mu|^{-\alpha}$ is bounded for some $\alpha \in (0, \frac{1}{2}]$ and some $\mu \in \rho(A_0)$, where the rate of the decay depends on α. Example 6.4 shows the sharpness of this result.

Our abstract results are applied in Section 7 to elliptic partial differential operators with (in general non-local) Robin boundary conditions on domains with possibly non-compact boundaries; the class of admissible unbounded domains includes, for instance, domains of waveguide-type as considered in [34,67]. In Section 8 we apply our abstract results to Schrödinger operators in $\mathbb{R}^n$ with δ-potentials of complex strength supported on (not necessarily bounded) hypersurfaces. We indicate also how our abstract methods can be combined with very recent norm estimates from [75] in order to obtain further spectral enclosures and to establish absence of non-real spectrum for 'weak' complex δ-interactions in space dimensions $n \geq 3$ for compact hypersurfaces. Finally, we apply our machinery to Schrödinger operators on the real line with non-Hermitian δ-interactions supported on infinitely many points in Section 9, and to Laplacians on finite (not necessarily compact) graphs with non-self-adjoint vertex couplings in Section 10. Each of these sections has the same structure: after the problem under consideration is explained, first a quasi (or ordinary) boundary triple and its Weyl function are provided; next a lemma on the decay of the Weyl function is proved, and then a main result on spectral properties and enclosures is formulated, which can be derived easily from that decay together with the abstract results in the first part of this paper in each particular situation. To illustrate the different types of boundary conditions and interactions, more specialized cases and explicit examples are included in Sections 7–10.

Finally, let us fix some notation. By $\sqrt{\cdot}$ we denote the branch of the complex square root such that $\operatorname{Im}\sqrt{\lambda} > 0$ for all $\lambda \in \mathbb{C} \setminus [0,\infty)$. Let us set $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{C}^\pm := \{\lambda \in \mathbb{C} : \pm \operatorname{Im}\lambda > 0\}$. Moreover, for any bounded, complex-valued function α we use the abbreviation $\|\alpha\|_\infty := \sup|\alpha|$. The space of bounded, everywhere defined operators from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, and we set $\mathcal{B}(\mathcal{H}_1) := \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$. The Schatten–von Neumann ideal that consists of all compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ whose singular values are p-summable is denoted by $\mathfrak{S}_p(\mathcal{H}_1, \mathcal{H}_2)$, and we set $\mathfrak{S}_p(\mathcal{H}_1) := \mathfrak{S}_p(\mathcal{H}_1, \mathcal{H}_1)$; see, e.g. [81] for a detailed study of the $\mathfrak{S}_p$-classes. Furthermore, for each densely defined operator A in a Hilbert space we write $\operatorname{Re} A := \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A := \frac{1}{2i}(A - A^*)$ for its real and imaginary part, respectively, and, if A is closed, we denote by $\rho(A)$ and $\sigma(A)$ its resolvent set and spectrum, respectively.

2. Quasi boundary triples and their Weyl functions

In this preparatory section we first recall the notion and some properties of quasi boundary triples and their Weyl functions from [22,24]. Moreover, we discuss some elementary estimates and decay properties of the Weyl function.

In the following let S be a densely defined, closed, symmetric operator in a Hilbert space $\mathcal{H}$.

Definition 2.1. Let $T \subset S^*$ be a linear operator in $\mathcal{H}$ such that $\overline{T} = S^*$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *quasi boundary triple* for $T \subset S^*$ if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings such that

(i) the abstract Green identity

$$(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) \tag{2.1}$$

holds for all $f, g \in \operatorname{dom} T$, where $(\cdot, \cdot)$ denotes the inner product both in $\mathcal{H}$ and $\mathcal{G}$;

(ii) the map $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ has dense range;
(iii) $A_0 := T \upharpoonright \ker \Gamma_0$ is a self-adjoint operator in $\mathcal{H}$.

If condition (ii) is replaced by the condition

(ii)' the map $\Gamma_0 : \operatorname{dom} T \to \mathcal{G}$ is onto,

then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *generalized boundary triple* for $T \subset S^*$.

The notion of quasi boundary triples was introduced in [22, Definition 2.1]. The concept of generalized boundary triples appeared first in [56, Definition 6.1]. It follows from [56, Lemma 6.1] that each generalized boundary triple is also a quasi boundary triple. We remark that the converse is in general not true. A quasi or generalized boundary triple reduces to an ordinary boundary triple if the map Γ in condition (ii) is onto (see [22, Corollary 3.2]). In this case T is closed and coincides with S^*, and A_0 in condition (iii) is automatically self-adjoint. For the convenience of the reader we recall the usual definition of ordinary boundary triples.

Definition 2.2. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called an *ordinary boundary triple* for S^* if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ are linear mappings such that

(i) the abstract Green identity

$$(S^* f, g) - (f, S^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) \tag{2.2}$$

holds for all $f, g \in \operatorname{dom} S^*$;
(ii) the map $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} S^* \to \mathcal{G} \times \mathcal{G}$ is onto.

We refer the reader to [22,24] for a detailed study of quasi boundary triples, to [52,56] for generalized boundary triples and to [43,44,55,82,96] for ordinary boundary triples. For later purposes we recall the following result, which is useful to determine the adjoint and a (quasi) boundary triple for a given symmetric operator; see [22, Theorem 2.3].

Theorem 2.3. *Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces and let T be a linear operator in $\mathcal{H}$. Assume that $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings such that the following conditions hold:*

(i) *the abstract Green identity*

$$(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)$$

holds for all $f, g \in \operatorname{dom} T$;
(ii) *the map $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ has dense range and $\ker \Gamma_0 \cap \ker \Gamma_1$ is dense in $\mathcal{H}$;*

(iii) $T \upharpoonright \ker \Gamma_0$ *is an extension of a self-adjoint operator* A_0.

Then the restriction

$$S := T \upharpoonright (\ker \Gamma_0 \cap \ker \Gamma_1)$$

is a densely defined closed symmetric operator in $\mathcal{H}$, $\overline{T} = S^*$, *and* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *is a quasi boundary triple for* $T \subset S^*$ *with* $A_0 = T \upharpoonright \ker \Gamma_0$. *If, in addition, the operator* T *is closed or, equivalently, the map* $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ *is onto, then* $T = S^*$ *and* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *is an ordinary boundary triple for* S^* *with* $A_0 = T \upharpoonright \ker \Gamma_0$.

In the following let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$. Since $A_0 = T \upharpoonright \ker \Gamma_0$ is self-adjoint, we have $\mathbb{C} \setminus \mathbb{R} \subset \rho(A_0)$, and for each $\lambda \in \rho(A_0)$ the direct sum decomposition

$$\operatorname{dom} T = \operatorname{dom} A_0 \dotplus \ker(T - \lambda) = \ker \Gamma_0 \dotplus \ker(T - \lambda)$$

holds. In particular, the restriction of the map Γ_0 to $\ker(T - \lambda)$ is injective. This allows the following definition.

Definition 2.4. The *γ-field* γ and the *Weyl function* M corresponding to the quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are defined by

$$\lambda \mapsto \gamma(\lambda) := \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1}, \qquad \lambda \in \rho(A_0),$$

and

$$\lambda \mapsto M(\lambda) := \Gamma_1 \gamma(\lambda), \qquad \lambda \in \rho(A_0),$$

respectively.

The values $\gamma(\lambda)$ of the γ-field are operators defined on the dense subspace $\operatorname{ran} \Gamma_0 \subset \mathcal{G}$ which map onto $\ker(T - \lambda) \subset \mathcal{H}$. The values $M(\lambda)$ of the Weyl function are densely defined operators in $\mathcal{G}$ mapping $\operatorname{ran} \Gamma_0$ into $\operatorname{ran} \Gamma_1$. In particular, if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized or ordinary boundary triple, then $\gamma(\lambda)$ and $M(\lambda)$ are defined on $\mathcal{G} = \operatorname{ran} \Gamma_0$, and it can be shown that $\gamma(\lambda) \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and $M(\lambda) \in \mathcal{B}(\mathcal{G})$ in this case.

Next we list some important properties of the γ-field and the Weyl function corresponding to a quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, which can be found in [22, Proposition 2.6] or [24, Propositions 6.13 and 6.14]. These properties are well known for the γ-field and Weyl function corresponding to a generalized or ordinary boundary triple. Let $\lambda, \mu \in \rho(A_0)$. Then the adjoint operator $\gamma(\lambda)^*$ is bounded and satisfies

$$\gamma(\lambda)^* = \Gamma_1 (A_0 - \overline{\lambda})^{-1} \in \mathcal{B}(\mathcal{H}, \mathcal{G}); \tag{2.3}$$

hence also $\gamma(\lambda)$ is bounded and $\overline{\gamma(\lambda)} = \gamma(\lambda)^{**} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. One has the useful identity

$$\gamma(\lambda) = \bigl(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\bigr)\gamma(\mu) = (A_0 - \mu)(A_0 - \lambda)^{-1}\gamma(\mu) \tag{2.4}$$

for $\lambda, \mu \in \rho(A_0)$, which implies

$$\overline{\gamma(\lambda)} = \bigl(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\bigr)\overline{\gamma(\mu)} = (A_0 - \mu)(A_0 - \lambda)^{-1}\overline{\gamma(\mu)}. \tag{2.5}$$

With the help of the functional calculus of the self-adjoint operator A_0 one can conclude from (2.5) that

$$\bigl\|\overline{\gamma(\overline{\lambda})}\bigr\| = \bigl\|\overline{\gamma(\lambda)}\bigr\| \qquad \text{for all } \ \lambda \in \rho(A_0). \tag{2.6}$$

The values $M(\lambda)$ of the Weyl function satisfy $M(\lambda) \subset M(\overline{\lambda})^*$ and, in particular, the operators $M(\lambda)$ are closable. In general, the operators $M(\lambda)$ and their closures $\overline{M(\lambda)}$ are not bounded. However, if $M(\lambda_0)$ is bounded for some $\lambda_0 \in \rho(A_0)$, then $M(\lambda)$ is bounded for all $\lambda \in \rho(A_0)$; see Lemma 2.5 below. The function $\lambda \mapsto M(\lambda)$ is holomorphic in the sense that for any fixed $\mu \in \rho(A_0)$ it can be written as the sum of the possibly unbounded operator $\operatorname{Re} M(\mu)$ and a $\mathcal{B}(\mathcal{G})$-valued holomorphic function,

$$M(\lambda) = \operatorname{Re} M(\mu) + \gamma(\mu)^*\bigl[(\lambda - \operatorname{Re}\mu) + (\lambda - \mu)(\lambda - \overline{\mu})(A_0 - \lambda)^{-1}\bigr]\overline{\gamma(\mu)}$$

for all $\lambda \in \rho(A_0)$. In particular, $\operatorname{Im} M(\lambda)$ is a bounded operator for each $\lambda \in \rho(A_0)$.

Further, for every $x \in \operatorname{ran}\Gamma_0$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\bigl(M(\lambda)x\bigr) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\bigl(\gamma(\mu)^*\bigl[(\lambda - \operatorname{Re}\mu) + (\lambda - \mu)(\lambda - \overline{\mu})(A_0 - \lambda)^{-1}\bigr]\overline{\gamma(\mu)}x\bigr)$$

for all $\lambda \in \rho(A_0)$ and all $n \in \mathbb{N}$, and hence the nth strong derivative $M^{(n)}(\lambda)$ (viewed as an operator defined on $\operatorname{ran}\Gamma_0$) admits a continuous extension $\overline{M^{(n)}(\lambda)} \in \mathcal{B}(\mathcal{G})$. It satisfies

$$\overline{M^{(n)}(\lambda)} = n!\,\gamma(\overline{\lambda})^*(A_0 - \lambda)^{-(n-1)}\overline{\gamma(\lambda)}, \qquad \lambda \in \rho(A_0),\ n \in \mathbb{N}; \tag{2.7}$$

see [28, Lemma 2.4 (iii)].

The Weyl function also satisfies (see [22, Proposition 2.6 (v)])

$$M(\lambda) - M(\mu) = (\lambda - \mu)\gamma(\overline{\mu})^*\gamma(\lambda), \tag{2.8}$$

and with $\mu = \overline{\lambda}$ and the relation $M(\overline{\lambda}) \subset M(\lambda)^*$ it follows that

$$\operatorname{Im} M(\lambda) = (\operatorname{Im}\lambda)\gamma(\lambda)^*\gamma(\lambda) \qquad \text{and} \qquad \overline{\operatorname{Im} M(\lambda)} = (\operatorname{Im}\lambda)\gamma(\lambda)^*\overline{\gamma(\lambda)}. \tag{2.9}$$

In the case when the values of M are bounded operators we provide a simple bound for the norms $\|\overline{M(\lambda)}\|$ in the next lemma.

Lemma 2.5. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M. Assume that $M(\lambda)$ is bounded for one $\lambda \in \rho(A_0)$. Then $M(\lambda)$ is bounded for all $\lambda \in \rho(A_0)$, and the estimate*

$$\|\overline{M(\lambda)}\| \leq \left(1 + \frac{|\lambda - \mu|}{|\operatorname{Im} \mu|} + \frac{|\lambda - \mu||\lambda - \overline{\mu}|}{|\operatorname{Im} \lambda| \cdot |\operatorname{Im} \mu|}\right) \|\overline{M(\mu)}\| \tag{2.10}$$

holds for all $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$.

Proof. It follows from (2.8), the relation $\overline{\gamma(\lambda)} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and (2.3) that $M(\lambda)$ is bounded for all $\lambda \in \rho(A_0)$ if it is bounded for one $\lambda \in \rho(A_0)$. Moreover, from the second identity in (2.9) we conclude that

$$\begin{aligned} \|\overline{\gamma(\overline{\mu})}\| &= \left\|\gamma(\overline{\mu})^* \overline{\gamma(\overline{\mu})}\right\|^{1/2} \\ &= \frac{\left\|\overline{\operatorname{Im} M(\overline{\mu})}\right\|^{1/2}}{|\operatorname{Im} \overline{\mu}|^{1/2}} = \frac{\left\|\overline{\operatorname{Im} M(\mu)}\right\|^{1/2}}{|\operatorname{Im} \mu|^{1/2}}, \qquad \mu \in \mathbb{C} \setminus \mathbb{R}, \end{aligned} \tag{2.11}$$

where we have used that $\overline{M(\overline{\mu})} = M(\mu)^*$. If we replace $\gamma(\lambda)$ on the right-hand side of (2.8) with the right-hand side of (2.4), we obtain the representation

$$M(\lambda) = M(\mu) + (\lambda - \mu)\gamma(\overline{\mu})^* \gamma(\overline{\mu}) + (\lambda - \mu)(\lambda - \overline{\mu})\gamma(\overline{\mu})^*(A_0 - \lambda)^{-1}\gamma(\overline{\mu}). \tag{2.12}$$

By combining (2.11) and (2.12), for $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ we obtain the estimate

$$\begin{aligned} \|\overline{M(\lambda)}\| &\leq \|\overline{M(\mu)}\| + \Big(|\lambda - \mu| + |\lambda - \mu||\lambda - \overline{\mu}|\|(A_0 - \lambda)^{-1}\|\Big)\|\overline{\gamma(\overline{\mu})}\|^2 \\ &\leq \|\overline{M(\mu)}\| + \left(|\lambda - \mu| + \frac{|\lambda - \mu||\lambda - \overline{\mu}|}{|\operatorname{Im} \lambda|}\right) \frac{\|\overline{\operatorname{Im} M(\mu)}\|}{|\operatorname{Im} \mu|} \\ &\leq \left(1 + \frac{|\lambda - \mu|}{|\operatorname{Im} \mu|} + \frac{|\lambda - \mu||\lambda - \overline{\mu}|}{|\operatorname{Im} \lambda| \cdot |\operatorname{Im} \mu|}\right)\|\overline{M(\mu)}\|. \quad \square \end{aligned}$$

Decay properties of the Weyl function play an important role in this paper. The next lemma shows that a decay of the Weyl function along a non-real ray implies a uniform decay in certain sectors.

Lemma 2.6. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M. Assume that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$ and fix $\varphi \in (-\pi, 0) \cup (0, \pi)$. Then for every interval $[\psi_1, \psi_2] \subset (-\pi, 0)$ or $[\psi_1, \psi_2] \subset (0, \pi)$ one has*

$$\|\overline{M(re^{i\psi})}\| = \mathrm{O}\Big(\|\overline{M(re^{i\varphi})}\|\Big) \qquad \text{as } r \to \infty \text{ uniformly in } \psi \in [\psi_1, \psi_2]. \tag{2.13}$$

In particular, if $\|\overline{M(re^{i\varphi})}\| \to 0$ *as* $r \to \infty$*, then* $\|\overline{M(re^{i\psi})}\| \to 0$ *as* $r \to \infty$ *uniformly in* $\psi \in [\psi_1, \psi_2]$.

Proof. Let $\mu = re^{i\varphi}$ and $\lambda = re^{i\psi}$ with $\psi \in [\psi_1, \psi_2]$ and $r > 0$. Then

$$|\lambda - \mu| = r\Big|e^{i\frac{\psi+\varphi}{2}}\Big(e^{i\frac{\psi-\varphi}{2}} - e^{-i\frac{\psi-\varphi}{2}}\Big)\Big| = 2r\Big|\sin\Big(\frac{\psi-\varphi}{2}\Big)\Big|$$

and

$$|\lambda - \overline{\mu}| = 2r\Big|\sin\Big(\frac{\psi+\varphi}{2}\Big)\Big|.$$

Now (2.10) yields

$$\|\overline{M(re^{i\psi})}\| \le \left(1 + \frac{2|\sin\frac{\psi-\varphi}{2}|}{|\sin\varphi|} + \frac{4|\sin\frac{\psi-\varphi}{2}||\sin\frac{\psi+\varphi}{2}|}{|\sin\psi|\cdot|\sin\varphi|}\right)\|\overline{M(re^{i\varphi})}\|, \tag{2.14}$$

which shows (2.13) since the expression in the brackets on the right-hand side of (2.14) is uniformly bounded in $\psi \in [\psi_1, \psi_2]$. □

In the context of the previous lemma we remark that $\lambda \mapsto \|\overline{M(\lambda)}\|$ decays at most as $|\lambda|^{-1}$ since $\lambda \mapsto -(M(\lambda)x, x)^{-1}$ grows at most linearly as it is a Nevanlinna function for every $x \in \operatorname{ran}\Gamma_0$. We also recall from [29, Lemma 2.3] that for $x \in \operatorname{ran}\Gamma_0 \setminus \{0\}$ the function

$$\lambda \mapsto \big(M(\lambda)x, x\big)$$

is strictly increasing on each interval in $\rho(A_0)\cap\mathbb{R}$; moreover, if A_0 is bounded from below and

$$\big(M(\lambda)x, x\big) \to 0 \qquad \text{as } \lambda \to -\infty$$

for all $x \in \operatorname{ran}\Gamma_0$, then

$$\big(M(\lambda)x, x\big) > 0, \qquad x \in \operatorname{ran}\Gamma_0 \setminus \{0\},\ \lambda < \min\sigma(A_0). \tag{2.15}$$

In the next proposition the case when the self-adjoint operator $A_0 = T \upharpoonright \ker\Gamma_0$ is bounded from below and $\|\overline{M(\lambda)}\| \to 0$ as $\lambda \to -\infty$ is considered. Here the extension

$$A_1 := T \upharpoonright \ker\Gamma_1 \tag{2.16}$$

is investigated. Observe that the abstract Green identity (2.2) yields that A_1 is symmetric in $\mathcal{H}$, but in the setting of quasi boundary triples or generalized boundary triples A_1 is not necessarily self-adjoint (in contrast to the case of ordinary boundary triples).

Proposition 2.7. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M and suppose that $A_1 = T \upharpoonright \ker \Gamma_1$ is self-adjoint and that A_0 and A_1 are bounded from below. Further, assume that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$ and that $\|\overline{M(\lambda)}\| \to 0$ as $\lambda \to -\infty$. Then*

$$\min \sigma(A_0) \leq \min \sigma(A_1). \tag{2.17}$$

Proof. The assumption $\|\overline{M(\lambda)}\| \to 0$ as $\lambda \to -\infty$ implies that (2.15) holds for all $x \in \operatorname{ran} \Gamma_0 \setminus \{0\}$. Fix $\lambda \in \mathbb{R}$ such that $\lambda < \min \sigma(A_0)$ and $\lambda < \min \sigma(A_1)$. It follows from [27, Theorem 3.8] and (2.15) that

$$\begin{aligned}
\big((A_1 - \lambda)^{-1} f, f\big) &= \big((A_0 - \lambda)^{-1} f, f\big) - \big(M(\lambda)^{-1}\gamma(\lambda)^* f, \gamma(\lambda)^* f\big) \\
&= \big((A_0 - \lambda)^{-1} f, f\big) - \big(M(\lambda)M(\lambda)^{-1}\gamma(\lambda)^* f, M(\lambda)^{-1}\gamma(\lambda)^* f\big) \\
&\leq \big((A_0 - \lambda)^{-1} f, f\big)
\end{aligned}$$

for $f \in \mathcal{H}$. Since $(A_1 - \lambda)^{-1}$ and $(A_0 - \lambda)^{-1}$ are bounded non-negative operators, we conclude that

$$\max \sigma\big((A_1 - \lambda)^{-1}\big) \leq \max \sigma\big((A_0 - \lambda)^{-1}\big)$$

and hence

$$\min \sigma(A_0 - \lambda) \leq \min \sigma(A_1 - \lambda),$$

which is equivalent to (2.17). □

3. Sectorial extensions of symmetric operators

Let S be a densely defined, closed, symmetric operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$. For a linear operator B in $\mathcal{G}$ we define the operator $A_{[B]}$ in $\mathcal{H}$ by

$$A_{[B]} f = Tf, \qquad \operatorname{dom} A_{[B]} = \big\{f \in \operatorname{dom} T : \Gamma_0 f = B\Gamma_1 f\big\}, \tag{3.1}$$

where the boundary condition $\Gamma_0 f = B\Gamma_1 f$ is understood in the sense that $\Gamma_1 f \in \operatorname{dom} B$ and $\Gamma_0 f = B\Gamma_1 f$ holds. Clearly, $A_{[B]}$ is a restriction of T and hence of S^*. Moreover, $A_{[B]}$ is an extension of S since $S = T \upharpoonright (\ker \Gamma_0 \cap \ker \Gamma_1)$ by [22, Proposition 2.2]. Recall that in the special case of an ordinary boundary triple there is a one-to-one correspondence between closed linear relations B in $\mathcal{G}$ and closed extensions $A_{[B]}$ of S that are restrictions of S^* via (3.1); for proper relations B the definition of $A_{[B]}$ has to be interpreted accordingly. For generalized and quasi boundary triples one has to impose additional assumptions on B to guarantee that $A_{[B]}$ is closed. In this and the following sections we study the operators $A_{[B]}$ thoroughly; in particular, we are interested in their spectral properties.

In the next theorem it is shown that under additional assumptions on B and the Weyl function M that corresponds to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ the operator $A_{[B]}$ is sectorial. Recall first that the *numerical range*, $W(A)$, of a linear operator A is defined as

$$W(A) := \big\{(Af, f) : f \in \operatorname{dom} A,\ \|f\| = 1\big\},$$

and that A is called *sectorial* if $W(A)$ is contained in a sector of the form

$$\big\{z \in \mathbb{C} : \operatorname{Re} z \geq \eta, |\operatorname{Im} z| \leq \kappa(\operatorname{Re} z - \eta)\big\} \tag{3.2}$$

for some $\eta \in \mathbb{R}$ and $\kappa > 0$. An operator A is called *m-sectorial* if $W(A)$ is contained in a sector (3.2) and the complement of (3.2) has a non-trivial intersection with $\rho(A)$. In this case the spectrum of A is contained in the closure of $W(A)$; see, e.g. [125, Propositions 2.8 and 3.19]. Note that if A is m-sectorial, then $-A$ generates an analytic semigroup; see, e.g. [95, Theorem IX.1.24].

Theorem 3.1. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M such that $A_1 = T \upharpoonright \ker \Gamma_1$ is self-adjoint and bounded from below and $\rho(A_0) \cap (-\infty, \min \sigma(A_1)) \neq \emptyset$. Moreover, suppose that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$ and that*

$$M(\eta) \geq 0 \quad \textit{for some } \eta < \min \sigma(A_1),\ \eta \in \rho(A_0). \tag{3.3}$$

Let B be a closable operator in $\mathcal{G}$ and assume that there exists $b \in \mathbb{R}$ such that

(i) $\operatorname{Re}(Bx, x) \leq b\|x\|^2$ *for all* $x \in \operatorname{dom} B$;
(ii) $b\big\|\overline{M(\eta)}\big\| < 1$;
(iii) $\operatorname{ran} \overline{M(\eta)}^{1/2} \subset \operatorname{dom} B$.

Then the operator $A_{[B]}$ is sectorial and the numerical range $W(A_{[B]})$ is contained in the sector

$$\mathcal{S}_\eta(B) := \Big\{z \in \mathbb{C} : \operatorname{Re} z \geq \eta,\ |\operatorname{Im} z| \leq \kappa_B(\eta)\big(\operatorname{Re} z - \eta\big)\Big\}, \tag{3.4}$$

where

$$\kappa_B(\eta) := \frac{\big\|\operatorname{Im}\big(\overline{M(\eta)}^{1/2} B \overline{M(\eta)}^{1/2}\big)\big\|}{1 - b\big\|\overline{M(\eta)}\big\|}. \tag{3.5}$$

In particular, if $\rho(A_{[B]}) \cap (\mathbb{C} \setminus \mathcal{S}_\eta(B)) \neq \emptyset$, then the operator $A_{[B]}$ is m-sectorial and $\sigma(A_{[B]})$ is contained in the sector $\mathcal{S}_\eta(B)$.

Proof. Let $\eta < \min \sigma(A_1)$ be such that $\eta \in \rho(A_0)$ and $M(\eta) \geq 0$, which exists by (3.3). Moreover, let $f \in \operatorname{dom} A_{[B]}$ with $\|f\| = 1$. Based on the decomposition

$$\operatorname{dom} T = \operatorname{dom} A_1 \dot{+} \ker(T - \eta) = \ker \Gamma_1 \dot{+} \ker(T - \eta)$$

we can write f in the form $f = f_1 + f_\eta$ with $f_1 \in \ker \Gamma_1 = \operatorname{dom} A_1$ and $f_\eta \in \ker(T - \eta)$. This yields

$$\begin{aligned}(A_{[B]}f, f) &= \big(T(f_1 + f_\eta), f_1 + f_\eta\big) \\ &= (A_1 f_1, f_1) + (T f_1, f_\eta) + (T f_\eta, f_\eta) + (T f_\eta, f_1) \\ &= (A_1 f_1, f_1) + (T f_1, f_\eta) + \eta\big[\|f_\eta\|^2 + (f_\eta, f_1)\big].\end{aligned} \tag{3.6}$$

Making use of the abstract Green identity (2.1) we obtain

$$\begin{aligned}(T f_1, f_\eta) &= (f_1, T f_\eta) + (\Gamma_1 f_1, \Gamma_0 f_\eta) - (\Gamma_0 f_1, \Gamma_1 f_\eta) \\ &= \eta(f_1, f_\eta) - (\Gamma_0 f_1, \Gamma_1 f_\eta).\end{aligned} \tag{3.7}$$

Moreover, since $f \in \operatorname{dom} A_{[B]}$ and $f_1 \in \ker \Gamma_1$, we have $\Gamma_1 f_\eta \in \operatorname{dom} B$ and

$$\Gamma_0 f_1 = B\Gamma_1 f - \Gamma_0 f_\eta = B\Gamma_1 f_\eta - \Gamma_0 f_\eta. \tag{3.8}$$

Combining (3.7) and (3.8) we can rewrite the right-hand side of (3.6) in the form

$$\begin{aligned}(A_{[B]}f, f) &= (A_1 f_1, f_1) + \eta(f_1, f_\eta) - \big(B\Gamma_1 f_\eta - \Gamma_0 f_\eta, \Gamma_1 f_\eta\big) + \eta\big[\|f_\eta\|^2 + (f_\eta, f_1)\big] \\ &= (A_1 f_1, f_1) + \eta\big[\|f_\eta\|^2 + 2\operatorname{Re}(f_\eta, f_1)\big] - \big(B\Gamma_1 f_\eta - \Gamma_0 f_\eta, \Gamma_1 f_\eta\big).\end{aligned}$$

Next we use

$$\|f_\eta\|^2 + 2\operatorname{Re}(f_\eta, f_1) = \|f\|^2 - \|f_1\|^2 = 1 - \|f_1\|^2$$

and the definition of $M(\eta)$ to obtain

$$\begin{aligned}(A_{[B]}f, f) &= (A_1 f_1, f_1) + \eta - \eta\|f_1\|^2 \\ &\qquad - \big(BM(\eta)\Gamma_0 f_\eta, M(\eta)\Gamma_0 f_\eta\big) + \big(\Gamma_0 f_\eta, M(\eta)\Gamma_0 f_\eta\big) \\ &= \big((A_1 - \eta)f_1, f_1\big) + \eta \\ &\qquad - \big(B\overline{M(\eta)}\Gamma_0 f_\eta, \overline{M(\eta)}\Gamma_0 f_\eta\big) + \big\|\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\big\|^2;\end{aligned} \tag{3.9}$$

recall that $\overline{M(\eta)}$ is a bounded, self-adjoint, non-negative operator. Using assumption (i) we obtain

$$
\begin{aligned}
\operatorname{Re}\big(B\overline{M(\eta)}\Gamma_0 f_\eta, \overline{M(\eta)}\Gamma_0 f_\eta\big) &\le b\big\|\overline{M(\eta)}\Gamma_0 f_\eta\big\|^2 \\
&\le b\big\|\overline{M(\eta)}^{1/2}\big\|^2\big\|\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\big\|^2 \\
&= b\big\|\overline{M(\eta)}\big\|\,\big\|\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\big\|^2.
\end{aligned}
\tag{3.10}
$$

From this, (3.9) and the fact that $\eta < \min \sigma(A_1)$ we conclude that

$$
\begin{aligned}
\operatorname{Re}(A_{[B]}f, f) &\ge \eta - \operatorname{Re}\big(B\overline{M(\eta)}\Gamma_0 f_\eta, \overline{M(\eta)}\Gamma_0 f_\eta\big) + \big\|\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\big\|^2 \\
&\ge \eta + \big(1 - b\big\|\overline{M(\eta)}\big\|\big)\big\|\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\big\|^2.
\end{aligned}
\tag{3.11}
$$

This, together with assumption (ii), implies that

$$
\operatorname{Re}(A_{[B]}f, f) \ge \eta. \tag{3.12}
$$

Moreover, it follows with assumption (iii) that the operator $B\overline{M(\eta)}^{1/2}$ is everywhere defined and closable since B is closable. Hence

$$
B\overline{M(\eta)}^{1/2} \in \mathcal{B}(\mathcal{G}) \qquad \text{and} \qquad \overline{M(\eta)}^{1/2}B\overline{M(\eta)}^{1/2} \in \mathcal{B}(\mathcal{G}). \tag{3.13}
$$

With (3.9) we obtain that

$$
\begin{aligned}
\big|\operatorname{Im}(A_{[B]}f, f)\big| &= \big|\operatorname{Im}\big(B\overline{M(\eta)}\Gamma_0 f_\eta, \overline{M(\eta)}\Gamma_0 f_\eta\big)\big| \\
&= \Big|\operatorname{Im}\Big(\overline{M(\eta)}^{1/2}B\overline{M(\eta)}^{1/2}\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta, \overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\Big)\Big| \\
&= \Big|\Big(\operatorname{Im}\Big(\overline{M(\eta)}^{1/2}B\overline{M(\eta)}^{1/2}\Big)\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta, \overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\Big)\Big| \\
&\le \big\|\operatorname{Im}\big(\overline{M(\eta)}^{1/2}B\overline{M(\eta)}^{1/2}\big)\big\|\,\big\|\overline{M(\eta)}^{1/2}\Gamma_0 f_\eta\big\|^2.
\end{aligned}
$$

This, together with (3.11), implies that

$$
\big|\operatorname{Im}(A_{[B]}f, f)\big| \le \frac{\big\|\operatorname{Im}\big(\overline{M(\eta)}^{1/2}B\overline{M(\eta)}^{1/2}\big)\big\|}{1 - b\big\|\overline{M(\eta)}\big\|}\Big(\operatorname{Re}(A_{[B]}f, f) - \eta\Big). \tag{3.14}
$$

The inequalities (3.12) and (3.14) show that the numerical range of $A_{[B]}$ is contained in the sector $\mathcal{S}_\eta(B)$, and hence the operator $A_{[B]}$ is sectorial. The last statement of the theorem is well known; see, e.g. [125, Proposition 3.19]. □

Remark 3.2. In Theorem 3.1 it is not assumed explicitly that the self-adjoint extension $A_0 = T \upharpoonright \ker \Gamma_0$ is bounded from below. However, the operator $B = 0$ satisfies assumptions (i)–(iii) in Theorem 3.1 with $b = 0$, which yields $\kappa_B(\eta) = 0$. Thus the spectrum of the operator $A_0 = A_{[0]}$ is contained in $[\eta, \infty)$ and therefore A_0 is bounded from below by η.

Theorem 3.1 provides explicit sufficient conditions for the extension $A_{[B]}$ in (3.1) to be sectorial. However, in applications it is essential to ensure that $A_{[B]}$ is m-sectorial, i.e. to guarantee that $\rho(A_{[B]}) \cap (\mathbb{C} \setminus \mathcal{S}_\eta(B)) \neq \emptyset$. We consider one particular situation in the next proposition, but deal in more detail with this question in the next section.

In the next proposition we specialize Theorem 3.1 to the situation of an ordinary boundary triple, where we can actually prove that the operator $A_{[B]}$ is m-sectorial; to the best of our knowledge the assertion is new. We remark that in the following proposition it is possible to choose $b = \max \sigma(\operatorname{Re} B)$.

Proposition 3.3. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triple for S^* with corresponding Weyl function M and assume that A_1 is bounded from below and that $\rho(A_0) \cap (-\infty, \min \sigma(A_1)) \neq \emptyset$. Moreover, assume that*

$$M(\eta) \geq 0 \qquad \text{for some } \eta < \min \sigma(A_1),\ \eta \in \rho(A_0).$$

Let $B \in \mathcal{B}(\mathcal{G})$, let $b \in \mathbb{R}$ be such that $\operatorname{Re}(Bx, x) \leq b\|x\|^2$ for all $x \in \mathcal{G}$, and assume that $b\|M(\eta)\| < 1$. Then the operator $A_{[B]}$ is m-sectorial and we have

$$\sigma(A_{[B]}) \subset \overline{W(A_{[B]})} \subset \Big\{ z \in \mathbb{C} : \operatorname{Re} z \geq \eta,\ |\operatorname{Im} z| \leq \kappa_B(\eta)\big(\operatorname{Re} z - \eta\big) \Big\}, \tag{3.15}$$

where

$$\kappa_B(\eta) := \frac{\big\|\operatorname{Im}\big(M(\eta)^{1/2} B M(\eta)^{1/2}\big)\big\|}{1 - b\big\|M(\eta)\big\|}.$$

Proof. The fact that $A_{[B]}$ is sectorial and the second inclusion in (3.15) follow directly from Theorem 3.1. To prove that $A_{[B]}$ is m-sectorial we show that $\eta \in \rho(A_{[B]})$. Without loss of generality we can assume that $b \geq 0$. Observe that $M(\eta)^{1/2}$ is well defined since $M(\eta) \geq 0$ by assumption. For $x \in \mathcal{G}$ with $\|x\| = 1$ we have

$$\begin{aligned} \operatorname{Re}\big(M(\eta)^{1/2} B M(\eta)^{1/2} x, x\big) &= \operatorname{Re}\big(B M(\eta)^{1/2} x, M(\eta)^{1/2} x\big) \\ &\leq b\big\|M(\eta)^{1/2} x\big\|^2 = b\big(M(\eta)x, x\big) \leq b\big\|M(\eta)\big\|, \end{aligned}$$

which implies that

$$\begin{aligned} \sigma\big(M(\eta)^{1/2} B M(\eta)^{1/2}\big) &\subset \overline{W\big(M(\eta)^{1/2} B M(\eta)^{1/2}\big)} \\ &\subset \big\{ z \in \mathbb{C} : \operatorname{Re} z \leq b\|M(\eta)\| \big\}. \end{aligned}$$

Since $b\|M(\eta)\| < 1$, this yields

$$1 \in \rho\big(M(\eta)^{1/2} B M(\eta)^{1/2}\big)$$

and hence $1 \in \rho(BM(\eta))$. Now [56, Proposition 1.6] implies that $\eta \in \rho(A_{[B]})$, and therefore $A_{[B]}$ is m-sectorial, which also proves the first inclusion in (3.15). □

4. Sufficient conditions for closed extensions with non-empty resolvent set

Let S be a densely defined, closed, symmetric operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$. In this section we provide some abstract sufficient conditions on the (boundary) operator B in $\mathcal{G}$ such that the operator $A_{[B]}$ defined in (3.1) is closed and has a non-empty resolvent set.

Theorem 4.1. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let B be a closable operator in $\mathcal{G}$ and assume that there exists $\lambda_0 \in \rho(A_0)$ such that the following conditions are satisfied:*

(i) $1 \in \rho\bigl(B\overline{M(\lambda_0)}\bigr)$;
(ii) $B\bigl(\operatorname{ran} \overline{M(\lambda_0)} \cap \operatorname{dom} B\bigr) \subset \operatorname{ran} \Gamma_0$;
(iii) $\operatorname{ran} \Gamma_1 \subset \operatorname{dom} B$;
(iv) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or* $\lambda_0 \in \rho(A_1)$.

Then the operator

$$A_{[B]} f = Tf, \qquad \operatorname{dom} A_{[B]} = \bigl\{ f \in \operatorname{dom} T : \Gamma_0 f = B\Gamma_1 f \bigr\}, \tag{4.1}$$

is a closed extension of S in $\mathcal{H}$ such that $\lambda_0 \in \rho(A_{[B]})$, and

$$(A_{[B]} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\bigl(I - BM(\lambda)\bigr)^{-1} B\gamma(\overline{\lambda})^* \tag{4.2}$$

holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.

Further, let B' be a linear operator in $\mathcal{G}$ that satisfies (i)–(iv) *with B replaced by B' and λ_0 replaced by $\overline{\lambda_0}$, and assume that*

$$(Bx, y) = (x, B'y) \qquad \text{for all } x \in \operatorname{dom} B,\ y \in \operatorname{dom} B'. \tag{4.3}$$

Then $A_{[B']}$ is closed and

$$A_{[B']} = A_{[B]}^*. \tag{4.4}$$

In particular, $\overline{\lambda_0} \in \rho(A_{[B']})$.

Remark 4.2. In the special case when the operator B in Theorem 4.1 is symmetric and the assumptions (i) and (ii) hold for some $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ the result reduces to [29, Theorem 2.6], where self-adjointness of $A_{[B]}$ was shown; cf. also [29, Theorem 2.4]. In this sense Theorem 4.1 can be seen as a generalization of the considerations in [29, Section 2] to non-self-adjoint extensions.

Before we prove Theorem 4.1, we formulate some corollaries. If $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triple, then $\operatorname{ran}\Gamma_0 = \mathcal{G}$ and $M(\lambda_0) \in \mathcal{B}(\mathcal{G})$. Hence in this case the above theorem reads as follows.

Corollary 4.3. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let B be a closable operator in $\mathcal{G}$ and assume that there exists $\lambda_0 \in \rho(A_0)$ such that the following conditions are satisfied:*

(i) $1 \in \rho(BM(\lambda_0))$;
(ii) $\operatorname{ran}\Gamma_1 \subset \operatorname{dom} B$.

Then the operator $A_{[B]}$ in (4.1) is a closed extension of S such that $\lambda_0 \in \rho(A_{[B]})$, and the resolvent formula (4.2) holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.

Further, let B' be a linear operator in $\mathcal{G}$ that satisfies (i) *and* (ii) *with B replaced by B' and λ_0 replaced by $\overline{\lambda_0}$, and assume that* (4.3) *holds. Then $A_{[B']}$ is closed and $A_{[B']} = A^*_{[B]}$. In particular, $\overline{\lambda_0} \in \rho(A_{[B']})$.*

In the special case when $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ in Theorem 4.1 or Corollary 4.3 is an ordinary boundary triple the condition $\operatorname{ran}\Gamma_1 \subset \operatorname{dom} B$ implies $\operatorname{dom} B = \mathcal{G}$. Since B is assumed to be closable, it follows that B is closed and hence $B \in \mathcal{B}(\mathcal{G})$. In this case the statements in Theorem 4.1 and Corollary 4.3 are well known.

In the next corollary we return to the general situation of a quasi boundary triple, but we assume that B is bounded and everywhere defined on $\mathcal{G}$.

Corollary 4.4. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M. Let $B \in \mathcal{B}(\mathcal{G})$ and assume that there exists $\lambda_0 \in \rho(A_0)$ such that the following conditions are satisfied:*

(i) $1 \in \rho\big(B\overline{M(\lambda_0)}\big)$;
(ii) $B\big(\operatorname{ran}\overline{M(\lambda_0)}\big) \subset \operatorname{ran}\Gamma_0$;
(iii) $B(\operatorname{ran}\Gamma_1) \subset \operatorname{ran}\Gamma_0$ *or* $\lambda_0 \in \rho(A_1)$.

Then the operator $A_{[B]}$ in (4.1) is a closed extension of S such that $\lambda_0 \in \rho(A_{[B]})$, and the resolvent formula (4.2) holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.

Further, if conditions (i)–(iii) *are satisfied also for B^* instead of B and λ_0 replaced by $\overline{\lambda_0}$, then $A_{[B^*]} = A^*_{[B]}$. In particular, $\overline{\lambda_0} \in \rho(A_{[B^*]})$.*

Note that if in Corollary 4.4 the triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triple, then assumptions (ii) and (iii) are automatically satisfied.

In the next two corollaries a set of conditions is provided which guarantee that condition (i) in Theorem 4.1 is satisfied; here Corollary 4.6 is a special case of Corollary 4.5 for bounded B. In contrast to the previous results it is also assumed that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$ and that the set $\rho(A_0) \cap \mathbb{R}$ is non-empty.

Corollary 4.5. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M, and assume that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$. Let B be a closable operator in $\mathcal{G}$ and assume that there exist $b \in \mathbb{R}$ and $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ such that the following conditions are satisfied:*

(i) $\operatorname{Re}(Bx, x) \leq b\|x\|^2$ *for all* $x \in \operatorname{dom} B$*;*
(ii) $M(\lambda_0) \geq 0$ *and* $b\|\overline{M(\lambda_0)}\| < 1$*;*
(iii) $\operatorname{ran} \overline{M(\lambda_0)}^{1/2} \subset \operatorname{dom} B$*;*
(iv) $B\big(\operatorname{ran} \overline{M(\lambda_0)}\big) \subset \operatorname{ran} \Gamma_0$*;*
(v) $\operatorname{ran} \Gamma_1 \subset \operatorname{dom} B$*;*
(vi) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or* $\lambda_0 \in \rho(A_1)$.

Then the operator $A_{[B]}$ in (4.1) *is a closed extension of S such that $\lambda_0 \in \rho(A_{[B]})$, and the resolvent formula* (4.2) *holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.*

Further, let B' be a linear operator in $\mathcal{G}$ that satisfies (i)–(vi) *with B replaced by B' and assume that* (4.3) *holds. Then $A_{[B']}$ is closed and $A_{[B']} = A^*_{[B]}$. In particular, $\lambda_0 \in \rho(A_{[B']})$.*

For $B \in \mathcal{B}(\mathcal{G})$, Corollary 4.5 reads as follows.

Corollary 4.6. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M, and assume that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$. Let $B \in \mathcal{B}(\mathcal{G})$ and $b \in \mathbb{R}$ such that*

$$\operatorname{Re}(Bx, x) \leq b\|x\|^2 \qquad \textit{for all } x \in \mathcal{G}$$

and assume that for some $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ the following conditions are satisfied:

(i) $M(\lambda_0) \geq 0$ *and* $b\|\overline{M(\lambda_0)}\| < 1$*;*
(ii) $B\big(\operatorname{ran} \overline{M(\lambda_0)}\big) \subset \operatorname{ran} \Gamma_0$*;*
(iii) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or* $\lambda_0 \in \rho(A_1)$.

Then the operator $A_{[B]}$ in (4.1) *is a closed extension of S such that $\lambda_0 \in \rho(A_{[B]})$, and the resolvent formula* (4.2) *holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.*

Further, if conditions (i)–(iii) *are satisfied also for B^* instead of B, then $A_{[B^*]} = A^*_{[B]}$. In particular, $\lambda_0 \in \rho(A_{[B^*]})$.*

Proof of Corollary 4.5. It suffices to show that assumptions (i)–(iii) in Corollary 4.5 imply assumption (i) in Theorem 4.1. The assumption (ii) in Theorem 4.1 is satisfied since the inclusion

$$\operatorname{ran}\overline{M(\lambda_0)} \subset \operatorname{ran}\overline{M(\lambda_0)}^{1/2} \subset \operatorname{dom} B$$

holds by (iii) in Corollary 4.5, and hence (iv) in Corollary 4.5 coincides with (ii) in Theorem 4.1; the assumptions (iii) and (iv) in Theorem 4.1 coincide with (v) and (vi) in Corollary 4.5.

In order to show (i) in Theorem 4.1 we use a similar idea as in the proof of Proposition 3.3, but we have to be more careful with operator domains. Note first that a negative b in (i) and (ii) in Corollary 4.5 can always be replaced by 0; hence without loss of generality we can assume that $b \geq 0$. For $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ such that $M(\lambda_0) \geq 0$ we have $\overline{M(\lambda_0)} \geq 0$. As in (3.13) in the proof of Theorem 3.1 the operator

$$B\overline{M(\lambda_0)}^{1/2} \tag{4.5}$$

is defined on all of $\mathcal{G}$ by (iii) and is closable since B is closable. Hence

$$B\overline{M(\lambda_0)}^{1/2} \in \mathcal{B}(\mathcal{G}) \qquad \text{and} \qquad \overline{M(\lambda_0)}^{1/2} B\overline{M(\lambda_0)}^{1/2} \in \mathcal{B}(\mathcal{G}). \tag{4.6}$$

Then for $x \in \mathcal{G}$ with $\|x\| = 1$ we conclude from assumption (i) that

$$\begin{aligned}\operatorname{Re}\big(\overline{M(\lambda_0)}^{1/2} B\overline{M(\lambda_0)}^{1/2} x, x\big) &= \operatorname{Re}\big(B\overline{M(\lambda_0)}^{1/2} x, \overline{M(\lambda_0)}^{1/2} x\big) \\ &\leq b\big\|\overline{M(\lambda_0)}^{1/2} x\big\|^2 = b\big(\overline{M(\lambda_0)}x, x\big) \leq b\big\|\overline{M(\lambda_0)}\big\|.\end{aligned}$$

Thus

$$\begin{aligned}\sigma\big(\overline{M(\lambda_0)}^{1/2} B\overline{M(\lambda_0)}^{1/2}\big) &\subset \overline{W\big(\overline{M(\lambda_0)}^{1/2} B\overline{M(\lambda_0)}^{1/2}\big)} \\ &\subset \big\{z \in \mathbb{C} : \operatorname{Re} z \leq b\big\|\overline{M(\lambda_0)}\big\|\big\},\end{aligned}$$

and hence assumption (ii) implies that

$$1 \in \rho\big(\overline{M(\lambda_0)}^{1/2} B\overline{M(\lambda_0)}^{1/2}\big).$$

This shows that also $1 \in \rho(B\overline{M(\lambda_0)})$ and therefore (i) in Theorem 4.1 holds. □

Now we finally turn to the proof of Theorem 4.1. We note that the arguments in Steps 2, 4 and 5 are similar to those in the proof of [29, Theorem 2.4], where the case when B is symmetric was treated. For the convenience of the reader we provide a self-contained and complete proof.

Proof of Theorem 4.1. The proof of Theorem 4.1 consists of six separate steps. During the first four steps of the proof we assume that the first condition in (iv) is satisfied. In Step 5 of the proof we show that the second condition in (iv) and assumptions (ii) and

(iii) imply the first condition in (iv). Finally, in Step 6 we prove the statements about $A_{[B']}$.

Step 1. We claim that $\ker(A_{[B]}-\lambda_0)=\{0\}$. To this end, let $f\in\ker(A_{[B]}-\lambda_0)$. Then f satisfies the equation $Tf=\lambda_0 f$ and the abstract boundary condition $\Gamma_0 f=B\Gamma_1 f$. It follows that

$$\Gamma_0 f = B\Gamma_1 f = BM(\lambda_0)\Gamma_0 f = B\overline{M(\lambda_0)}\Gamma_0 f,$$

that is, $\Gamma_0 f\in\ker(I-B\overline{M(\lambda_0)})$. From this and assumption (i) of the theorem it follows that $\Gamma_0 f=0$ and, thus, $f\in\ker(A_0-\lambda_0)$. Since $\lambda_0\in\rho(A_0)$, we obtain that $f=0$. Therefore we have $\ker(A_{[B]}-\lambda_0)=\{0\}$.

Step 2. Next we show that

$$\operatorname{ran}(A_{[B]}-\lambda_0)=\mathcal{H} \tag{4.7}$$

holds. In order to do so, we first verify the inclusion

$$\operatorname{ran}\big(B\gamma(\overline{\lambda_0})^*\big)\subset\operatorname{ran}\big(I-BM(\lambda_0)\big). \tag{4.8}$$

Note that the product $B\gamma(\overline{\lambda_0})^*$ on the left-hand side of (4.8) is defined on all of $\mathcal{H}$ since $\gamma(\overline{\lambda_0})^*=\Gamma_1(A_0-\lambda_0)^{-1}$ by (2.3) and $\operatorname{ran}\Gamma_1\subset\operatorname{dom}B$ by condition (iii). For the inclusion in (4.8) consider $\psi=B\gamma(\overline{\lambda_0})^*f$ for some $f\in\mathcal{H}$. From (2.3) and the first condition in (iv) we obtain that $\psi\in\operatorname{ran}\Gamma_0$. Making use of assumption (i) we see that

$$\varphi:=\big(I-B\overline{M(\lambda_0)}\big)^{-1}\psi\in\operatorname{dom}\big(B\overline{M(\lambda_0)}\big) \tag{4.9}$$

is well defined. Hence

$$\varphi=B\overline{M(\lambda_0)}\varphi+\psi,$$

and since $\overline{M(\lambda_0)}\varphi\in\operatorname{ran}\overline{M(\lambda_0)}\cap\operatorname{dom}B$, it follows from (ii) and $\psi\in\operatorname{ran}\Gamma_0$ that $\varphi\in\operatorname{ran}\Gamma_0=\operatorname{dom}M(\lambda_0)$. Thus we conclude from (4.9) that

$$\big(I-BM(\lambda_0)\big)\varphi=\psi,$$

which shows the inclusion (4.8).

To verify (4.7), let $f\in\mathcal{H}$ and consider

$$h:=(A_0-\lambda_0)^{-1}f+\gamma(\lambda_0)\big(I-BM(\lambda_0)\big)^{-1}B\gamma(\overline{\lambda_0})^*f. \tag{4.10}$$

Observe that h is well defined since $\operatorname{dom}\gamma(\lambda_0)=\operatorname{dom}M(\lambda_0)\supset\operatorname{ran}(I-BM(\lambda_0))^{-1}$ and the product of $(I-BM(\lambda_0))^{-1}$ and $B\gamma(\overline{\lambda_0})^*$ makes sense by (4.8). It is clear that $h\in\operatorname{dom}T$. Moreover, from $\operatorname{dom}A_0=\ker\Gamma_0$, the definitions of the γ-field and Weyl function, and (2.3) we conclude that

$$\Gamma_0 h = \big(I - BM(\lambda_0)\big)^{-1} B\gamma(\overline{\lambda_0})^* f$$

and

$$\Gamma_1 h = \gamma(\overline{\lambda_0})^* f + M(\lambda_0)\big(I - BM(\lambda_0)\big)^{-1} B\gamma(\overline{\lambda_0})^* f.$$

Now it follows that

$$B\Gamma_1 h = \big(I - BM(\lambda_0)\big)^{-1} B\gamma(\overline{\lambda_0})^* f = \Gamma_0 h,$$

and therefore $h \in \operatorname{dom} A_{[B]}$. From the definition of h in (4.10) and $\operatorname{ran}\gamma(\lambda_0) = \ker(T-\lambda_0)$ we obtain that

$$(A_{[B]} - \lambda_0)h = (T - \lambda_0)h = f.$$

Hence we have proved (4.7). Moreover, since $h = (A_{[B]} - \lambda_0)^{-1} f$, we also conclude from (4.10) that

$$(A_{[B]} - \lambda_0)^{-1} f = (A_0 - \lambda_0)^{-1} f + \gamma(\lambda_0)\big(I - BM(\lambda_0)\big)^{-1} B\gamma(\overline{\lambda_0})^* f. \tag{4.11}$$

Step 3. We verify that $A_{[B]}$ is closed and that $\lambda_0 \in \rho(A_{[B]})$. Since B is closable by assumption and $\gamma(\overline{\lambda_0})^* \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, it follows that $B\gamma(\overline{\lambda_0})^*$ is closable and hence closed, so that

$$B\gamma(\overline{\lambda_0})^* \in \mathcal{B}(\mathcal{H}, \mathcal{G}). \tag{4.12}$$

The operators $\gamma(\lambda_0)$ and $(I - BM(\lambda_0))^{-1}$ in (4.11) are bounded by (2.3) and assumption (i), respectively. Therefore (4.11) shows that the operator $(A_{[B]} - \lambda_0)^{-1}$ is bounded. Since $(A_{[B]} - \lambda_0)^{-1}$ is defined on $\mathcal{H}$ by (4.7), it follows that $A_{[B]}$ is closed and $\lambda_0 \in \rho(A_{[B]})$.

Step 4. Now we prove the resolvent formula (4.2) for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$. We first observe that $I - BM(\lambda)$ is injective for $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$. In fact, let $\varphi \in \ker(I - BM(\lambda))$. Then $\varphi \in \operatorname{dom} M(\lambda) = \operatorname{ran}\Gamma_0$ and $f := \gamma(\lambda)\varphi$ belongs to $\ker(T - \lambda)$. Furthermore, $\Gamma_0 f = \varphi$, and from

$$B\Gamma_1 f = BM(\lambda)\Gamma_0 f = BM(\lambda)\varphi = \varphi = \Gamma_0 f$$

we conclude that $f \in \operatorname{dom} A_{[B]}$. Since $f \in \ker(T - \lambda)$, this implies that $f \in \ker(A_{[B]} - \lambda)$, and hence $f = 0$ as $\lambda \in \rho(A_{[B]})$ by assumption. It follows that $\varphi = \Gamma_0 f = 0$, and therefore $I - BM(\lambda)$ is injective.

Now let $f \in \mathcal{H}$, $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$, and set

$$k := (A_{[B]} - \lambda)^{-1} f - (A_0 - \lambda)^{-1} f. \tag{4.13}$$

With $g := (A_{[B]} - \lambda)^{-1} f \in \operatorname{dom} A_{[B]}$ we have $B\Gamma_1 g = \Gamma_0 g = \Gamma_0 k$. Since $k \in \ker(T - \lambda)$, it is also clear that $M(\lambda)\Gamma_0 k = \Gamma_1 k$. Moreover, $\Gamma_1(g - k) = \gamma(\overline{\lambda})^* f$ by (2.3), and therefore

$$\big(I - BM(\lambda)\big)\Gamma_0 k = \Gamma_0 g - BM(\lambda)\Gamma_0 k = B\Gamma_1 g - B\Gamma_1 k = B\gamma(\overline{\lambda})^* f$$

yields $\Gamma_0 k = (I - BM(\lambda))^{-1} B\gamma(\overline{\lambda})^* f$. Since $k \in \ker(T - \lambda)$, we have

$$k = \gamma(\lambda)\Gamma_0 k = \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\gamma(\overline{\lambda})^* f,$$

which, together with (4.13), yields (4.2) for $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.

Step 5. Now assume that $\lambda_0 \in \rho(A_1)$, i.e. the second condition in (iv) holds. We claim that in this situation $B(\operatorname{ran}\Gamma_1) \subset \operatorname{ran}\Gamma_0$ follows. In fact, suppose that $g \in \operatorname{ran}\Gamma_1$. Then $g \in \operatorname{dom} B$ by condition (iii). Since $\operatorname{ran}\Gamma_1 = \operatorname{ran} M(\lambda_0) \subset \operatorname{ran}(\overline{M(\lambda_0)})$ in the present situation by [22, Proposition 2.6 (iii)], we conclude from (ii) that $Bg \in \operatorname{ran}\Gamma_0$.

Step 6. Now let B' be as in the last part of the statement of the theorem. By assumption (iii) for B and B', both operators are densely defined. Hence relation (4.3) implies that B' is also closable. It follows from Steps 1–5 that $A_{[B']}$ is closed and that $\overline{\lambda}_0 \in \rho(A_{[B']})$. Let $f \in \operatorname{dom} A_{[B]}$ and $g \in \operatorname{dom} A_{[B']}$. Then $\Gamma_1 f \in \operatorname{dom} B$, $\Gamma_1 g \in \operatorname{dom} B'$ and

$$\Gamma_0 f = B\Gamma_1 f \qquad \text{and} \qquad \Gamma_0 g = B'\Gamma_1 g.$$

Hence Green's identity (2.1) and the relation (4.3) yield

$$\begin{aligned}(A_{[B]}f, g) - (f, A_{[B']}g) &= (Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)\\ &= (\Gamma_1 f, B'\Gamma_1 g) - (B\Gamma_1 f, \Gamma_1 g) = 0,\end{aligned}$$

which implies that

$$A_{[B']} \subset A_{[B]}^*. \tag{4.14}$$

Since $\lambda_0 \in \rho(A_{[B]})$, we have $\overline{\lambda}_0 \in \rho(A_{[B]}^*)$. This, together with $\overline{\lambda}_0 \in \rho(A_{[B']})$ and (4.14), proves the relation in (4.4). $\square$

In the next proposition we consider Schatten–von Neumann properties of certain resolvent differences (see the end of the introduction for the definition of the classes $\mathfrak{S}_p$). For the self-adjoint case parts of the results of the following proposition can be found in [27, Theorem 3.17].

Proposition 4.7. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let B be a closable operator in $\mathcal{G}$ and assume that there exists $\lambda_0 \in \rho(A_0)$ such that conditions* (i)–(iv) *in Theorem 4.1 are satisfied. Moreover, assume that*

$$\gamma(\lambda_1)^* \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G}) \tag{4.15}$$

for some $\lambda_1 \in \rho(A_0)$ *and some* $p > 0$. *Then*

$$(A_{[B]} - \lambda)^{-1} - (A_0 - \lambda)^{-1} \in \mathfrak{S}_p(\mathcal{H}) \tag{4.16}$$

for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$. *If, in addition,* A_1 *is self-adjoint, then*

$$(A_{[B]} - \lambda)^{-1} - (A_1 - \lambda)^{-1} \in \mathfrak{S}_p(\mathcal{H}) \tag{4.17}$$

for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_1)$.

Proof. By Theorem 4.1, the resolvent formula (4.2) holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$, and it can also be written in the form

$$(A_{[B]} - \lambda)^{-1} - (A_0 - \lambda)^{-1} = \overline{\gamma(\lambda)}\big(I - B\overline{M(\lambda)}\big)^{-1} B\gamma(\overline{\lambda})^*. \tag{4.18}$$

Moreover, it follows from (4.15) and [27, Proposition 3.5 (ii)] that $\gamma(\lambda)^* \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G})$ for all $\lambda \in \rho(A_0)$ and, hence, also $\overline{\gamma(\lambda)} = \gamma(\lambda)^{**} \in \mathfrak{S}_p(\mathcal{G}, \mathcal{H})$ for all $\lambda \in \rho(A_0)$.

To prove (4.16), let first $\lambda = \lambda_0$ be given as in the assumptions of the proposition. Since $B\gamma(\overline{\lambda})^* \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ can be shown as in (4.12) and $(I - B\overline{M(\lambda)})^{-1} \in \mathcal{B}(\mathcal{G})$ holds by assumption (i) of Theorem 4.1, it is clear that the right-hand side of (4.18) belongs to the Schatten–von Neumann ideal $\mathfrak{S}_p(\mathcal{H})$, which proves (4.16) for $\lambda = \lambda_0$. With the help of [27, Lemma 2.2] this property extends to all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.

Assume now, in addition, that A_1 is self-adjoint and fix some $\lambda \in \rho(A_{[B]}) \cap \rho(A_0) \cap \rho(A_1)$. Note that by [27, Theorem 3.8] the identity

$$(A_1 - \lambda)^{-1} - (A_0 - \lambda)^{-1} = -\overline{\gamma(\lambda)} M(\lambda)^{-1} \gamma(\overline{\lambda})^* \tag{4.19}$$

is true. It follows from [24, Proposition 6.14 (iii)] that the operator $M(\lambda)^{-1}$ is closable, and [22, Proposition 2.6 (iii)] implies that

$$\operatorname{ran}\big(\gamma(\overline{\lambda})^*\big) \subset \operatorname{ran} \Gamma_1 = \operatorname{ran} M(\lambda).$$

Thus, the operator $M(\lambda)^{-1}\gamma(\overline{\lambda})^*$ is everywhere defined and closable and hence closed, so that $M(\lambda)^{-1}\gamma(\overline{\lambda})^* \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Since $\overline{\gamma(\lambda)} \in \mathfrak{S}_p(\mathcal{G}, \mathcal{H})$ by the first part of the proof, the identity (4.19) implies that

$$(A_1 - \lambda)^{-1} - (A_0 - \lambda)^{-1} \in \mathfrak{S}_p(\mathcal{H}). \tag{4.20}$$

From (4.16) and (4.20) we conclude that (4.17) holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0) \cap \rho(A_1)$, and again with the help of [27, Lemma 2.2] this property extends to all $\lambda \in \rho(A_{[B]}) \cap \rho(A_1)$. $\square$

In the case when B is bounded and everywhere defined the assertion of the previous proposition improves as follows.

Proposition 4.8. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let $B \in \mathcal{B}(\mathcal{G})$ and assume that there exists $\lambda_0 \in \rho(A_0)$ such that conditions* (i)–(iii) *in Corollary 4.4 are satisfied. Further, assume that*

$$\gamma(\lambda_1)^* \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G}) \tag{4.21}$$

for some $\lambda_1 \in \rho(A_0)$ and some $p > 0$. Then

$$(A_{[B]} - \lambda)^{-1} - (A_0 - \lambda)^{-1} \in \mathfrak{S}_{\frac{p}{2}}(\mathcal{H}) \tag{4.22}$$

for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$. If, in addition, A_1 is self-adjoint and

$$M(\lambda_2)^{-1}\gamma(\overline{\lambda_2})^* \in \mathfrak{S}_q(\mathcal{H}, \mathcal{G})$$

for some $\lambda_2 \in \rho(A_0) \cap \rho(A_1)$ and some $q > 0$, then

$$(A_{[B]} - \lambda)^{-1} - (A_1 - \lambda)^{-1} \in \mathfrak{S}_r(\mathcal{H}) \qquad \textit{with} \quad r = \max\left\{\frac{p}{2}, \left(\frac{1}{p} + \frac{1}{q}\right)^{-1}\right\} \tag{4.23}$$

for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_1)$.

Proof. By Corollary 4.4 the resolvent formula (4.18) holds for all λ in the non-empty set $\rho(A_{[B]}) \cap \rho(A_0)$. As in the proof of Proposition 4.7 we conclude that $\gamma(\lambda)^* \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G})$ and $\overline{\gamma(\lambda)} \in \mathfrak{S}_p(\mathcal{G}, \mathcal{H})$ for all $\lambda \in \rho(A_0)$. Since $B \in \mathcal{B}(\mathcal{G})$, the operator $(I - B\overline{M(\lambda)})^{-1}B$ is also in $\mathcal{B}(\mathcal{G})$, and hence standard properties of Schatten–von Neumann ideals imply that the right-hand side of (4.18) belongs to the Schatten–von Neumann ideal $\mathfrak{S}_{\frac{p}{2}}(\mathcal{H})$.

Assume now that A_1 is self-adjoint and that $M(\lambda_2)^{-1}\gamma(\overline{\lambda_2})^* \in \mathfrak{S}_q(\mathcal{H}, \mathcal{G})$ for some $\lambda_2 \in \rho(A_0) \cap \rho(A_1)$. From the first part of the proof we have that $\overline{\gamma(\lambda_2)} \in \mathfrak{S}_p(\mathcal{G}, \mathcal{H})$. Using the identity (4.19), standard properties of Schatten–von Neumann classes and [27, Lemma 2.2] we obtain that

$$(A_1 - \lambda)^{-1} - (A_0 - \lambda)^{-1} \in \mathfrak{S}_{(1/p+1/q)^{-1}}(\mathcal{H}) \tag{4.24}$$

for all $\lambda \in \rho(A_0) \cap \rho(A_1)$. From (4.22) and (4.24) we conclude that (4.23) holds for $\lambda \in \rho(A_{[B]}) \cap \rho(A_0) \cap \rho(A_1)$, and again [27, Lemma 2.2] shows that this property extends to all $\lambda \in \rho(A_{[B]}) \cap \rho(A_1)$. $\square$

Remark 4.9. Propositions 4.7 and 4.8 can also be formulated for abstract operator ideals (see [27] and [121] for more details). In particular, they remain true for the so-called weak Schatten–von Neumann ideals $\mathfrak{S}_{p,\infty}$ and $\mathfrak{S}_{p,\infty}^{(0)}$ instead of $\mathfrak{S}_p$, where the ideals $\mathfrak{S}_{p,\infty}$ and $\mathfrak{S}_{p,\infty}^{(0)}$ consist of those compact operators whose singular values s_k satisfy $s_k = \mathrm{O}(k^{-1/p})$ and $s_k = \mathrm{o}(k^{-1/p})$, respectively, as $k \to \infty$; cf. [81].

5. Consequences of the decay of the Weyl function

In this section we continue the theme from Section 4. In addition to the assumptions of the previous section we now assume that the Weyl function M decays as $\operatorname{dist}(\lambda, \sigma(A_0)) \to \infty$. In the first theorem we deal with a situation where A_0 is bounded from below. Recall from (2.15) that in this case a decay assumption of the form $\|\overline{M(\lambda)}\| \to 0$ as $\lambda \to -\infty$ implies that $\overline{M(\lambda)}$ is a non-negative operator in $\mathcal{G}$ for all $\lambda < \min \sigma(A_0)$. The following theorem is now a consequence of Corollary 4.5; cf. [29, Theorem 2.8] for the special case when B is symmetric. Recall that a linear operator A in a Hilbert space is called *dissipative* (resp., *accumulative*) if $W(A) \subset \overline{\mathbb{C}^+}$ (resp., $W(A) \subset \overline{\mathbb{C}^-}$), and *maximal dissipative* (resp., *maximal accumulative*) if $W(A) \subset \overline{\mathbb{C}^+}$ and $\rho(A) \cap \mathbb{C}^- \neq \emptyset$ (resp., $W(A) \subset \overline{\mathbb{C}^-}$ and $\rho(A) \cap \mathbb{C}^+ \neq \emptyset$).

Theorem 5.1. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M. Assume that A_0 is bounded from below, that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$ and that*

$$\|\overline{M(\lambda)}\| \to 0 \quad as \quad \lambda \to -\infty. \tag{5.1}$$

Let B be a closable operator in $\mathcal{G}$ and assume that there exists $b \in \mathbb{R}$ such that

(i) $\operatorname{Re}(Bx, x) \leq b\|x\|^2$ *for all* $x \in \operatorname{dom} B$;
(ii) $\operatorname{ran} \overline{M(\lambda)}^{1/2} \subset \operatorname{dom} B$ *for all* $\lambda < \min \sigma(A_0)$;
(iii) $B\big(\operatorname{ran} \overline{M(\lambda)}\big) \subset \operatorname{ran} \Gamma_0$ *for all* $\lambda < \min \sigma(A_0)$;
(iv) $\operatorname{ran} \Gamma_1 \subset \operatorname{dom} B$;
(v) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or* $\rho(A_1) \cap (-\infty, \min \sigma(A_0)) \neq \emptyset$.

Then the operator

$$A_{[B]} f = Tf, \qquad \operatorname{dom} A_{[B]} = \big\{ f \in \operatorname{dom} T : \Gamma_0 f = B\Gamma_1 f \big\}, \tag{5.2}$$

is a closed extension of S in $\mathcal{H}$ and

$$\big\{ \lambda < \min \sigma(A_0) : b\|\overline{M(\lambda)}\| < 1 \big\} \subset \rho(A_{[B]}). \tag{5.3}$$

In particular, there exists $\mu \leq \min \sigma(A_0)$ such that $(-\infty, \mu) \subset \rho(A_{[B]})$. Moreover, the resolvent formula

$$(A_{[B]} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\gamma(\overline{\lambda})^* \tag{5.4}$$

holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$. If, in addition, B is symmetric (dissipative, accumulative, respectively), then $A_{[B]}$ is self-adjoint and bounded from below (maximal accumulative, maximal dissipative, respectively).

Further, let B' *be a linear operator in* $\mathcal{G}$ *that satisfies* (i)–(v) *with* B *replaced by* B' *and assume that*

$$(Bx, y) = (x, B'y) \qquad \text{for all } x \in \operatorname{dom} B,\ y \in \operatorname{dom} B'. \tag{5.5}$$

Then $A_{[B']} = A^*_{[B]}$ *and the left-hand side of* (5.3) *is contained in* $\rho(A_{[B']})$.

Proof. First note that it can be shown in the same way as in Step 5 in the proof of Theorem 4.1 that the second condition in (v) and (ii)–(iv) imply the first condition in (v). Further, the assumption (5.1) implies $M(\lambda) \geq 0$ for every $\lambda < \min \sigma(A_0)$; see (2.15). It follows from Corollary 4.5 that $A_{[B]}$ is a closed extension of S in $\mathcal{H}$ and that every point $\lambda < \min \sigma(A_0)$ with the property $b\|\overline{M(\lambda)}\| < 1$ belongs to $\rho(A_{[B]})$. Note that such λ exist due to the decay condition (5.1). Condition (5.1) and relation (5.3) also imply that there exists $\mu \leq \min \sigma(A_0)$ with

$$(-\infty, \mu) \subset \rho(A_{[B]}). \tag{5.6}$$

The resolvent formula (5.4) and the assertions on $A_{[B']}$ are immediate from Corollary 4.5.

It remains to show that $A_{[B]}$ is self-adjoint (maximal accumulative, maximal dissipative, respectively) if B is symmetric (dissipative, accumulative, respectively). For this let $f \in \operatorname{dom} A_{[B]}$ and observe that the abstract Green identity (2.1) yields

$$\begin{aligned} \operatorname{Im}(A_{[B]}f, f) &= \frac{1}{2i}\big((Tf, f) - (f, Tf)\big) = \frac{1}{2i}\big((\Gamma_1 f, \Gamma_0 f) - (\Gamma_0 f, \Gamma_1 f)\big) \\ &= \frac{1}{2i}\Big((\Gamma_1 f, B\Gamma_1 f) - (B\Gamma_1 f, \Gamma_1 f)\Big) = -\operatorname{Im}(B\Gamma_1 f, \Gamma_1 f). \end{aligned} \tag{5.7}$$

If B is symmetric (dissipative, accumulative), then $\operatorname{Im}(Bx, x)$ is zero (non-negative, non-positive, respectively) for all $x \in \operatorname{dom} B$, and it follows from (5.7) that $A_{[B]}$ is symmetric (accumulative, dissipative, respectively). Now (5.6) implies that $A_{[B]}$ is self-adjoint and bounded from below (maximal accumulative, maximal dissipative, respectively). □

In the case when $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triple, Theorem 5.1 simplifies in the following way.

Corollary 5.2. *Let* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *be a generalized boundary triple for* $T \subset S^*$ *with corresponding Weyl function* M. *Assume that* A_0 *is bounded from below and that*

$$\|M(\lambda)\| \to 0 \quad \text{as} \quad \lambda \to -\infty.$$

Let B *be a closable operator in* $\mathcal{G}$ *and assume that there exists* $b \in \mathbb{R}$ *such that*

(i) $\operatorname{Re}(Bx, x) \le b\|x\|^2$ *for all* $x \in \operatorname{dom} B$*;*
(ii) $\operatorname{ran} M(\lambda)^{1/2} \subset \operatorname{dom} B$ *for all* $\lambda < \min \sigma(A_0)$*;*
(iii) $\operatorname{ran} \Gamma_1 \subset \operatorname{dom} B$*.*

Then the operator $A_{[B]}$ *in* (5.2) *is a closed extension of* S *in* $\mathcal{H}$ *and*

$$\big\{\lambda < \min \sigma(A_0) : b\|M(\lambda)\| < 1\big\} \subset \rho(A_{[B]}). \tag{5.8}$$

In particular, there exists $\mu \le \min \sigma(A_0)$ *such that* $(-\infty, \mu) \subset \rho(A_{[B]})$*. Moreover, the resolvent formula* (5.4) *holds for all* $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$*. If, in addition,* B *is symmetric (dissipative, accumulative, respectively), then* $A_{[B]}$ *is self-adjoint and bounded from below (maximal accumulative, maximal dissipative, respectively).*

Further, let B' *be a linear operator in* $\mathcal{G}$ *that satisfies* (i)–(iii) *with* B *replaced by* B' *and assume that* (5.5) *holds. Then* $A_{[B']} = A_{[B]}^*$ *and the left-hand side of* (5.8) *is contained in* $\rho(A_{[B']})$*.*

Remark 5.3. Note that for an ordinary boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ condition (iv) in Theorem 5.1 (condition (iii) in Corollary 5.2) implies that $B \in \mathcal{B}(\mathcal{G})$. In this situation the conditions (ii), (iii), and the first condition in (v) in Theorem 5.1 (condition (ii) in Corollary 5.2) are automatically satisfied. We shall formulate a corollary on spectral enclosures in the case of an ordinary boundary triple in Corollary 5.7 below.

Let us formulate another corollary of Theorem 5.1 (in particular, of the inclusion in (5.3)).

Corollary 5.4. *Let all assumptions of Theorem 5.1 be satisfied and assume that* $b \le 0$ *in* (i) *of Theorem 5.1. Then the closed operator* $A_{[B]}$ *in* (5.2) *satisfies*

$$\big(-\infty, \min \sigma(A_0)\big) \subset \rho(A_{[B]}).$$

We now turn to situations where the rate of decay of the Weyl function for $\lambda \to -\infty$ is known in more detail. In such cases we derive spectral estimates for the operator $A_{[B]}$, which refine the inclusion (5.3) in Theorem 5.1. The following proposition provides a first, easy step towards this. Here we assume that b in Theorem 5.1 (i) is positive; the case $b \le 0$ is treated in Corollary 5.4 above. The proposition is a generalization of [29, Theorem 2.8 (b)] to the non-self-adjoint setting.

Proposition 5.5. *Let* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *be a quasi boundary triple for* $T \subset S^*$ *with corresponding Weyl function* M*. Assume that* A_0 *is bounded from below, that* $M(\lambda)$ *is bounded for one (and hence for all)* $\lambda \in \rho(A_0)$ *and that there exist* $\beta \in (0, 1]$*,* $C > 0$ *and* $\mu \le \min \sigma(A_0)$ *such that*

$$\big\|\overline{M(\lambda)}\big\| \le \frac{C}{(\mu - \lambda)^\beta} \qquad \textit{for all } \ \lambda < \mu. \tag{5.9}$$

Moreover, let B be a closable operator in $\mathcal{G}$, let $b > 0$, and assume that conditions (i)–(v) *in Theorem 5.1 are satisfied. Then the operator $A_{[B]}$ in* (5.2) *is closed and satisfies*

$$\left(-\infty, \mu - (Cb)^{1/\beta}\right) \subset \rho(A_{[B]}). \tag{5.10}$$

Proof. That $A_{[B]}$ is closed follows from Theorem 5.1. Consider $\lambda < \mu - (Cb)^{1/\beta}$. Then $(\mu - \lambda)^\beta > Cb$ and hence

$$b\left\|\overline{M(\lambda)}\right\| \leq b\frac{C}{(\mu - \lambda)^\beta} < 1.$$

Now Theorem 5.1 yields that $\lambda \in \rho(A_{[B]})$. $\square$

In the next theorem we study the m-sectorial case discussed in Theorem 3.1 in more detail and obtain refined estimates for the numerical range of $A_{[B]}$. Roughly speaking, if the Weyl function decays for $\lambda \to -\infty$, then there exists an $\eta_* \in \mathbb{R}$ such that the assumptions in Theorem 3.1 are satisfied for every $\eta < \eta_*$ and hence

$$\sigma(A_{[B]}) \subset \overline{W(A_{[B]})} \subset \bigcap_{\eta \in (-\infty, \eta_*)} \mathcal{S}_\eta(B).$$

In the particular case when $\operatorname{Im} B$ is bounded and the Weyl function satisfies a decay condition as in Proposition 5.5, we use this fact to obtain an extension of Proposition 5.5 including estimates for the non-real spectrum.

Theorem 5.6. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M and suppose that A_1 is self-adjoint and that A_0 and A_1 are bounded from below. Further, assume that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$ and that there exist $\beta \in (0, 1]$, $C > 0$ and $\mu \leq \min \sigma(A_0)$ such that*

$$\left\|\overline{M(\lambda)}\right\| \leq \frac{C}{(\mu - \lambda)^\beta} \qquad \text{for every } \lambda < \mu. \tag{5.11}$$

Moreover, let B be a closable linear operator in $\mathcal{G}$ and let $b \in \mathbb{R}$ such that conditions (i)–(iv) *in Theorem 5.1 are satisfied. Then the operator $A_{[B]}$ in* (5.2) *is m-sectorial and, in particular, the inclusion $\sigma(A_{[B]}) \subset \overline{W(A_{[B]})}$ holds.*

Assume, in addition, that $\operatorname{dom} B^* \supset \operatorname{dom} B$ and that $\operatorname{Im} B$ is bounded. Then the following assertions are true.

(a) *If $b > 0$, then for every $\xi < \mu - (Cb)^{1/\beta}$,*

$$W(A_{[B]}) \subset \left\{z \in \mathbb{C} : \operatorname{Re} z \geq \mu - (Cb)^{1/\beta},\ |\operatorname{Im} z| \leq K_\xi (\operatorname{Re} z - \xi)^{1-\beta}\right\}, \tag{5.12}$$

where

$$K_\xi = \frac{2C\|\overline{\operatorname{Im} B}\|}{1 - \frac{Cb}{(\mu-\xi)^\beta}}.$$

(b) *If* $b = 0$, *then*

$$W(A_{[B]}) \subset \Big\{ z \in \mathbb{C} : \operatorname{Re} z \geq \mu,\ |\operatorname{Im} z| \leq K'_\beta (\operatorname{Re} z - \mu)^{1-\beta} \Big\}, \tag{5.13}$$

where

$$K'_\beta = \begin{cases} \dfrac{C\|\overline{\operatorname{Im} B}\|}{\beta^\beta (1-\beta)^{1-\beta}} & \text{if } 0 < \beta < 1, \\[2ex] C\|\overline{\operatorname{Im} B}\| & \text{if } \beta = 1, \end{cases} \tag{5.14}$$

and the convention $0^0 = 1$ *is used in* (5.13) *when* $\beta = 1$ *and* $\operatorname{Re} z = \mu$. *Moreover,* K'_β *satisfies* $C\|\overline{\operatorname{Im} B}\| \leq K'_\beta \leq 2C\|\overline{\operatorname{Im} B}\|$.

(c) *If* $b < 0$, *then*

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq \mu,\ |\operatorname{Im} z| \leq \frac{2C\|\overline{\operatorname{Im} B}\|(\operatorname{Re} z - \mu)}{(\operatorname{Re} z - \mu)^\beta - Cb} \right\}. \tag{5.15}$$

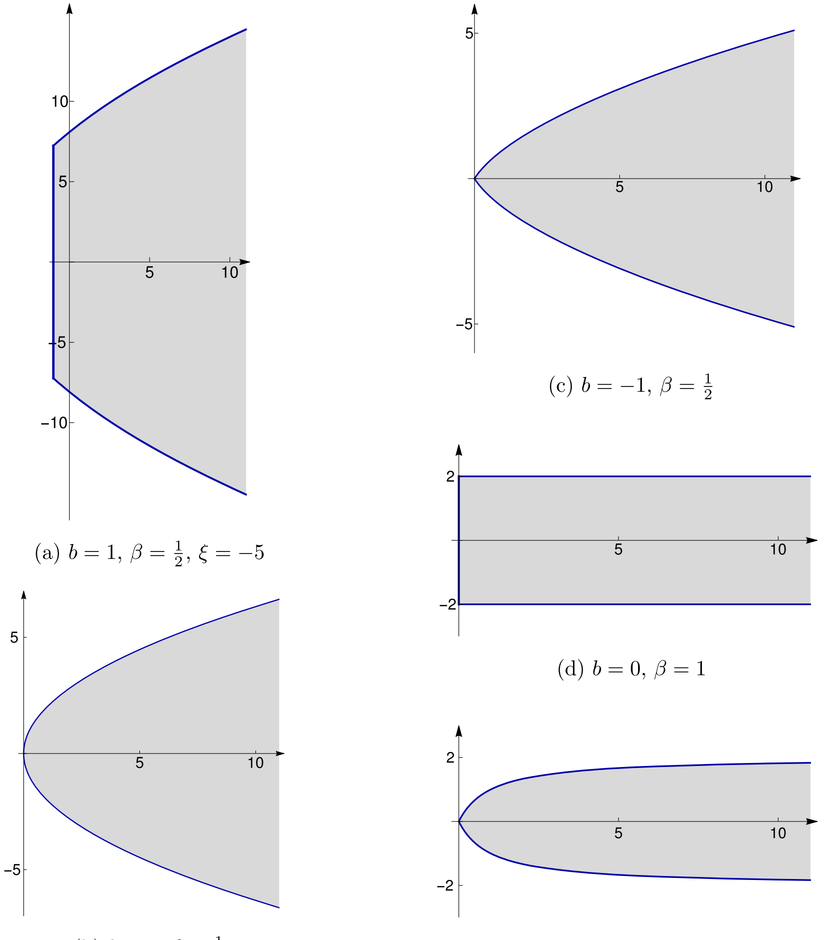
See Fig. 1 for plots of the regions given by the right-hand sides of (5.12), (5.13), (5.15). Notice that in Theorem 5.6 (a) we get, in fact, a family of enclosures in parabola-type regions that depend on the choice of the parameter ξ. By intersecting all these regions with respect to $\xi \in (-\infty, \mu - (Cb)^{1/\beta})$ one gets a finer enclosure for the numerical range of $A_{[B]}$.

Proof. Note first, that the conditions of Theorem 3.1 are satisfied; we point out, particularly, that by (5.11) and (2.15) we have $M(\lambda) \geq 0$ for each $\lambda < \min \sigma(A_0) \leq \min \sigma(A_1)$ (see Proposition 2.7), and there exists $\eta < \mu$ such that $b\|\overline{M(\eta)}\| < 1$. Hence, $A_{[B]}$ is sectorial. Since A_1 is self-adjoint and bounded from below and the assumptions (i)–(iv) in Theorem 5.1 hold, the latter yields $\eta \in \rho(A_{[B]})$. Thus $A_{[B]}$ is m-sectorial and hence $\sigma(A_{[B]}) \subset \overline{W(A_{[B]})}$.

For the rest of the proof assume that $\operatorname{dom} B^* \supset \operatorname{dom} B$ and that $\operatorname{Im} B$ is bounded. For every $\lambda < \min \sigma(A_0)$ we have $\operatorname{ran} \overline{M(\lambda)}^{1/2} \subset \operatorname{dom} B \subset \operatorname{dom} B^*$ by condition (ii) of Theorem 5.1; in particular, $B\overline{M(\lambda)}^{1/2} \in \mathcal{B}(\mathcal{G})$ and $B^*\overline{M(\lambda)}^{1/2} \in \mathcal{B}(\mathcal{G})$. Hence

$$\begin{aligned} \big(\overline{M(\lambda)}^{1/2} B \overline{M(\lambda)}^{1/2}\big)^* &= \big(B\overline{M(\lambda)}^{1/2}\big)^* \overline{M(\lambda)}^{1/2} = \Big(\big(\overline{M(\lambda)}^{1/2} B^*\big)^*\Big)^* \overline{M(\lambda)}^{1/2} \\ &= \overline{\overline{M(\lambda)}^{1/2} B^*} \overline{M(\lambda)}^{1/2} = \overline{M(\lambda)}^{1/2} B^* \overline{M(\lambda)}^{1/2}. \end{aligned}$$

This implies that



(a) $b = 1$, $\beta = \frac{1}{2}$, $\xi = -5$

(b) $b = 0$, $\beta = \frac{1}{2}$

(c) $b = -1$, $\beta = \frac{1}{2}$

(d) $b = 0$, $\beta = 1$

(e) $b = -1$, $\beta = 1$

Fig. 1. The plots show the regions given by the right-hand sides of (5.12), (5.13), (5.15) with $\mu = 0$, $C = 1$ and $\|\overline{\operatorname{Im} B}\| = 1$ for the following cases: $\beta = \frac{1}{2}$ in (a)–(c) ($b > 0$, $b = 0$, $b < 0$, respectively) and $\beta = 1$ in (d), (e) ($b = 0$, $b < 0$, respectively).

$$
\begin{aligned}
\big\|\operatorname{Im}\big(\overline{M(\lambda)}^{1/2} B \overline{M(\lambda)}^{1/2}\big)\big\| &= \frac{1}{2}\Big\|\overline{M(\lambda)}^{1/2} B \overline{M(\lambda)}^{1/2} - \big(\overline{M(\lambda)}^{1/2} B \overline{M(\lambda)}^{1/2}\big)^*\Big\| \\
&= \frac{1}{2}\Big\|\overline{M(\lambda)}^{1/2} (B - B^*) \overline{M(\lambda)}^{1/2}\Big\| \\
&\leq \big\|\overline{\operatorname{Im} B}\big\|\,\big\|\overline{M(\lambda)}\big\|,
\end{aligned}
\tag{5.16}
$$

where we have used that $\operatorname{Im} B$ is a bounded operator defined on the dense subspace $\operatorname{dom} B$ of $\mathcal{G}$. Let $z \in W(A_{[B]})$. It follows from Theorem 3.1 and (5.16) that, for every $\eta < \min \sigma(A_0)$ for which $b\|\overline{M(\eta)}\| < 1$, the inequalities

$$\operatorname{Re} z \geq \eta, \qquad |\operatorname{Im} z| \leq \frac{\|\overline{\operatorname{Im} B}\| \, \|\overline{M(\eta)}\|}{1 - b\|\overline{M(\eta)}\|}(\operatorname{Re} z - \eta) \tag{5.17}$$

hold.

(a) Assume that $b > 0$. For every $\eta < \mu - (Cb)^{1/\beta}$ we have $\eta < \min \sigma(A_0)$ and, by (5.11),

$$b\|\overline{M(\eta)}\| \leq \frac{Cb}{(\mu - \eta)^\beta} < 1. \tag{5.18}$$

Hence (5.17) is true for each such η. For the real part of z this yields

$$\operatorname{Re} z \geq \mu - (Cb)^{1/\beta}. \tag{5.19}$$

To estimate $|\operatorname{Im} z|$ further, note that the function

$$\Big(-\infty, \frac{1}{b}\Big) \ni t \mapsto \frac{\|\overline{\operatorname{Im} B}\| t}{1 - bt}$$

is strictly increasing and that $\|\overline{M(\eta)}\| \leq \frac{C}{(\mu-\eta)^\beta} < \frac{1}{b}$ for all $\eta < \mu - (Cb)^{1/\beta}$ by (5.18). Hence (5.17) yields

$$|\operatorname{Im} z| \leq \frac{\|\overline{\operatorname{Im} B}\| \frac{C}{(\mu-\eta)^\beta}}{1 - \frac{Cb}{(\mu-\eta)^\beta}}(\operatorname{Re} z - \eta) \qquad \text{for all } \eta < \mu - (Cb)^{1/\beta}. \tag{5.20}$$

Now let $\xi < \mu - (Cb)^{1/\beta}$ be arbitrary. Then (5.19) implies that $\operatorname{Re} z > \xi$. Choose $\eta := 2\xi - \operatorname{Re} z$, which satisfies $\eta < 2\xi - \xi = \xi$. From (5.20) and $\xi < \mu$ we obtain the inequality

$$|\operatorname{Im} z| \leq \frac{C\|\overline{\operatorname{Im} B}\|}{1 - \frac{Cb}{(\mu-\xi)^\beta}} \cdot \frac{\operatorname{Re} z - \eta}{(\xi - \eta)^\beta} = \frac{C\|\overline{\operatorname{Im} B}\|}{1 - \frac{Cb}{(\mu-\xi)^\beta}} \cdot \frac{2(\operatorname{Re} z - \xi)}{(\operatorname{Re} z - \xi)^\beta},$$

which, together with (5.19), shows (5.12).

(b), (c) Assume now that $b \leq 0$. For every $\eta < \mu$ we have $\eta < \min \sigma(A_0)$ and $b\|\overline{M(\eta)}\| \leq 0$. Hence (5.17) is true for $\eta < \mu$, which, in particular, shows that

$$\operatorname{Re} z \geq \mu. \tag{5.21}$$

Note that $t \mapsto \frac{\|\overline{\operatorname{Im} B}\| t}{1-bt}$ is strictly increasing on $(0, \infty)$. Hence (5.17) and (5.11) imply that

$$|\operatorname{Im} z| \leq \frac{\|\overline{\operatorname{Im} B}\| \, \|\overline{M(\eta)}\|}{1 - b\|\overline{M(\eta)}\|}(\operatorname{Re} z - \eta) \leq \frac{\|\overline{\operatorname{Im} B}\| \frac{C}{(\mu-\eta)^\beta}}{1 - \frac{Cb}{(\mu-\eta)^\beta}}(\operatorname{Re} z - \eta). \tag{5.22}$$

Assume first that $\operatorname{Re} z > \mu$. Now we distinguish the two cases $b = 0$ and $b < 0$. First let $b = 0$ and $\beta \in (0, 1)$. We choose

$$\eta := \frac{1}{1-\beta}(\mu - \beta \operatorname{Re} z),$$

which yields

$$\operatorname{Re} z - \eta = \frac{1}{1-\beta}(\operatorname{Re} z - \mu) \qquad \text{and} \qquad \mu - \eta = \frac{\beta}{1-\beta}(\operatorname{Re} z - \mu);$$

in particular, we have $\eta < \mu$. Hence (5.22) implies that

$$\begin{aligned} |\operatorname{Im} z| &\leq C\big\|\overline{\operatorname{Im} B}\big\| \frac{\operatorname{Re} z - \eta}{(\mu - \eta)^\beta} = C\big\|\overline{\operatorname{Im} B}\big\| \frac{\frac{1}{1-\beta}(\operatorname{Re} z - \mu)}{\big[\frac{\beta}{1-\beta}(\operatorname{Re} z - \mu)\big]^\beta} \\ &= \frac{C\big\|\overline{\operatorname{Im} B}\big\|}{\beta^\beta(1-\beta)^{1-\beta}}(\operatorname{Re} z - \mu)^{1-\beta}, \end{aligned}$$

which shows that z is contained in the right-hand side of (5.13). Taking the limit $\beta \nearrow 1$ we obtain this inclusion also for the case when $\beta = 1$. The estimates for K'_β follow from the fact that the function $f(\beta) = \beta^\beta(1-\beta)^{1-\beta}$, $\beta \in (0, 1)$ has a unique minimum at $\beta = \frac{1}{2}$ and that $f(\beta) \to 1$ as $\beta \searrow 0$ or $\beta \nearrow 1$.

Now let $b < 0$ (and still $\operatorname{Re} z > \mu$). We choose $\eta := 2\mu - \operatorname{Re} z$, which yields

$$\operatorname{Re} z - \eta = 2(\operatorname{Re} z - \mu) \qquad \text{and} \qquad \mu - \eta = \operatorname{Re} z - \mu.$$

Therefore (5.22) implies that

$$|\operatorname{Im} z| \leq \frac{2C\big\|\overline{\operatorname{Im} B}\big\|}{(\operatorname{Re} z - \mu)^\beta - Cb}(\operatorname{Re} z - \mu),$$

and hence z is contained in the right-hand side of (5.15). Since the numerical range $W(A_{[B]})$ is a convex set, the inclusions (5.13) and (5.15) hold also for z with $\operatorname{Re} z = \mu$. □

Next we formulate a variant of Theorem 5.6 for the special case when $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple. In this case the assumptions in Theorem 5.1 imply that B is a bounded operator in $\mathcal{G}$; cf. Remark 5.3.

Corollary 5.7. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triple for S^* with corresponding Weyl function M and suppose that the self-adjoint operators A_0 and A_1 are bounded from below. Further, assume that there exist $\beta \in (0, 1]$, $C > 0$ and $\mu \leq \min \sigma(A_0)$ such that*

$$\big\|M(\lambda)\big\| \leq \frac{C}{(\mu - \lambda)^\beta} \qquad \textit{for every } \lambda < \mu.$$

Let $B \in \mathcal{B}(\mathcal{G})$ be a bounded, everywhere defined operator in $\mathcal{G}$ and let $b \in \mathbb{R}$ be such that $\operatorname{Re}(Bx, x) \leq b\|x\|^2$ for all $x \in \mathcal{G}$. Then the operator $A_{[B]}$ in (5.2) is m-sectorial and, in particular, the inclusion $\sigma(A_{[B]}) \subset \overline{W(A_{[B]})}$ holds. Moreover, the assertions in Theorem 5.6 (a), (b) and (c) are true.

In the following theorem we drop the assumption that A_0 is bounded from below, but we assume that $B \in \mathcal{B}(\mathcal{G})$. We remark that the condition (5.1) does no longer make sense if A_0 is not bounded from below. Therefore we replace it by the more appropriate condition (5.23) below.

Theorem 5.8. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M. Assume that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$ and that*

$$\big\|\overline{M(re^{i\varphi})}\big\| \to 0 \qquad \text{as } r \to \infty \tag{5.23}$$

for some fixed $\varphi \in (-\pi, 0) \cup (0, \pi)$. Let $B \in \mathcal{B}(\mathcal{G})$ be such that

(i) $B\big(\operatorname{ran} \overline{M(\lambda)}\big) \subset \operatorname{ran} \Gamma_0$ *for all $\lambda \in \rho(A_0)$;*
(ii) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or A_1 is self-adjoint.*

Then the operator $A_{[B]}$ in (5.2) is closed, the resolvent formula (5.4) holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$, and

$$\big\{\lambda \in \rho(A_0) : \big\|B\overline{M(\lambda)}\big\| < 1\big\} \subset \rho(A_{[B]}). \tag{5.24}$$

In particular, for every interval $[\psi_1, \psi_2] \subset (-\pi, 0)$ or $[\psi_1, \psi_2] \subset (0, \pi)$ there exists $R_{[\psi_1,\psi_2]} > 0$ such that

$$\big\{re^{i\psi} : r \geq R_{[\psi_1,\psi_2]},\, \psi \in [\psi_1, \psi_2]\big\} \subset \rho(A_{[B]}). \tag{5.25}$$

Moreover, if B is self-adjoint (accumulative, dissipative, respectively), then $A_{[B]}$ is self-adjoint (maximal dissipative, maximal accumulative, respectively).

Further, if conditions (i) and (ii) are satisfied also for the adjoint operator B^* instead of B, then $A_{[B^*]} = A_{[B]}^*$.

Proof. Let $\lambda \in \rho(A_0)$ with $\|B\overline{M(\lambda)}\| < 1$; such λ exist by (5.23). Then

$$1 \in \rho\big(B\overline{M(\lambda)}\big).$$

It follows from this and the assumptions of the current theorem that Corollary 4.4 can be applied. Thus $A_{[B]}$ is closed, the resolvent formula (5.4) holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$, (5.24) is valid, and the statement on $A_{[B^*]}$ follows. The relation (5.25) follows from (5.23), Lemma 2.6 and (5.24). If B is symmetric (accumulative, dissipative, respectively), then it follows as in the proof of Theorem 5.1 that $A_{[B]}$ is symmetric (dissipative, accumulative, respectively). This, together with (5.25), implies the remaining assertions. □

The next proposition complements Proposition 5.5. Here we require a decay condition on the Weyl function on a set $G \subset \rho(A_0)$ that is sufficiently large. In later sections this is applied to, e.g. all of $\rho(A_0)$ or to certain sectors in the complex plane.

Proposition 5.9. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M and assume that $M(\lambda)$ is bounded for one (and hence for all) $\lambda \in \rho(A_0)$. Further, let $B \in \mathcal{B}(\mathcal{G})$ such that*

(i) $B\big(\operatorname{ran} \overline{M(\lambda)}\big) \subset \operatorname{ran} \Gamma_0$ *for all* $\lambda \in \rho(A_0)$;
(ii) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or* A_1 *is self-adjoint.*

Let $G \subset \rho(A_0)$ be a set such that there exist $\lambda_n \in G$, $n \in \mathbb{N}$, with

$$\operatorname{dist}(\lambda_n, \sigma(A_0)) \to \infty \quad \text{as } n \to \infty.$$

Then the following assertions hold.

(a) *If there exist $\beta \in (0,1]$ and $C > 0$ such that*

$$\big\|\overline{M(\lambda)}\big\| \le \frac{C}{\big(\operatorname{dist}(\lambda, \sigma(A_0))\big)^{\beta}} \qquad \text{for all } \lambda \in G,$$

then $A_{[B]}$ is a closed extension of S and

$$\sigma(A_{[B]}) \cap G \subset \Big\{ z \in G : \operatorname{dist}(z, \sigma(A_0)) \le \big(C\|B\|\big)^{1/\beta} \Big\}. \tag{5.26}$$

(b) *If there exist $\beta \in (0,1]$, $C > 0$ and $\mu \le \min \sigma(A_0)$ such that*

$$\big\|\overline{M(\lambda)}\big\| \le \frac{C}{|\lambda - \mu|^{\beta}} \qquad \text{for all } \lambda \in G, \tag{5.27}$$

then $A_{[B]}$ is a closed extension of S and

$$\sigma(A_{[B]}) \cap G \subset \Big\{ z \in G : |z - \mu| \le \big(C\|B\|\big)^{1/\beta} \Big\}. \tag{5.28}$$

Proof. We prove only assertion (a); the proof of the second assertion is analogous. Assume first that condition (i) and the first condition in (ii) are satisfied. By the assumption on G, there exists $\lambda \in G$ such that $\operatorname{dist}(\lambda, \sigma(A_0)) > (C\|B\|)^{1/\beta}$. Then

$$\big\|B\overline{M(\lambda)}\big\| \le \|B\|\big\|\overline{M(\lambda)}\big\| < \frac{\big(\operatorname{dist}(\lambda,\sigma(A_0))\big)^\beta}{C}\cdot\frac{C}{\big(\operatorname{dist}(\lambda,\sigma(A_0))\big)^\beta} = 1$$

implies that $1 \in \rho(B\overline{M(\lambda)})$. It follows from Theorem 4.1 that $A_{[B]}$ is closed with $\lambda \in \rho(A_{[B]})$. If the condition (i) together with the second condition in (ii) is satisfied then $\rho(A_0)\cap\rho(A_1) \neq \emptyset$ and for each $\lambda \in \rho(A_0)\cap\rho(A_1)$ we have $\operatorname{ran}\Gamma_1 = \operatorname{ran} M(\lambda) \subset \operatorname{ran}\overline{M(\lambda)}$; see [22, Proposition 2.6 (iii)]. Hence, for each such λ we have $B(\operatorname{ran}\Gamma_1) \subset \operatorname{ran}\Gamma_0$ by (i), that is, the first condition of (ii) is satisfied as well. □

In the special case $G = \rho(A_0)$ and $A_0 \ge 0$ with $\mu = 0$ in (5.27), Proposition 5.9 (b) reads as follows.

Corollary 5.10. *Let the assumptions be as in Proposition 5.9 and assume, in addition, that A_0 is non-negative and that there exist $\beta \in (0,1]$ and $C > 0$ such that*

$$\big\|\overline{M(\lambda)}\big\| \le \frac{C}{|\lambda|^\beta} \qquad \text{for all } \lambda \in \rho(A_0).$$

Then

$$\sigma(A_{[B]}) \cap \rho(A_0) \subset \Big\{z \in \rho(A_0) : |z| \le \big(C\|B\|\big)^{1/\beta}\Big\}.$$

6. Sufficient conditions for decay of the Weyl function

In this section we consider conditions on the quasi boundary triple that ensure an asymptotic behaviour of the Weyl function M as required in the results of the previous section. We emphasize that these results are also new in the settings of ordinary and generalized boundary triples. For the next theorem some notation for sectors in the complex plane is needed. For $z_0 \in \overline{\mathbb{C}^+}$ and $\theta \in \big(0, \frac{\pi}{2}\big)$ we define the closed sector $\mathbb{S}_{z_0,\theta}$ in $\mathbb{C}^+$ by

$$\mathbb{S}_{z_0,\theta} := \Big\{z \in \mathbb{C} : z \neq z_0,\ \arg(z - z_0) \in \Big[\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta\Big]\Big\} \cup \{z_0\} \tag{6.1}$$

and we denote the corresponding complex conjugate sector in $\mathbb{C}^-$ by $\mathbb{S}^*_{z_0,\theta}$, that is,

$$\mathbb{S}^*_{z_0,\theta} := \big\{z \in \mathbb{C} : \overline{z} \in \mathbb{S}_{z_0,\theta}\big\}.$$

Furthermore, for $w_0 \in \mathbb{R}$ and $\nu \in (0,\pi)$ we set

$$\mathbb{U}_{w_0,\nu} := \big\{z \in \mathbb{C} : z \neq w_0,\ \arg(z - w_0) \in [\nu, 2\pi - \nu]\big\} \cup \{w_0\}; \tag{6.2}$$

see Fig. 2.

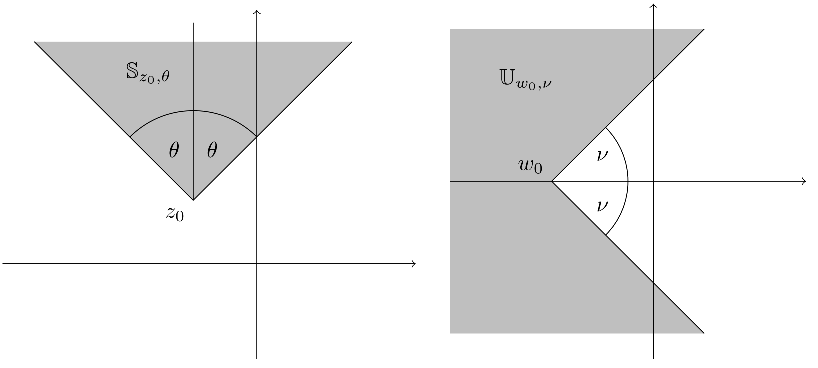


Fig. 2. The sectors $\mathbb{S}_{z_0,\theta}$ and $\mathbb{U}_{w_0,\nu}$, defined in (6.1) and (6.2), respectively.

In the proof of the next theorem we need the following fact from the functional calculus for self-adjoint operators, which is found, e.g. in [125, Theorem 5.9]: for a self-adjoint operator A and measurable functions $\Phi, \Psi : \sigma(A) \to \mathbb{C}$ one has

$$\overline{\Phi(A)\Psi(A)} = (\Phi\Psi)(A). \tag{6.3}$$

If Ψ is bounded on $\sigma(A)$, then the closure on the left-hand side is not needed.

Theorem 6.1. *Let S be a densely defined, closed, symmetric operator in a Hilbert space $\mathcal{H}$ and let $\Pi = \{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M. Moreover, assume that*

$$\Gamma_1|A_0 - \mu|^{-\alpha} : \mathcal{H} \supset \operatorname{dom}(\Gamma_1|A_0 - \mu|^{-\alpha}) \to \mathcal{G} \tag{6.4}$$

is bounded for some $\mu \in \rho(A_0)$ and some $\alpha \in \left(0, \frac{1}{2}\right]$. Then the following assertions hold.

(a) *$M(\lambda)$ is bounded for all $\lambda \in \rho(A_0)$.*
(b) *For all $z_0 \in \overline{\mathbb{C}^+} \cap \rho(A_0)$ and all $\theta \in (0, \frac{\pi}{2})$ there exists $C = C(\Pi, \alpha, \mu, z_0, \theta) > 0$ such that*

$$\left\|\overline{M(\lambda)}\right\| \le \frac{C}{\big(\operatorname{dist}(\lambda, \sigma(A_0))\big)^{1-2\alpha}} \tag{6.5}$$

for all $\lambda \in \mathbb{S}_{z_0,\theta} \cup \mathbb{S}^*_{z_0,\theta}$.
(c) *If A_0 is bounded from below, then for all $w_0 < \min \sigma(A_0)$ and all $\nu \in (0, \pi)$ there exists $D = D(\Pi, \alpha, \mu, w_0, \nu) > 0$ such that*

$$\left\|\overline{M(\lambda)}\right\| \le \frac{D}{\big(\operatorname{dist}(\lambda, \sigma(A_0))\big)^{1-2\alpha}} \tag{6.6}$$

for all $\lambda \in \mathbb{U}_{w_0,\nu}$.

Proof. Let us first observe that $\Gamma_1|A_0-\mu|^{-\alpha}$ is densely defined. Indeed, with the functions $\Phi(t) := (t-\mu)^{-1}$ and $\Psi(t) := (t-\mu)|t-\mu|^{-\alpha}$ we can use (6.3) and (2.3) to write

$$\Gamma_1|A_0-\mu|^{-\alpha} = \Gamma_1(\Phi\Psi)(A_0) \supset \Gamma_1\Phi(A_0)\Psi(A_0) = \gamma(\overline{\mu})^*\Psi(A_0).$$

Since $\gamma(\overline{\mu})^* \in \mathcal{B}(\mathcal{H},\mathcal{G})$ and $\operatorname{dom}\Psi(A_0) = \operatorname{dom}|A_0-\mu|^{1-\alpha}$ is dense in $\mathcal{H}$, it follows that $\Gamma_1|A_0-\mu|^{-\alpha}$ is densely defined. By assumption (6.4) we therefore have

$$\overline{\Gamma_1|A_0-\mu|^{-\alpha}} \in \mathcal{B}(\mathcal{H},\mathcal{G}). \tag{6.7}$$

Note that also $\gamma(\mu)^*|A_0-\mu|^{1-\alpha}$ is densely defined since $\gamma(\mu)^* \in \mathcal{B}(\mathcal{H},\mathcal{G})$ and $|A_0-\mu|^{1-\alpha}$ is self-adjoint. Moreover, set

$$\begin{aligned} \Phi_1(t) &:= (t-\overline{\mu})^{-1}, & \Psi_1(t) &:= |t-\mu|^{1-\alpha}, \\ \Phi_2(t) &:= |t-\mu|^{-\alpha}, & \Psi_2(t) &:= |t-\mu|(t-\overline{\mu})^{-1}, \end{aligned} \qquad t \in \sigma(A_0),$$

and note that $\Phi_1\Psi_1 = \Phi_2\Psi_2$ and that Ψ_2 is bounded. We obtain from (2.3), (6.3) and (6.7) that

$$\begin{aligned} \gamma(\mu)^*|A_0-\mu|^{1-\alpha} &= \Gamma_1(A_0-\overline{\mu})^{-1}|A_0-\mu|^{1-\alpha} \\ &= \Gamma_1\Phi_1(A_0)\Psi_1(A_0) \subset \Gamma_1(\Phi_1\Psi_1)(A_0) = \Gamma_1(\Phi_2\Psi_2)(A_0) \\ &= \Gamma_1\Phi_2(A_0)\Psi_2(A_0) \subset \overline{\Gamma_1|A_0-\mu|^{-\alpha}}\Psi_2(A_0) \in \mathcal{B}(\mathcal{H},\mathcal{G}). \end{aligned}$$

Thus $\gamma(\mu)^*|A_0-\mu|^{1-\alpha}$ is bounded and densely defined. In particular,

$$|A_0-\mu|^{1-\alpha}\overline{\gamma(\mu)} = \left[\gamma(\mu)^*|A_0-\mu|^{1-\alpha}\right]^* \in \mathcal{B}(\mathcal{G},\mathcal{H}), \tag{6.8}$$

where we have used again that $\gamma(\mu)^* \in \mathcal{B}(\mathcal{H},\mathcal{G})$. Let $\lambda \in \rho(A_0)$ and define the functions

$$\begin{aligned} \Phi_3(t) &:= \frac{t-\mu}{t-\lambda}|t-\mu|^{\alpha-1}, & \Psi_3(t) &:= |t-\mu|^{1-\alpha}, \\ \Phi_4(t) &:= |t-\mu|^{-\alpha}, & \Psi_4(t) &:= \frac{t-\mu}{t-\lambda}|t-\mu|^{2\alpha-1}, \end{aligned} \qquad t \in \sigma(A_0),$$

which satisfy $\Phi_3 = \Phi_4\Psi_4$. The functions Φ_3, Φ_4 and Ψ_4 are bounded on $\sigma(A_0)$ and $\operatorname{ran}\overline{\gamma(\mu)} \subset \operatorname{dom}\Psi_3(A_0)$ by (6.8). Hence for each $g \in \operatorname{dom} M(\lambda) = \operatorname{ran}\Gamma_0$ we have (where we use (2.5) in the second equality)

$$\begin{aligned} M(\lambda)g &= \Gamma_1 \overline{\gamma(\lambda)} g = \Gamma_1 (A_0 - \mu)(A_0 - \lambda)^{-1} \overline{\gamma(\mu)} g \\ &= \Gamma_1 (\Phi_3 \Psi_3)(A_0) \overline{\gamma(\mu)} g = \Gamma_1 \overline{\Phi_3(A_0) \Psi_3(A_0)}\, \overline{\gamma(\mu)} g \\ &= \Gamma_1 \Phi_3(A_0) \Psi_3(A_0) \overline{\gamma(\mu)} g \\ &= \Gamma_1 \Phi_4(A_0) \Psi_4(A_0) \Psi_3(A_0) \overline{\gamma(\mu)} g \\ &= \bigl[\overline{\Gamma_1 |A_0 - \mu|^{-\alpha}} \bigr] \Psi_4(A_0) \bigl[|A_0 - \mu|^{1-\alpha} \overline{\gamma(\mu)} \bigr] g. \end{aligned} \tag{6.9}$$

According to (6.7) and (6.8) the terms in the square brackets are bounded and everywhere defined operators, which are independent of λ. Since Ψ_4 is bounded on $\sigma(A_0)$, it follows that $M(\lambda)$ is a bounded, densely defined operator, and assertion (a) is proved.

Relations (6.9) and (6.8) imply that

$$\bigl\| \overline{M(\lambda)} \bigr\| \leq \bigl\| \overline{\Gamma_1 |A_0 - \mu|^{-\alpha}} \bigr\|^2 \| \Psi_4(A_0) \|.$$

Assertions (b) and (c) follow from suitable estimates of $\|\Psi_4(A_0)\|$. Let E be the spectral measure for the operator A_0. For all $\lambda \in \rho(A_0)$ and all $f \in \mathcal{H}$ we have

$$\begin{aligned} \|\Psi_4(A_0) f\|^2 &= \int_{\sigma(A_0)} \frac{|t-\mu|^{4\alpha}}{|t-\lambda|^2}\, \mathrm{d}\bigl(E(t)f, f\bigr) \\ &= \int_{\sigma(A_0)} \frac{|t-\mu|^{4\alpha}}{|t-\lambda|^{4\alpha}} \cdot \frac{1}{|t-\lambda|^{2-4\alpha}}\, \mathrm{d}\bigl(E(t)f, f\bigr) \\ &\leq \frac{1}{\bigl(\mathrm{dist}(\lambda, \sigma(A_0))\bigr)^{2-4\alpha}} \int_{\sigma(A_0)} \frac{|t-\mu|^{4\alpha}}{|t-\lambda|^{4\alpha}}\, \mathrm{d}\bigl(E(t)f, f\bigr). \end{aligned} \tag{6.10}$$

In order to prove (b), fix $z_0 \in \overline{\mathbb{C}^+} \cap \rho(A_0)$ and $\theta \in (0, \pi/2)$. It remains to estimate the integrand of the last integral in (6.10) uniformly in $\lambda \in \mathbb{S}_{z_0,\theta}$ and $t \in \sigma(A_0)$. To this end set $d_{z_0,\theta} := \mathrm{dist}(\mathbb{S}_{z_0,\theta}, \sigma(A_0)) > 0$. Let $\lambda \in \mathbb{S}_{z_0,\theta}$, i.e.

$$\operatorname{Im} \lambda \geq \operatorname{Im} z_0 \qquad \text{and} \qquad |\operatorname{Re}(\lambda - z_0)| \leq \tan\theta \cdot \operatorname{Im}(\lambda - z_0).$$

If $\lambda \neq z_0$, then

$$\begin{aligned} \frac{|t-\mu|^2}{|t-\lambda|^2} &= \frac{(t - \operatorname{Re}\mu)^2 + (\operatorname{Im}\mu)^2}{|t-\lambda|^2} \\ &\leq \frac{3\bigl[(t-\operatorname{Re}\lambda)^2 + (\operatorname{Re}\lambda - \operatorname{Re} z_0)^2 + (\operatorname{Re} z_0 - \operatorname{Re}\mu)^2\bigr] + (\operatorname{Im}\mu)^2}{|t-\lambda|^2} \end{aligned}$$

$$
\begin{aligned}
&\le 3 + 3\frac{\big(\mathrm{Re}(\lambda - z_0)\big)^2}{\big(\mathrm{Im}(\lambda - z_0)\big)^2} + \frac{3\big(\mathrm{Re}(z_0 - \mu)\big)^2 + (\mathrm{Im}\,\mu)^2}{d_{z_0,\theta}^2} \\
&\le 3 + 3\tan^2\theta + \frac{3\big(\mathrm{Re}(z_0 - \mu)\big)^2 + (\mathrm{Im}\,\mu)^2}{d_{z_0,\theta}^2},
\end{aligned}
$$

where the right-hand side is independent of λ and t; by continuity this estimate extends to $\lambda = z_0$. The case $\lambda \in \mathbb{S}_{z_0,\theta}^*$ can be treated analogously. From this, together with (6.9) and (6.10), the claim of (b) follows.

To prove (c), let $w_0 < \min \sigma(A_0)$ and $\nu \in (0, \pi)$; note that $\mathrm{dist}(\mathbb{U}_{w_0,\nu}, \sigma(A_0)) > 0$. Let first $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < w_0$. Then with $m := \min \sigma(A_0)$ the integrand of the last integral in (6.10) can be estimated using

$$
\begin{aligned}
\frac{|t-\mu|^2}{|t-\lambda|^2} &\le \frac{3\big[(t - \mathrm{Re}\,\lambda)^2 + (\mathrm{Re}\,\lambda - m)^2 + (m - \mathrm{Re}\,\mu)^2\big] + (\mathrm{Im}\,\mu)^2}{(t - \mathrm{Re}\,\lambda)^2 + (\mathrm{Im}\,\lambda)^2} \\
&\le 3 + 3 + \frac{3(m - \mathrm{Re}\,\mu)^2 + (\mathrm{Im}\,\mu)^2}{(m - w_0)^2},
\end{aligned}
$$

where we have used $t - \mathrm{Re}\,\lambda \ge m - \mathrm{Re}\,\lambda \ge m - w_0 > 0$. If $\nu \ge \pi/2$, this and (6.10) lead to a uniform estimate of $\Psi_4(A_0)$ in $\mathbb{U}_{w_0,\nu}$. If $\nu \in (0, \pi/2)$, then

$$
\mathbb{U}_{w_0,\nu} = \{z \in \mathbb{C} : \mathrm{Re}\,z < w_0\} \cup \mathbb{S}_{w_0,\theta,} \cup \mathbb{S}_{w_0,\theta,}^*
$$

with $\theta = \pi/2 - \nu$, and a uniform estimate of the last integral in (6.10) for $\lambda \in \mathbb{U}_{w_0,\nu}$ follows from the previous consideration and item (b). The proof is complete. $\square$

Remark 6.2. Suppose that the assumptions of Theorem 6.1 are satisfied for $\alpha = \frac{1}{2}$. It follows from Theorem 6.1 that $M(\lambda)$ is bounded for every $\lambda \in \rho(A_0)$ and that $\|\overline{M(\lambda)}\|$ is uniformly bounded on each sector $\mathbb{S}_{z_0,\theta}$ as in the theorem. In addition, we can show (see below) that for each $\mathbb{S}_{z_0,\theta}$ as in the theorem,

$$
\overline{M(\lambda)}g \to 0 \qquad \text{as } \lambda \to \infty \text{ in } \mathbb{S}_{z_0,\theta}, \quad g \in \mathcal{G}. \tag{6.11}
$$

Similarly, if A_0 is bounded from below, then $M(\lambda)$ is bounded for every $\lambda \in \rho(A_0)$ and $\|\overline{M(\lambda)}\|$ is uniformly bounded on each sector $\mathbb{U}_{w_0,\nu}$ as in the theorem, and for each such $\mathbb{U}_{w_0,\nu}$,

$$
\overline{M(\lambda)}g \to 0 \qquad \text{as } \lambda \to \infty \text{ in } \mathbb{U}_{w_0,\nu}, \quad g \in \mathcal{G}. \tag{6.12}
$$

To prove (6.11) set

$$
f := |A_0 - \mu|^{1/2}\overline{\gamma(\mu)}g
$$

and observe that by (6.9) it is sufficient to show that

$$\|\Psi_4(A_0)f\|^2 = \int\limits_{\sigma(A_0)} \frac{|t-\mu|^2}{|t-\lambda|^2}\,\mathrm{d}\big(E(t)f,f\big) \to 0 \qquad \text{as } \lambda\to\infty \text{ in } \mathbb{S}_{z_0,\theta}.$$

It was shown in the proof of Theorem 6.1 that the integrand is uniformly bounded for $\lambda \in \mathbb{S}_{z_0,\theta}$ and $t \in \sigma(A_0)$. Moreover, the measure $(E(\cdot)f,f)$ is finite and the integrand converges to 0 as $\lambda \to \infty$ for each fixed $t \in \sigma(A_0)$. Hence the dominated convergence theorem implies that $\|\Psi_4(A_0)f\| \to 0$ as $\lambda \to \infty$ in $\mathbb{S}_{z_0,\theta}$, which proves (6.11). The same argument also shows (6.12).

Corollary 6.3. *Let $\Pi = \{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M and assume that the operator in (6.4) is bounded for some $\mu \in \rho(A_0)$ and some $\alpha \in \big(0, \frac{1}{2}\big]$. Then the following assertions hold.*

(a) *For all $z_0 \in \overline{\mathbb{C}^+} \cap \rho(A_0)$ and all $\theta \in (0, \frac{\pi}{2})$ there exist $C_1 = C_1(\Pi, \alpha, \mu, z_0, \theta)$ and $C_2 = C_2(\Pi, \alpha, \mu, z_0, \theta)$ such that*

$$\big\|\overline{\gamma(\lambda)}\big\| \le \frac{C_1}{\big(\mathrm{dist}(\lambda, \sigma(A_0))\big)^{1-\alpha}}, \tag{6.13}$$

$$\big\|\overline{M^{(n)}}(\lambda)\big\| \le \frac{C_2\, n!}{\big(\mathrm{dist}(\lambda, \sigma(A_0))\big)^{n+1-2\alpha}} \tag{6.14}$$

for all $\lambda \in \mathbb{S}_{z_0,\theta} \cup \mathbb{S}^*_{z_0,\theta}$.

(b) *If A_0 is bounded from below, then for all $w_0 < \min \sigma(A_0)$ and all $\nu \in (0, \pi)$ there exist $D_1 = D_1(\Pi, \alpha, \mu, w_0, \nu)$ and $D_2 = D_2(\Pi, \alpha, \mu, w_0, \nu)$ such that*

$$\big\|\overline{\gamma(\lambda)}\big\| \le \frac{D_1}{\big(\mathrm{dist}(\lambda, \sigma(A_0))\big)^{1-\alpha}}, \tag{6.15}$$

$$\big\|\overline{M^{(n)}}(\lambda)\big\| \le \frac{D_2\, n!}{\big(\mathrm{dist}(\lambda, \sigma(A_0))\big)^{n+1-2\alpha}} \tag{6.16}$$

for all $\lambda \in \mathbb{U}_{w_0,\nu}$.

Proof. (a) First we prove (6.13). Let $z_0 \in \overline{\mathbb{C}^+} \cap \rho(A_0)$ and $\theta \in \big(0, \frac{\pi}{2}\big)$. For $\lambda \in \mathbb{S}_{z_0,\theta}$ with $\operatorname{Im}\lambda \ge 1$ we have

$$\begin{aligned}
\mathrm{dist}(\lambda, \sigma(A_0)) &\le |\lambda - z_0| + \mathrm{dist}\big(z_0, \sigma(A_0)\big) \\
&\le \frac{\operatorname{Im}\lambda}{\cos\theta} + \mathrm{dist}(z_0, \sigma(A_0)) \\
&\le \left(\frac{1}{\cos\theta} + \mathrm{dist}(z_0, \sigma(A_0))\right) \operatorname{Im}\lambda.
\end{aligned}$$

This, (2.6), (2.11) and (6.5) imply that

$$\|\overline{\gamma(\lambda)}\| = \frac{\|\overline{\operatorname{Im} M(\lambda)}\|^{1/2}}{(\operatorname{Im}\lambda)^{1/2}} \le \frac{\|\overline{M(\lambda)}\|^{1/2}}{(\operatorname{Im}\lambda)^{1/2}}$$

$$\le \frac{C^{1/2}\big(\operatorname{dist}(\lambda,\sigma(A_0))\big)^{1/2}}{(\operatorname{Im}\lambda)^{1/2}\big(\operatorname{dist}(\lambda,\sigma(A_0))\big)^{1-\alpha}} \le \frac{C^{1/2}\big[\frac{1}{\cos\theta}+\operatorname{dist}(z_0,\sigma(A_0))\big]^{1/2}}{\big(\operatorname{dist}(\lambda,\sigma(A_0))\big)^{1-\alpha}}$$

for $\lambda \in \mathbb{S}_{z_0,\theta}$ with $\operatorname{Im}\lambda \ge 1$. Since $\|\overline{\gamma(\overline{\lambda})}\| = \|\overline{\gamma(\lambda)}\|$, see (2.6), and γ is bounded on the set $\{z \in \mathbb{S}_{z_0,\theta} \cup \mathbb{S}^*_{z_0,\theta} : |\operatorname{Im} z| \le 1\}$, the inequality (6.13) is proved.

The inequality in (6.14) is obtained from (6.13) and (2.7) as follows:

$$\|\overline{M^{(n)}}(\lambda)\| \le n!\|\gamma(\overline{\lambda})^*\|\,\|(A_0-\lambda)^{-(n-1)}\|\,\|\overline{\gamma(\lambda)}\|$$

$$\le \frac{n!\,C_1^2}{\big(\operatorname{dist}(\lambda,\sigma(A_0))\big)^{1-\alpha+n-1+1-\alpha}}.$$

(b) Now assume that A_0 is bounded from below and set $m := \min\sigma(A_0)$. Let $w_0 < m$ and, without loss of generality, $\nu \in \big(0,\frac{\pi}{2}\big)$. Let $x \in \mathcal{G}$ and $u \in \mathcal{H}$ and define the function

$$f(z) := (m-z)^{1-\alpha}\big(\overline{\gamma(z)}x,u\big), \qquad z \in \mathbb{C} \;\text{ with }\; \operatorname{Re} z \le w_0,$$

where the function $\zeta \mapsto \zeta^{1-\alpha}$ is defined with a cut on the negative half-line. The already proved item (a) implies that (6.13) is valid for $z \in \mathbb{S}_{w_0,\theta}$ with $\theta := \frac{\pi}{2}-\nu$ and some $D_1 > 0$. In particular, it is true for $z \in \mathbb{C}$ with $\operatorname{Re} z = w_0$, which yields that

$$|f(z)| \le |m-z|^{1-\alpha}\|\overline{\gamma(z)}\|\,\|x\|\,\|u\| \le |m-z|^{1-\alpha}\frac{D_1\|x\|\,\|u\|}{\big(\operatorname{dist}(z,\sigma(A_0))\big)^{1-\alpha}} = D_1\|x\|\,\|u\|$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z = w_0$. Since by (2.4) the function f grows at most like a power of z on the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z \le w_0\}$, the Phragmén–Lindelöf principle (see, e.g. [47, Corollary VI.4.2]) implies that

$$|f(z)| \le D_1\|x\|\,\|u\| \qquad \text{for all }\; z \in \mathbb{C} \;\text{ with }\; \operatorname{Re} z \le w_0.$$

It follows from this that

$$\|\overline{\gamma(z)}\| \le \frac{D_1}{|m-z|^{1-\alpha}} \qquad \text{for all }\; z \in \mathbb{C} \;\text{ with }\; \operatorname{Re} z \le w_0.$$

If we combine this with (6.13) with $z_0 = w_0$ and $\theta = \frac{\pi}{2}-\nu$, we obtain (6.15). The estimate (6.16) follows from (6.15) in the same way as in (a). □

The following example shows that Theorem 6.1 is sharp in a certain sense.

Example 6.4. Let $\alpha \in \left(0, \frac{1}{2}\right]$ and let μ be the Borel measure on $\mathbb{R}$ that has support $[e, \infty)$, is absolutely continuous and has density

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = \frac{1}{t^{1-2\alpha}(\ln t)^2}, \qquad t \in [e, \infty).$$

Moreover, define

$$M(\lambda) := \int_e^\infty \frac{1}{t-\lambda}\,\mathrm{d}\mu(t), \qquad \lambda \in \mathbb{C} \setminus [e, \infty).$$

This function is the Weyl function of the following ordinary boundary triple

$$\begin{aligned}
&\mathcal{H} = L^2(\mu), \quad \mathcal{G} = \mathbb{C},\\
&\operatorname{dom} T = \left\{ f \in \mathcal{H} : \exists\, c_f \in \mathbb{C} \text{ such that } tf(t) - c_f \in \mathcal{H} \right\},\\
&(Tf)(t) = tf(t) - c_f,\\
&\Gamma_0 f = c_f, \qquad \Gamma_1 f = \int_e^\infty f(t)\,\mathrm{d}\mu(t);
\end{aligned}$$

note that c_f is uniquely determined by f since the measure μ is infinite. The operator A_0 is the multiplication operator by the independent variable. The mapping in (6.4) with $\mu = 0$ is bounded since for $f \in \mathcal{H}$ with compact support we have

$$\Gamma_1 A_0^{-\alpha} f = \int_e^\infty f(t)\, t^{-\alpha} \mathrm{d}\mu(t) \le \|f\|_{\mathcal{H}} \left[\int_e^\infty \frac{1}{t^{2\alpha} t^{1-2\alpha} (\ln t)^2}\,\mathrm{d}t \right]^{1/2}$$

and the last integral converges. Hence Theorem 6.1 yields that

$$M(\lambda) = \mathrm{O}\left(\frac{1}{|\lambda|^{1-2\alpha}} \right), \qquad \lambda \to -\infty.$$

One can show that the actual asymptotic behaviour of M is

$$M(\lambda) \sim \frac{C}{|\lambda|^{1-2\alpha} (\ln |\lambda|)^2}, \qquad \lambda \to -\infty,$$

with a positive constant C.

Hence, apart from the logarithmic factor, Theorem 6.1 yields the correct asymptotic behaviour. Using Krein's inverse spectral theorem (see, e.g. [92]) one can rewrite this example as a Krein–Feller operator: $-D_m D_x$ with some mass distribution m so that the measure μ becomes the principal spectral measure of the string.

The next corollary is an immediate consequence of Theorem 6.1.

Corollary 6.5. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M and assume that the operator in (6.4) is bounded for some $\alpha \in \big(0, \frac{1}{2}\big]$ and some $\mu \in \rho(A_0)$. Then M satisfies*

$$\int_1^\infty \frac{\big\|\operatorname{Im} \overline{M(iy)}\big\|}{y^\gamma} \mathrm{d}y < \infty \tag{6.17}$$

for every $\gamma > 2\alpha$.

Condition (6.17) says that the function M belongs to the *Kac class* $\mathbf{N}_\gamma$ (see, e.g. [93] for the scalar case). Assume that M satisfies (6.17) for some $\gamma \in (0, 2)$ and consider the integral representation

$$M(\lambda) = A + \lambda B + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) \mathrm{d}\Sigma(t),$$

where A and $B \geq 0$ are bounded symmetric operators and Σ is an operator-valued measure (see, e.g. [116] or [23, §3.4]). Often the measure Σ plays the role of a spectral measure. For each $\varphi \in \operatorname{ran} \Gamma_0$ we have

$$(M(\lambda)\varphi, \varphi) = (A\varphi, \varphi) + \lambda(B\varphi, \varphi) + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) \mathrm{d}\big(\Sigma(t)\varphi, \varphi\big).$$

It follows from [130, Lemma 3.1] and its proof that $(B\varphi, \varphi) = 0$ and that

$$\int_{\mathbb{R}} \frac{1}{1 + |t|^\gamma} \, \mathrm{d}\big(\Sigma(t)\varphi, \varphi\big) \leq C\|\varphi\|^2,$$

with some $C > 0$, which does not depend on φ. Hence $B = 0$ and

$$\int_{\mathbb{R}} \frac{1}{1 + |t|^\gamma} \, \mathrm{d}\Sigma(t)$$

is a bounded operator.

7. Elliptic operators with non-local Robin boundary conditions

In this section we apply the results of the previous sections to elliptic differential operators on domains whose boundaries are not necessarily compact. Our main focus is on operators subject to non-self-adjoint boundary conditions. For some recent investigations of non-self-adjoint elliptic operators we refer the reader to [40,41,76,86,115].

Let $\Omega \subset \mathbb{R}^n$ be a domain that is *uniformly regular*[1] in the sense of [38, p. 366] and [74, page 72]; see also [20,39]. This includes, e.g. domains with compact C^∞-smooth boundaries or compact, smooth perturbations of half-spaces. Moreover, the class of uniformly regular unbounded domains includes certain quasi-conical and quasi-cylindrical domains in the sense of [57, Definition X.6.1]. Non-self-adjoint elliptic operators with Robin boundary conditions on such domains have been investigated recently in connection with non-Hermitian quantum waveguides and layers; see, e.g. [34–36,113]. Further, let

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + a \tag{7.1}$$

be a differential expression on Ω, where we assume that $a_{jk} \in C^\infty(\overline{\Omega})$ are bounded, have bounded, uniformly continuous derivatives on $\overline{\Omega}$ and satisfy $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \overline{\Omega}$, $1 \le j,k \le n$, and that $a \in L^\infty(\Omega)$ is real-valued; cf. [20, (S1)–(S5) in Chapter 4]. Moreover, we assume that $\mathcal{L}$ is uniformly elliptic, i.e. there exists $E > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \ge E \sum_{k=1}^{n} \xi_k^2, \qquad \xi = (\xi_1, \dots \xi_n)^\top \in \mathbb{R}^n, x \in \overline{\Omega}.$$

In the following we denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$ the Sobolev spaces of order $s \ge 0$ on Ω and $\partial\Omega$, respectively. For $f \in C_0^\infty(\overline{\Omega})$, where $C_0^\infty(\overline{\Omega})$ denotes the set of $C^\infty(\overline{\Omega})$-functions with compact support, let

$$\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} := \sum_{j,k=1}^{n} a_{jk}\nu_j \frac{\partial f}{\partial x_k}\Big|_{\partial\Omega}$$

denote the conormal derivative of f at $\partial\Omega$ with respect to $\mathcal{L}$, where $\nu = (\nu_1, \dots, \nu_n)^\top$ is the unit normal vector field at $\partial\Omega$ pointing outwards. Then Green's identity

$$(\mathcal{L}f, g) - (f, \mathcal{L}g) = \left(f|_{\partial\Omega}, \frac{\partial g}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right) - \left(\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, g|_{\partial\Omega} \right) \tag{7.2}$$

holds for all $f, g \in C_0^\infty(\overline{\Omega})$, where the inner products are in $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. Recall that the pair of mappings

[1] This means that $\partial\Omega$ is C^∞-smooth and that there exists a covering of $\partial\Omega$ by open sets Ω_j, $j \in \mathbb{N}$, and $n_0 \in \mathbb{N}$ such that at most n_0 of the Ω_j have a non-empty intersection, and a family of C^∞-homeomorphisms

$$\varphi_j : \Omega_j \cap \Omega \to \mathcal{B}_1 \cap \{x_n > 0\}, \qquad \text{where} \quad \mathcal{B}_r = \{x \in \mathbb{R}^n : \|x\| < r\},$$

such that $\varphi_j : \Omega_j \cap \partial\Omega \to \mathcal{B}_1 \cap \{x_n = 0\}$, the derivatives of φ_j, $j \in \mathbb{N}$, and their inverses are uniformly bounded, and $\bigcup_j \varphi_j^{-1}(\mathcal{B}_{1/2})$ covers a uniform neighbourhood of $\partial\Omega$.

$$C_0^\infty(\overline{\Omega}) \ni f \mapsto \left\{ f|_{\partial\Omega};\ \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \right\} \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

extends by continuity to a bounded map from $H^2(\Omega)$ onto $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$; see, e.g. [74, Theorem 3.9]. The extended trace and conormal derivative are again denoted by $f|_{\partial\Omega}$ and $\frac{\partial f}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}$, respectively. Moreover, Green's identity (7.2) extends to all $f, g \in H^2(\Omega)$; see [74, Theorem 4.4].

In order to construct a quasi boundary triple, let us define the operators S and T in $L^2(\Omega)$ via

$$Sf = \mathcal{L}f, \qquad \operatorname{dom} S = \left\{ f \in H^2(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = 0 \right\}, \tag{7.3}$$

and

$$Tf = \mathcal{L}f, \qquad \operatorname{dom} T = H^2(\Omega). \tag{7.4}$$

Moreover, we define boundary mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to L^2(\partial\Omega)$ by

$$\Gamma_0 f = \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \qquad \Gamma_1 f = f|_{\partial\Omega} \qquad \text{for } f \in \operatorname{dom} T.$$

The assertions of the following proposition can be found in [29, Propositions 3.1 and 3.2].

Proposition 7.1. *The operator S in (7.3) is closed, symmetric and densely defined with $\overline{T} = S^*$ for T in (7.4), and the triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $T \subset S^*$ with the following properties.*

(i) $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top = H^{1/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$.

(ii) A_0 *is the* Neumann operator

$$A_{\mathrm{N}} f = \mathcal{L}f, \qquad \operatorname{dom} A_{\mathrm{N}} = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = 0 \right\},$$

and A_1 *is the* Dirichlet operator

$$A_{\mathrm{D}} f = \mathcal{L}f, \qquad \operatorname{dom} A_{\mathrm{D}} = \left\{ f \in H^2(\Omega) : f|_{\partial\Omega} = 0 \right\}.$$

Both operators, A_{N} and A_{D}, are self-adjoint and bounded from below.

(iii) *For $\lambda \in \rho(A_{\mathrm{N}})$, the associated γ-field satisfies*

$$\gamma(\lambda) \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = f \qquad \textit{for all } f \in \ker(T - \lambda), \tag{7.5}$$

and the associated Weyl function is given by the Neumann-to-Dirichlet map*,*

$$M(\lambda)\frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = f|_{\partial\Omega} \qquad \textit{for all } f \in \ker(T-\lambda). \tag{7.6}$$

Moreover, $M(\lambda)$ is a bounded, non-closed operator in $L^2(\partial\Omega)$ with domain $H^{1/2}(\partial\Omega)$ such that $\operatorname{ran}\overline{M(\lambda)} \subset H^1(\partial\Omega)$.

In order to apply the results of Section 5 to the quasi boundary triple in Proposition 7.1 we prove estimates for the Weyl function in certain sectors using Theorem 6.1.

Lemma 7.2. *Let $\mathbb{U}_{w_0,\nu}$ be defined as in (6.2). Then for each $w_0 < \min\sigma(A_{\mathrm{N}})$, $\nu \in (0,\pi)$ and $\beta \in (0,\frac{1}{2})$ there exists $C = C(\mathcal{L},\Omega,w_0,\nu,\beta) > 0$ such that*

$$\big\|\overline{M(\lambda)}\big\| \le \frac{C}{\big(\operatorname{dist}(\lambda,\sigma(A_{\mathrm{N}}))\big)^{\beta}} \qquad \textit{for all } \lambda \in \mathbb{U}_{w_0,\nu}. \tag{7.7}$$

Proof. Let $\mu = \min\sigma(A_{\mathrm{N}}) - 1$. Then $A_{\mathrm{N}} - \mu$ is a positive, self-adjoint operator in $L^2(\Omega)$ and $\Lambda := (A_{\mathrm{N}} - \mu)^{1/2}$ in $L^2(\Omega)$ is well defined, self-adjoint and positive. It can be seen with the help of the quadratic form associated with A_{N} that $\operatorname{dom}\Lambda = H^1(\Omega)$ and that the $H^1(\Omega)$-norm is equivalent to the graph norm $\|\Lambda\cdot\|_{L^2(\Omega)}$. Thus the identity operator provides an isomorphism between $H^1(\Omega)$ and $(\operatorname{dom}\Lambda, \|\Lambda\cdot\|_{L^2(\Omega)})$ as well as, trivially, between $L^2(\Omega)$ and $(\operatorname{dom}\Lambda^0, \|\Lambda^0\cdot\|_{L^2(\Omega)})$. By interpolation (see, e.g. [110, Theorems 5.1 and 7.7]), the identity operator is also an isomorphism between $H^s(\Omega)$ and $(\operatorname{dom}\Lambda^s, \|\Lambda^s\cdot\|_{L^2(\Omega)})$ for each $s \in (0,1)$. In particular, $\operatorname{dom}(A_{\mathrm{N}}-\mu)^{s/2} = \operatorname{dom}\Lambda^s = H^s(\Omega)$ for each $s \in (0,1)$. It follows from the closed graph theorem that $(A_{\mathrm{N}}-\mu)^{-s/2}$ is bounded as an operator from $L^2(\Omega)$ to $H^s(\Omega)$ for each such s. Since the trace map is bounded from $H^s(\Omega)$ to $L^2(\partial\Omega)$ for each $s \in (\frac{1}{2},1)$ by [74, Theorem 3.7], it follows that $f \mapsto ((A_{\mathrm{N}}-\mu)^{-s/2}f)|_{\partial\Omega}$ is bounded from $L^2(\Omega)$ to $L^2(\partial\Omega)$ for each $s \in (\frac{1}{2},1)$. In particular, the operator

$$\Gamma_1(A_{\mathrm{N}}-\mu)^{-\alpha} : L^2(\Omega) \supset \operatorname{dom}\big(\Gamma_1(A_{\mathrm{N}}-\mu)^{-\alpha}\big) \to L^2(\partial\Omega) \tag{7.8}$$

is bounded for each $\alpha \in (\frac{1}{4},\frac{1}{2})$. By Theorem 6.1 for each $w_0 < \min\sigma(A_{\mathrm{N}})$, each $\nu \in (0,\pi)$ and each $\alpha \in (\frac{1}{4},\frac{1}{2})$ there exists $C = C(\mathcal{L},\Omega,w_0,\nu,\alpha) > 0$ such that

$$\big\|\overline{M(\lambda)}\big\| \le \frac{C}{\big(\operatorname{dist}(\lambda,\sigma(A_{\mathrm{N}}))\big)^{1-2\alpha}}$$

holds for all $\lambda \in \mathbb{U}_{w_0,\nu}$. From this the claim of the lemma follows. □

Remark 7.3. Along the negative real axis the result of Lemma 7.2 can be slightly improved. It was proved in [29, Proposition 3.2 (iv)] (using techniques from [4]) that for each $\mu < \min\sigma(A_{\mathrm{N}})$ there exists $C = C(\mathcal{L},\Omega,\mu)$ such that

$$\big\|\overline{M(\lambda)}\big\| \le \frac{C}{(\mu-\lambda)^{1/2}} \qquad \text{for all } \lambda < \mu. \tag{7.9}$$

In the next theorem we apply Lemma 7.2, Remark 7.3 and the results from Section 5 to obtain m-sectorial (self-adjoint, maximal dissipative, maximal accumulative) realizations of $\mathcal{L}$ subject to generalized Robin boundary conditions and also spectral enclosures for these realizations.

Theorem 7.4. *Let B be a closable operator in $L^2(\partial\Omega)$ such that*

$$H^{1/2}(\partial\Omega) \subset \operatorname{dom} B \qquad \textit{and} \qquad B\big(H^1(\partial\Omega)\big) \subset H^{1/2}(\partial\Omega). \tag{7.10}$$

Assume further that there exists $b \in \mathbb{R}$ such that

$$\operatorname{Re}(B\varphi, \varphi)_{L^2(\partial\Omega)} \le b\|\varphi\|^2_{L^2(\partial\Omega)} \qquad \textit{for all } \varphi \in \operatorname{dom} B. \tag{7.11}$$

Then the operator

$$A_{[B]}f = \mathcal{L}f, \qquad \operatorname{dom} A_{[B]} = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = Bf|_{\partial\Omega} \right\}, \tag{7.12}$$

in $L^2(\Omega)$ is m-sectorial, one has $\sigma(A_{[B]}) \subset \overline{W(A_{[B]})}$, the resolvent formula

$$(A_{[B]} - \lambda)^{-1} = (A_{\mathrm{N}} - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\gamma(\overline{\lambda})^* \tag{7.13}$$

holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_{\mathrm{N}})$, and the following assertions are true.

(i) *If B is symmetric, then $A_{[B]}$ is self-adjoint and bounded from below. If B is dissipative (accumulative, respectively), then $A_{[B]}$ is maximal accumulative (maximal dissipative, respectively).*
(ii) *If B' is a closable operator in $L^2(\partial\Omega)$ that satisfies (7.10) and (7.11) with B replaced by B' and*

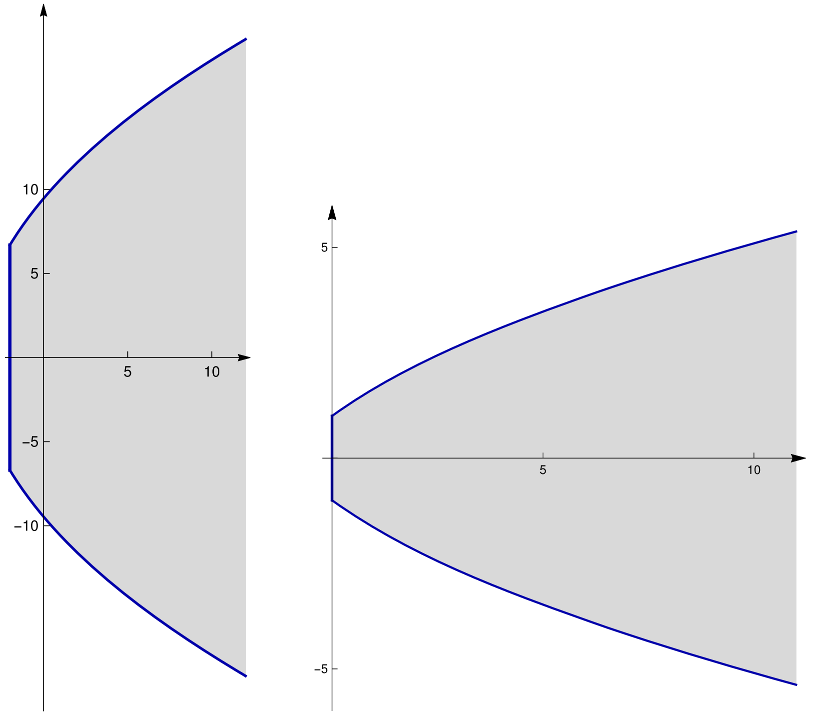
$$(B\varphi, \psi) = (\varphi, B'\psi) \qquad \textit{for all } \varphi \in \operatorname{dom} B,\ \psi \in \operatorname{dom} B' \tag{7.14}$$

holds, then $A_{[B']} = A^*_{[B]}$.

Moreover, the following spectral enclosures hold.

(iii) *If $b \le 0$, then $(-\infty, \min \sigma(A_{\mathrm{N}})) \subset \rho(A_{[B]})$.*
(iv) *If $\operatorname{dom} B^* \supset \operatorname{dom} B$, $\operatorname{Im} B$ is bounded and $b > 0$, then for each $\mu < \min \sigma(A_{\mathrm{N}})$ there exists $C > 0$ such that for each $\xi < \mu - (Cb)^2$ one has (see Fig. 3 (a))*

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \ge \mu - (Cb)^2,\ |\operatorname{Im} z| \le \frac{2C\big\|\overline{\operatorname{Im} B}\big\|}{1 - \frac{Cb}{(\mu-\xi)^{1/2}}} (\operatorname{Re} z - \xi)^{1/2} \right\}.$$



(a) $b = 1$ (b) $b = -1$

Fig. 3. The plots show the regions given in Theorem 7.4 (iv), (v), respectively, that contain $W(A_{[B]})$ for (a) $b > 0$ and (b) $b < 0$; it is assumed that $\min \sigma(A_\mathrm{N}) = 0$, $C\|\overline{\operatorname{Im} B}\| = 1$, $\mu = -1$ for both cases and $\xi = -4$ in (a).

(v) *If* $\operatorname{dom} B^* \supset \operatorname{dom} B$, $\operatorname{Im} B$ *is bounded and* $b \leq 0$*, then for each* $\mu < \min \sigma(A_\mathrm{N})$ *there exists* $C > 0$ *such that (see Fig. 3 (b))*

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq \min \sigma(A_\mathrm{N}),\ |\operatorname{Im} z| \leq \frac{2C\left\|\overline{\operatorname{Im} B}\right\|(\operatorname{Re} z - \mu)}{(\operatorname{Re} z - \mu)^{1/2} - Cb} \right\}.$$

(vi) *If* B *is bounded, then for each* $w_0 < \min \sigma(A_\mathrm{N})$*,* $\nu \in (0, \pi)$ *and* $\beta \in \left(0, \frac{1}{2}\right)$ *there exists* $C > 0$ *such that*

$$\sigma(A_{[B]}) \cap \mathbb{U}_{w_0,\nu} \subset \left\{ z \in \mathbb{U}_{w_0,\nu} : \operatorname{dist}(z, \sigma(A_\mathrm{N})) \leq (C\|B\|)^{1/\beta} \right\},$$

where $\mathbb{U}_{w_0,\nu}$ *is defined in* (6.2)*.*

Proof. Let B be a closable operator in $L^2(\partial\Omega)$ that satisfies (7.10) and (7.11) for some $b \in \mathbb{R}$. Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple in Proposition 7.1. It follows from Lemma 7.2 that (5.1) is valid for the corresponding Weyl function. The assumptions

(i) and (iv) and the second assumption in (v) of Theorem 5.1 are satisfied due to the assumptions of the present theorem and the fact that $A_{\mathrm{D}} = A_1$ is self-adjoint and bounded from below by Proposition 7.1. Assumption (iii) of Theorem 5.1 follows from the last assertion of Proposition 7.1 (iii) and (7.10). For assumption (ii) of Theorem 5.1 note that

$$\operatorname{ran} \overline{M(\lambda)}^{1/2} = H^{1/2}(\partial\Omega), \qquad \lambda < \min \sigma(A_{\mathrm{N}}),$$

which can be verified as in the proof of [29, Proposition 3.2 (iii)], and use (7.10). It follows from Proposition 7.1 that A_0 and A_1 are bounded from below. Thus Theorem 5.1 and Corollary 5.4 imply assertions (i)–(iii). Moreover, Theorem 5.6 and (7.9) yield that $A_{[B]}$ is m-sectorial and the assertions in items (iv) and (v); note that the estimate for $\operatorname{Re} z$ in (v) follows from taking the estimates $\operatorname{Re} z > \mu$ in Theorem 5.6 (b), (c) for all $\mu < \min \sigma(A_{\mathrm{N}})$. Finally, to prove item (vi) one combines Lemma 7.2 and Proposition 5.9 (a) with $G = \mathbb{U}_{w_0,\nu}$. □

Remark 7.5.

(i) The constants C in items (iv)–(vi) of the above theorem depend only on the differential expression $\mathcal{L}$ and the domain Ω and on μ in (iv), (v) and on w_0, ν, β in (vi); the constants are independent of the operator B.

(ii) In many cases (e.g. when Ω is bounded), one can define T in (7.4) on the larger domain

$$H^{3/2}_{\mathcal{L}}(\Omega) := \{f \in H^{3/2}(\Omega) : \mathcal{L}f \in L^2(\Omega)\};$$

see [22, §4.2]. In this case the extensions of the boundary mappings Γ_0 and Γ_1 to $H^{3/2}_{\mathcal{L}}(\Omega)$ give rise to a generalized boundary triple, and the second condition in (7.10) on B is not needed to guarantee that the assertions of Theorem 7.4 are true for the operator

$$A_{[B]}f = \mathcal{L}f, \qquad \operatorname{dom} A_{[B]} = \left\{ f \in H^{3/2}_{\mathcal{L}}(\Omega) : \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = Bf|_{\partial\Omega} \right\},$$

instead of (7.12). In particular, for every bounded operator B the statements (i)–(vi) in Theorem 7.4 are true. The second condition in (7.10) is needed to obtain the extra regularity $\operatorname{dom} A_{[B]} \subset H^2(\Omega)$; see also [1, Theorem 7.2] for a related result.

(iii) The assertions in (iv) and (v) of Theorem 7.4 imply that the spectrum of $A_{[B]}$ is contained in a parabola if $\operatorname{dom} B^* \supset \operatorname{dom} B$ and $\operatorname{Im} B$ is bounded. This is in accordance with [19, Theorem 5.14], where the Laplacian on a bounded domain with bounded B was studied. In that paper a setting with $H^{3/2}_{\mathcal{L}}(\Omega)$ as mentioned in the previous item of this remark was used.

(iv) Under the basic assumptions of Theorem 7.4 the operator $A_{[B]}$ is m-sectorial and hence $-A_{[B]}$ generates an analytic semigroup. For the Laplacian on a bounded domain Ω this was proved in [3] in the $H^{3/2}_{\mathcal{L}}(\Omega)$ setting as in (ii).

The next remark shows that the condition (7.10) can be relaxed when an adjoint pair of boundary operators that map $H^1(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$ is given. In this case the assumption $H^{1/2}(\partial\Omega) \subset \operatorname{dom} B$ is not needed.

Remark 7.6. Assume that B_0 and B_0' are linear operators in $L^2(\partial\Omega)$ which satisfy

$$(B_0\varphi, \psi) = (\varphi, B_0'\psi) \qquad \text{for all } \ \varphi \in \operatorname{dom} B_0, \ \psi \in \operatorname{dom} B_0', \tag{7.15}$$

and

$$H^1(\partial\Omega) \subset \operatorname{dom} B_0, \qquad B_0\big(H^1(\partial\Omega)\big) \subset H^{1/2}(\partial\Omega), \tag{7.16}$$

$$H^1(\partial\Omega) \subset \operatorname{dom} B_0', \qquad B_0'\big(H^1(\partial\Omega)\big) \subset H^{1/2}(\partial\Omega). \tag{7.17}$$

Then B_0 and B_0' have closable extensions B and B', respectively, that satisfy (7.10) and (7.14). Indeed, it follows from (7.16) and (7.17) that B_0 and B_0' are densely defined. Hence (7.15) shows that B_0 and B_0' are closable. This and the second condition in (7.17) imply that $B_0' \upharpoonright H^1(\partial\Omega)$ is bounded from $H^1(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. A duality argument as, e.g. in [27, Lemma 4.4] shows that the Banach space adjoint of $B_0' \upharpoonright H^1(\partial\Omega)$, which we denote by $\widetilde{B}$, is an extension of B_0 and a bounded mapping from $H^{-1/2}(\partial\Omega)$ to $H^{-1}(\partial\Omega)$. Interpolation (see, e.g. [110, Theorems 5.1 and 7.7]) implies that $B := \widetilde{B} \upharpoonright H^{1/2}(\partial\Omega)$ is bounded from $H^{1/2}(\partial\Omega)$ to $L^2(\partial\Omega)$. Hence $H^{1/2}(\partial\Omega) \subset \operatorname{dom} B$ and (7.10) is satisfied. In a similar way one constructs an extension B' of B_0' that satisfies $H^{1/2}(\partial\Omega) \subset \operatorname{dom} B'$. The relation (7.14) is obtained by continuity. We emphasize that in this situation replacing B by B_0 in the definition of $A_{[B]}$ does not change the domain of the operator.

If, for B, we choose a multiplication operator by some function α, we obtain classical Robin boundary conditions. We formulate this situation in the following corollary, which follows from Theorem 7.4 and Remark 7.6 with B_0' being the multiplication operator by $\overline{\alpha}$.

Corollary 7.7. *Let α be a measurable complex-valued function on $\partial\Omega$ such that*

$$\alpha\varphi \in H^{1/2}(\partial\Omega) \qquad \textit{for all } \ \varphi \in H^1(\partial\Omega) \tag{7.18}$$

and that

$$b := \sup(\operatorname{Re} \alpha) < \infty. \tag{7.19}$$

Then the operator

$$A_{[\alpha]} f = \mathcal{L} f, \qquad \operatorname{dom} A_{[\alpha]} = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = \alpha f|_{\partial\Omega} \right\},$$

in $L^2(\Omega)$ is m-sectorial, one has $\sigma(A_{[\alpha]}) \subset \overline{W(A_{[\alpha]})}$, and the resolvent formula

$$(A_{[\alpha]} - \lambda)^{-1} = (A_{\mathrm{N}} - \lambda)^{-1} + \gamma(\lambda)\big(I - \alpha M(\lambda)\big)^{-1} \alpha \gamma(\overline{\lambda})^*$$

holds for all $\lambda \in \rho(A_{[\alpha]}) \cap \rho(A_{\mathrm{N}})$. Moreover, the following assertions are true.

(i) $A_{[\overline{\alpha}]} = A_{[\alpha]}^*$.
(ii) *If α is real-valued, then $A_{[\alpha]}$ is self-adjoint and bounded from below. If* $\operatorname{Im}(\alpha(x)) \geq 0$ *(≤ 0, respectively) for almost all $x \in \partial\Omega$, then $A_{[\alpha]}$ is maximal accumulative (maximal dissipative, respectively).*
(iii) *If $b \leq 0$ in (7.19), then $(-\infty, \min \sigma(A_{\mathrm{N}})) \subset \rho(A_{[\alpha]})$.*

Further, if $\operatorname{Im} \alpha$ *is bounded, then the enclosures for $W(A_{[\alpha]})$ in Theorem 7.4* (iv) *and* (v) *hold with $\|\overline{\operatorname{Im} B}\|$ replaced by* $\sup |\operatorname{Im} \alpha|$. *If α is bounded, then also the enclosure in Theorem 7.4* (vi) *holds with $\|B\|$ replaced by* $\sup |\alpha|$.

Remark 7.8. Condition (7.18) says that α is a multiplier from $H^1(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$, in the notation of [119] written as

$$\alpha \in M\big(H^1(\partial\Omega) \to H^{1/2}(\partial\Omega)\big).$$

In certain situations there exist characterizations or sufficient conditions for this property. For example let

$$\Omega = \mathbb{R}^n_+ = \big\{ x = (x', x_n)^\top : x' \in \mathbb{R}^{n-1},\, x_n > 0 \big\}.$$

Then $\partial\Omega = \mathbb{R}^{n-1}$. The set of multipliers can be characterized using capacities; see [119, Theorem 3.2.2]. For the case $n = 2$ there is a simpler characterization and for $n > 2$ there are simpler sufficient conditions. To this end, let us recall some notation. Let $H^{s,p}(\mathbb{R}^{n-1})$ denote the (fractional) Sobolev space (or Bessel potential space) defined as

$$H^{s,p}(\mathbb{R}^{n-1}) = \big\{ u \in \mathcal{S}'(\mathbb{R}^{n-1}) : \mathcal{F} M^s \mathcal{F}^{-1} u \in L^p(\mathbb{R}^{n-1}) \big\}$$

where $\mathcal{S}'(\mathbb{R}^{n-1})$ is the space of tempered distributions, $\mathcal{F}$ is the $(n-1)$-dimensional Fourier transform, and M is the operator of multiplication by $\sqrt{1 + |\xi|^2}$; see, e.g. [58, §2.2.2 (iii)] or [119, §3.1.1]. Further, let $\eta \in C_0^\infty(\mathbb{R}^{n-1})$ be such that $\eta(x) = 1$ on the unit ball, and set $\eta_z(x) := \eta(x - z)$ for $z \in \mathbb{R}^{n-1}$. Let

$$H^{s,p}_{\mathrm{loc,unif}}(\mathbb{R}^{n-1}) = \Big\{ u \in \mathcal{S}'(\mathbb{R}^{n-1}) : \sup_{z \in \mathbb{R}^{n-1}} \|\eta_z u\|_{H^{s,p}(\mathbb{R}^{n-1})} < \infty \Big\},$$

a space of functions being in $H^{s,p}$ only locally but in a uniform way; see [119, p. 34]. We also set $H^s_{\mathrm{loc,unif}}(\mathbb{R}^{n-1}) := H^{s,2}_{\mathrm{loc,unif}}(\mathbb{R}^{n-1})$. When $n = 2$, one obtains from [119, Theorem 3.2.5] that α satisfies (7.18) if and only if

$$\alpha \in H^{\frac{1}{2}}_{\mathrm{loc,unif}}(\mathbb{R}). \tag{7.20}$$

In the case $n > 2$ we can use [119, Theorem 3.3.1 (ii)] to provide sufficient conditions: α satisfies (7.18) if

$$\begin{aligned} &\alpha \in H^{\frac{1}{2},p}_{\mathrm{loc,unif}}(\mathbb{R}^{n-1}) \ \text{ for some } p \in (2,4) && \text{when } \ n = 3, \\ &\alpha \in H^{\frac{1}{2},n-1}_{\mathrm{loc,unif}}(\mathbb{R}^{n-1}) && \text{when } \ n > 3. \end{aligned} \tag{7.21}$$

The implication in the case $n = 3$ can be shown as follows: if $\alpha \in H^{\frac{1}{2},p}_{\mathrm{loc,unif}}(\mathbb{R}^{n-1})$ and $p \in (2,4)$, then $\alpha \in M\big(H^{\frac{2}{p}}(\mathbb{R}^2) \to H^{\frac{1}{2}}(\mathbb{R}^2)\big)$ by [119, Theorem 3.3.1 (ii)], and since $H^1(\mathbb{R}^2)$ is continuously embedded in $H^{\frac{2}{p}}(\mathbb{R}^2)$, we therefore have $\alpha \in M\big(H^1(\mathbb{R}^2) \to H^{\frac{1}{2}}(\mathbb{R}^2)\big)$.

If Ω is a domain with smooth compact boundary, then one can characterize multipliers using charts to reduce the situation to the half-space case, i.e. α satisfies (7.18) if and only if $\alpha \in H^{\frac{1}{2}}(\partial\Omega)$ when $n = 2$; when $n > 2$, α satisfies (7.18) if (7.21) holds with $H^{\frac{1}{2},p}_{\mathrm{loc,unif}}(\mathbb{R}^{n-1})$ replaced by $H^{\frac{1}{2},p}(\partial\Omega)$.

Example 7.9. An example of an unbounded function α that satisfies (7.20) is

$$\alpha(x_1) = -\log\Big(\log\Big(1 + \frac{1}{|x_1|}\Big)\Big), \qquad x_1 \in (-1,1),$$

smoothly connected, e.g. to the zero function outside $\mathbb{R}\setminus(-2,2)$ or to periodically shifted copies of this function. That α belongs to $H^{\frac{1}{2}}_{\mathrm{loc,unif}}(\mathbb{R})$ can be seen from the fact that it is the trace of a function $f \in H^1(\mathbb{R} \times (0,\infty))$ that satisfies

$$f(x_1,x_2) = -\log\left(\log\left(1 + \frac{1}{\sqrt{x_1^2 + x_2^2}}\right)\right), \qquad x_1 \in (-1,1),\ x_2 \in (0,1).$$

Note that such a function α also satisfies (7.19) and hence Corollary 7.7 can be applied.

Let us consider an example in which the spectral estimates of the previous theorem can be made more explicit.

Example 7.10. Let $\Omega = \mathbb{R}^n_+ = \{(x', x_n)^\top : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, so that $\partial\Omega = \mathbb{R}^{n-1}$, and consider the negative Laplacian $\mathcal{L} = -\Delta$. Then $\sigma(A_{\mathrm{N}}) = [0,\infty)$ and the Weyl function of the quasi boundary triple in Proposition 7.1 can be calculated explicitly,

$$\overline{M(\lambda)} = (-\Delta_{\mathbb{R}^{n-1}} - \lambda)^{-1/2}, \qquad \lambda \in \mathbb{C} \setminus [0,\infty); \tag{7.22}$$

see, e.g. [87, (9.65)]. Here $-\Delta_{\mathbb{R}^{n-1}}$ denotes the self-adjoint Laplacian in $L^2(\mathbb{R}^{n-1})$. From (7.22) we obtain

$$\|\overline{M(\lambda)}\| = \frac{1}{\sqrt{\operatorname{dist}(\lambda, \mathbb{R}_+)}}, \qquad \lambda \in \mathbb{C} \setminus [0, \infty). \tag{7.23}$$

In particular, the estimate (7.9) is satisfied with $\mu = 0$ and $C = 1$. Hence we can use Theorem 5.6 to obtain a better inclusion for the numerical range. Let B be a closable operator that satisfies (7.10) and (7.11) such that $\operatorname{dom} B^* \supset \operatorname{dom} B$ and $\operatorname{Im} B$ is bounded. If $b > 0$, then for every $\xi < -b^2$ one has

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq -b^2,\ |\operatorname{Im} z| \leq \frac{2\|\overline{\operatorname{Im} B}\|}{1 - \frac{b}{\sqrt{|\xi|}}} (\operatorname{Re} z - \xi)^{1/2} \right\}.$$

If $b \leq 0$, then

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z > 0,\ |\operatorname{Im} z| \leq \frac{2\|\overline{\operatorname{Im} B}\| \operatorname{Re} z}{(\operatorname{Re} z)^{1/2} - b} \right\} \cup \{0\}. \tag{7.24}$$

Note that $\sigma(A_{[B]}) \subset \overline{W(A_{[B]})}$. If B is bounded, then we can use Proposition 5.9 (a) with $G = \mathbb{C} \setminus [0, \infty)$ to obtain the spectral enclosure

$$\sigma(A_{[B]}) \subset \{ z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{R}_+) \leq \|B\|^2 \}. \tag{7.25}$$

In the case of the Robin boundary condition, i.e. when B is a multiplication operator with a complex-valued function α, an enclosure alternative to (7.25) can be found in [72, Theorem 2], where the operator norm is replaced by an L^p-norm of α with a suitably chosen $p > 0$. Finally, we remark that for $b \leq 0$ and z close to the origin, the enclosure (7.24) is sharper than (7.25).

If the boundary $\partial\Omega$ of Ω is compact, then the differences of the resolvents of $A_{[B]}$ and A_{N} or A_{D}, respectively, belong to certain Schatten–von Neumann ideals as the following theorem shows. For the case of a bounded self-adjoint operator B in $L^2(\partial\Omega)$ the inclusions in (7.28) and (7.29) were proved in [27, Theorem 4.10 and Corollary 4.14]; cf. also [25,88].

Theorem 7.11. *Let $\partial\Omega$ be compact and let all assumptions of Theorem 7.4 be satisfied. Then*

$$(A_{[B]} - \lambda)^{-1} - (A_{\mathrm{N}} - \lambda)^{-1} \in \mathfrak{S}_p\big(L^2(\Omega)\big) \qquad \textit{for all } p > \frac{2(n-1)}{3} \tag{7.26}$$

and $\lambda \in \rho(A_{[B]}) \cap \rho(A_{\mathrm{N}})$, and

$$(A_{[B]} - \lambda)^{-1} - (A_{\mathrm{D}} - \lambda)^{-1} \in \mathfrak{S}_p\big(L^2(\Omega)\big) \qquad \text{for all } p > \frac{2(n-1)}{3} \tag{7.27}$$

and $\lambda \in \rho(A_{[B]}) \cap \rho(A_{\mathrm{D}})$. *If, in addition,* $B \in \mathcal{B}(L^2(\partial\Omega))$ *then*

$$(A_{[B]} - \lambda)^{-1} - (A_{\mathrm{N}} - \lambda)^{-1} \in \mathfrak{S}_p\big(L^2(\Omega)\big) \qquad \text{for all } p > \frac{n-1}{3} \tag{7.28}$$

and $\lambda \in \rho(A_{[B]}) \cap \rho(A_{\mathrm{N}})$, *and*

$$(A_{[B]} - \lambda)^{-1} - (A_{\mathrm{D}} - \lambda)^{-1} \in \mathfrak{S}_p\big(L^2(\Omega)\big) \qquad \text{for all } p > \frac{n-1}{2} \tag{7.29}$$

and $\lambda \in \rho(A_{[B]}) \cap \rho(A_{\mathrm{D}})$.

Proof. Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple in Proposition 7.1 and let γ be the corresponding γ-field. Clearly, $\gamma(\lambda)^* \in \mathcal{B}(L^2(\Omega), L^2(\partial\Omega))$, and it follows from (2.3) that $\operatorname{ran}\gamma(\lambda)^* = \operatorname{ran}(\Gamma_1 \upharpoonright \operatorname{dom} A_{\mathrm{N}}) = H^{3/2}(\partial\Omega)$ for all $\lambda \in \rho(A_{\mathrm{N}})$. Therefore we can conclude as in [25, Lemma 3.4] that

$$\gamma(\lambda)^* \in \mathfrak{S}_p\big(L^2(\Omega), L^2(\partial\Omega)\big) \quad \text{for all} \quad p > \frac{2(n-1)}{3} \tag{7.30}$$

and for each $\lambda \in \rho(A_{\mathrm{N}})$. Moreover, for $\lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}})$ we have the relations $M(\lambda)^{-1}\gamma(\overline{\lambda})^* \in \mathcal{B}(L^2(\Omega), L^2(\partial\Omega))$ and $\operatorname{ran} M(\lambda)^{-1}\gamma(\overline{\lambda})^* = H^{1/2}(\partial\Omega)$ since $M(\lambda)^{-1}$ maps $H^{3/2}(\partial\Omega)$ onto $H^{1/2}(\partial\Omega)$. It follows again as in [25, Lemma 3.4] that

$$M(\lambda)^{-1}\gamma(\overline{\lambda})^* \in \mathfrak{S}_q\big(L^2(\Omega), L^2(\partial\Omega)\big) \qquad \text{for all} \quad q > 2(n-1) \tag{7.31}$$

and for each $\lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}})$. From (7.30) we obtain with the help of Proposition 4.7 the assertions (7.26) and (7.27). For $B \in \mathcal{B}(L^2(\partial\Omega))$, Proposition 4.8, (7.30) and (7.31) yield (7.28) and (7.29). □

Remark 7.12. Note that the statement of Theorem 7.11 can be refined if we replace the usual Schatten–von Neumann classes $\mathfrak{S}_p$ by the weak Schatten–von Neumann classes $\mathfrak{S}_{p,\infty}$, which are discussed in Remark 4.9. In this case one can allow p to be equal to $2(n-1)/3$, $(n-1)/3$ or $(n-1)/2$, respectively; cf. [27, Section 4.2] and [28, Section 3].

8. Schrödinger operators with δ-interaction on hypersurfaces

In this section we provide some applications of the results in Sections 4, 5 and 6 to Schrödinger operators with δ-interaction supported on a smooth, not necessarily bounded hypersurface Σ in $\mathbb{R}^n$. To be more specific, we consider operators associated with the formal differential expression

$$-\Delta - \alpha\langle\,\cdot, \delta_\Sigma\rangle\delta_\Sigma,$$

where α is a complex constant or a complex-valued function on Σ, the strength of the δ-interaction. The spectral theory of such operators is a prominent subject in mathematical physics; see the review paper [62], the monograph [67], and the references therein. The largest part of the existing literature (see, e.g. [37,64,66,68,69,111,118]) is devoted to the case of a real interaction strength α. However, there has been recent interest in non-real α; see, e.g. [72,98].

In what follows, let Ω_+ be a uniformly regular, bounded or unbounded domain in $\mathbb{R}^n$ (see Section 7) with boundary $\Sigma := \partial\Omega_+$. Furthermore, let $\Omega_- = \mathbb{R}^n \setminus (\Omega_+ \cup \Sigma)$ be its complement in $\mathbb{R}^n$. We write $f = f_+ \oplus f_-$ for $f \in L^2(\mathbb{R}^n)$, where $f_\pm = f|_{\Omega_\pm}$. By the same reason as in Section 7, the trace and the normal derivative extend to continuous linear mappings

$$H^2(\Omega_\pm) \ni f_\pm \mapsto \left\{ f_\pm|_\Sigma;\ \frac{\partial f_\pm}{\partial \nu_\pm}\Big|_\Sigma \right\} \in H^{3/2}(\Sigma) \times H^{1/2}(\Sigma).$$

Both the above mappings are surjective onto $H^{3/2}(\Sigma) \times H^{1/2}(\Sigma)$. Furthermore, we introduce an operator T in $L^2(\mathbb{R}^n)$ by

$$Tf = (-\Delta f_+) \oplus (-\Delta f_-), \qquad \operatorname{dom} T = H^2(\mathbb{R}^n \setminus \Sigma) \cap H^1(\mathbb{R}^n). \tag{8.1}$$

On $\operatorname{dom} T$ we define boundary mappings Γ_0 and Γ_1 by

$$\Gamma_0 f = \frac{\partial f_+}{\partial \nu_+}\Big|_\Sigma + \frac{\partial f_-}{\partial \nu_-}\Big|_\Sigma, \qquad \Gamma_1 f = f|_\Sigma \qquad \text{for } f \in \operatorname{dom} T; \tag{8.2}$$

here $\frac{\partial f_\pm}{\partial \nu_\pm}\big|_\Sigma$ stand for the normal derivatives of $f = f_+ \oplus f_- \in \operatorname{dom} T$ on two opposite faces of Σ with the normals pointing outwards $\Omega_\pm$; note that the outer unit normal vector fields ν_- and ν_+ of Ω_- and Ω_+, respectively, satisfy $\nu_-(x) = -\nu_+(x)$ for all $x \in \Sigma$. Moreover, consider the symmetric operator S in $L^2(\mathbb{R}^n)$ defined as

$$Sf = -\Delta f, \qquad \operatorname{dom} S = H^2(\mathbb{R}^n) \cap H^1_0(\mathbb{R}^n \setminus \Sigma). \tag{8.3}$$

In the following proposition we state that $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $T \subset S^*$ and we formulate properties of this triple and of the associated γ-field and Weyl function. This proposition is analogous to Proposition 7.1 and can be proved in a similar way; see the proofs of [29, Propositions 3.1 and 3.2]. Note that in the case of a compact Σ, the statements and proofs of the next proposition and further details can be found in [26, §3] and [27, §3.1].

Proposition 8.1. *The operator S in (8.3) is closed, symmetric and densely defined with $S^* = \overline{T}$ for T in (8.1), and the triple $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $T \subset S^*$ with the following properties.*

(i) $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top = H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)$.

(ii) A_0 *is the* free Laplace operator

$$-\Delta_{\mathbb{R}^n} f = -\Delta f, \qquad \operatorname{dom}(-\Delta_{\mathbb{R}^n}) = H^2(\mathbb{R}^n),$$

and A_1 *is the orthogonal sum of the* Dirichlet Laplacians *on* Ω_+ *and* Ω_-*, respectively,*

$$-\Delta_{\mathrm{D}} f = -\Delta f, \qquad \operatorname{dom}(-\Delta_{\mathrm{D}}) = H^2(\mathbb{R}^n \setminus \Sigma) \cap H_0^1(\mathbb{R}^n \setminus \Sigma).$$

Both operators, $-\Delta_{\mathbb{R}^n}$ *and* $-\Delta_{\mathrm{D}}$*, are self-adjoint and non-negative in* $L^2(\mathbb{R}^n)$*.*

(iii) *For all* $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ *the associated* γ*-field satisfies*

$$\gamma(\lambda)\left(\frac{\partial f_+}{\partial \nu_+}\Big|_\Sigma + \frac{\partial f_-}{\partial \nu_-}\Big|_\Sigma\right) = f \qquad \textit{for all } f \in \ker(T-\lambda), \tag{8.4}$$

and the associated Weyl function is given by:

$$M(\lambda)\left(\frac{\partial f_+}{\partial \nu_+}\Big|_\Sigma + \frac{\partial f_-}{\partial \nu_-}\Big|_\Sigma\right) = f|_\Sigma \qquad \textit{for all } f \in \ker(T-\lambda). \tag{8.5}$$

Moreover, $M(\lambda)$ *is a bounded, non-closed operator in* $L^2(\Sigma)$ *with domain* $H^{1/2}(\Sigma)$ *such that* $\operatorname{ran}\overline{M(\lambda)} \subset H^1(\Sigma)$*.*

The following lemma ensures the decay of the Weyl function M in (8.5). For the definition of the exterior sector $\mathbb{U}_{w_0,\nu}$ we refer to (6.2).

Lemma 8.2. *Let* M *denote the Weyl function in* (8.5)*. Then for all* $w_0 < 0$*,* $\nu \in (0, \pi)$*, and* $\beta \in (0, \frac{1}{2})$ *there exists a constant* $C = C(\Sigma, \beta, w_0, \nu) > 0$ *such that*

$$\|\overline{M(\lambda)}\| \le \frac{C}{\big(\operatorname{dist}(\lambda, \mathbb{R}_+)\big)^\beta} \qquad \textit{for all } \lambda \in \mathbb{U}_{w_0,\nu}. \tag{8.6}$$

Proof. Let $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple in Proposition 8.1. Recall that $A_0 = -\Delta_{\mathbb{R}^n}$; in particular, $\sigma(A_0) = [0, \infty)$ and $\operatorname{dom}(A_0 + 1)^{\frac{s}{2}} = H^s(\mathbb{R}^n)$ for all $s > 0$ by the definition of the Sobolev spaces. Hence by the closed graph theorem, $(A_0 + 1)^{-s/2}$ is bounded as an operator from $L^2(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$ for each $s \ge 0$. Since the trace map is bounded from $H^s(\mathbb{R}^n)$ to $L^2(\Sigma)$ for each $s \in (\frac{1}{2}, 1)$, it follows that $f \mapsto ((A_0 + 1)^{-s/2} f)|_\Sigma$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\Sigma)$ for each $s \in (\frac{1}{2}, 1)$. Therefore the operator

$$\Gamma_1(A_0 + 1)^{-\alpha} : L^2(\mathbb{R}^n) \supset \operatorname{dom}\big(\Gamma_1(A_0 + 1)^{-\alpha}\big) \to L^2(\Sigma)$$

is bounded for each $\alpha \in (\frac{1}{4}, \frac{1}{2})$. By Theorem 6.1 it follows that for each $w_0 < 0$, each $\nu \in (0, \pi)$ and each $\alpha \in (\frac{1}{4}, \frac{1}{2})$ there exists $C = C(\Sigma, \beta, w_0, \nu) > 0$ such that

$$\|\overline{M(\lambda)}\| \le \frac{C}{\big(\operatorname{dist}(\lambda,\mathbb{R}_+)\big)^{1-2\alpha}}$$

holds for all $\lambda \in \mathbb{U}_{w_0,\nu}$. From this the claim of the lemma follows. □

Remark 8.3. It can be shown as in [26, Proposition 3.2 (iii)] that

$$M(\lambda) = \big(M_+(\lambda)^{-1} + M_-(\lambda)^{-1}\big)^{-1}, \qquad \lambda \in \mathbb{C}\setminus[0,\infty), \tag{8.7}$$

where M_+ and M_- are the Weyl functions from Section 7 for $-\Delta$ on Ω_+ and Ω_-, respectively. Remark 7.3 implies that for each $\mu < 0$ there exist $C_\pm > 0$ such that

$$\|\overline{M_\pm(\lambda)}\| \le \frac{C_\pm}{(\mu-\lambda)^{1/2}}, \qquad \lambda < \mu.$$

Since $M_\pm(\lambda) \ge 0$ for $\lambda \in (-\infty, 0)$, it follows from [10, Corollaries I.2.4 and I.3.2] that

$$\|\overline{M(\lambda)}\| \le \frac{1}{4}\Big(\|\overline{M_+(\lambda)}\| + \|\overline{M_-(\lambda)}\|\Big).$$

Hence for each $\mu < 0$ there exists $C = C(\Sigma,\mu) > 0$ such that

$$\|\overline{M(\lambda)}\| \le \frac{C}{(\mu-\lambda)^{1/2}} \qquad \text{for all } \ \lambda < \mu.$$

From Lemma 8.2, Remark 8.3 and the results of Section 5 we obtain the following consequences for Schrödinger operators with δ-potentials supported on Σ; cf. the proof of Theorem 7.4 and Corollary 7.7. Note that the assumptions of the next theorem allow certain classes of unbounded functions α; cf. Remark 7.8.

Theorem 8.4. *Let α be a measurable complex-valued function such that*

$$\alpha\varphi \in H^{1/2}(\Sigma) \qquad \textit{for all } \ \varphi \in H^1(\Sigma), \tag{8.8}$$

and that

$$b := \sup(\operatorname{Re}\alpha) < \infty.$$

Then the Schrödinger operator with δ-interaction of strength α supported on Σ,

$$\begin{aligned} A_{[\alpha]} f &= (-\Delta f_+) \oplus (-\Delta f_-), \\ \operatorname{dom} A_{[\alpha]} &= \left\{ f \in H^2(\mathbb{R}^n\setminus\Sigma) \cap H^1(\mathbb{R}^n) : \frac{\partial f_+}{\partial\nu_+}\Big|_\Sigma + \frac{\partial f_-}{\partial\nu_-}\Big|_\Sigma = \alpha f|_\Sigma \right\}, \end{aligned} \tag{8.9}$$

in $L^2(\mathbb{R}^n)$ is m-sectorial, one has $\sigma(A_{[\alpha]}) \subset \overline{W(A_{[\alpha]})}$, the resolvent formula

$$(A_{[\alpha]} - \lambda)^{-1} = (-\Delta_{\mathbb{R}^n} - \lambda)^{-1} + \gamma(\lambda)\big(I - \alpha M(\lambda)\big)^{-1}\alpha\gamma(\overline{\lambda})^* \tag{8.10}$$

holds for all $\lambda \in \rho(A_{[\alpha]}) \setminus \mathbb{R}_+$, and the following assertions are true.

(i) $A_{[\overline{\alpha}]} = A_{[\alpha]}^*$.
(ii) *If α is real-valued, then $A_{[\alpha]}$ is self-adjoint and bounded from below. If $\operatorname{Im}(\alpha(s)) \geq 0$ (≤ 0, respectively) for almost all $s \in \Sigma$, then $A_{[\alpha]}$ is maximal accumulative (maximal dissipative, respectively).*

Moreover, the following spectral enclosures hold.

(iii) *If $b \leq 0$, then $(-\infty, 0) \subset \rho(A_{[\alpha]})$.*
(iv) *If $\operatorname{Im}\alpha$ is bounded and $b > 0$, then for each $\mu < 0$ there exists $C > 0$ such that for each $\xi < \mu - (Cb)^2$,*

$$W(A_{[\alpha]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq \mu - (Cb)^2,\ |\operatorname{Im} z| \leq \frac{2C\|\operatorname{Im}\alpha\|_\infty}{1 - \frac{Cb}{(\mu-\xi)^{1/2}}}(\operatorname{Re} z - \xi)^{1/2} \right\}.$$

(v) *If $\operatorname{Im}\alpha$ is bounded and $b \leq 0$, then for each $\mu < 0$ there exists $C > 0$ such that*

$$W(A_{[\alpha]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq 0,\ |\operatorname{Im} z| \leq \frac{2C\|\operatorname{Im}\alpha\|_\infty(\operatorname{Re} z - \mu)}{(\operatorname{Re} z - \mu)^{1/2} - Cb} \right\}.$$

(vi) *If α is bounded, then for each $w_0 < 0$, $\nu \in (0, \pi)$ and $\beta \in \big(0, \frac{1}{2}\big)$ there exists $C > 0$ such that*

$$\sigma(A_{[\alpha]}) \cap \mathbb{U}_{w_0,\nu} \subset \left\{ z \in \mathbb{U}_{w_0,\nu} : \operatorname{dist}(z, \mathbb{R}_+) \leq (C\|\alpha\|_\infty)^{1/\beta} \right\},$$

where $\mathbb{U}_{w_0,\nu}$ is defined in (6.2).

Let us illustrate the obtained spectral estimates in an example.

Example 8.5. Consider the case

$$\Omega_\pm = \mathbb{R}^n_\pm = \big\{ x = (x', x_n)^\top \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, \pm x_n > 0 \big\},$$

that is, $\Sigma = \{(x', 0)^\top : x' \in \mathbb{R}^{n-1}\}$, which we identify with $\mathbb{R}^{n-1}$. It follows from (8.7) and (7.22) that

$$\overline{M(\lambda)} = \frac{1}{2}\big(-\Delta_{\mathbb{R}^{n-1}} - \lambda\big)^{-1/2}, \qquad \lambda \in \mathbb{C} \setminus [0, \infty),$$

and hence

$$\|\overline{M(\lambda)}\| = \frac{1}{2\sqrt{\operatorname{dist}(\lambda, \mathbb{R}_+)}}, \qquad \lambda \in \mathbb{C} \setminus [0, \infty).$$

In particular, the estimate (5.11) is satisfied with $\mu = 0$ and $C = 1/2$. In analogy to Example 7.10, this observation can be used to obtain several better enclosures for the spectrum and numerical range of the operator $A_{[\alpha]}$. Let α satisfy the conditions of Theorem 8.4 and let $\operatorname{Im}\alpha$ be bounded. If $b > 0$, then for every $\xi < -b^2/4$ one has

$$W(A_{[\alpha]}) \subset \left\{ z \in \mathbb{C}\colon \operatorname{Re} z \geq -\frac{b^2}{4}, \ |\operatorname{Im} z| \leq \frac{\|\operatorname{Im}\alpha\|_\infty}{1 - \frac{b}{2\sqrt{|\xi|}}} (\operatorname{Re} z - \xi)^{1/2} \right\}.$$

If $b \leq 0$, then

$$W(A_{[\alpha]}) \subset \left\{ z \in \mathbb{C}\colon \operatorname{Re} z > 0, |\operatorname{Im} z| \leq \frac{\|\operatorname{Im}\alpha\|_\infty \operatorname{Re} z}{(\operatorname{Re} z)^{1/2} - b/2} \right\} \cup \{0\}.$$

If, in addition, α is bounded, then by Proposition 5.9 (a) with $G = \mathbb{C} \setminus \mathbb{R}_+$ the spectrum of $A_{[\alpha]}$ satisfies the enclosure

$$\sigma(A_{[\alpha]}) \subset \left\{ z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{R}_+) \leq \frac{1}{4}\|\alpha\|_\infty^2 \right\}.$$

We now have a closer look at the special case of a compact hypersurface Σ and bounded α. For this case certain refined bounds for the function M from the recent work [75] are available and can be combined with the results in the abstract part of this paper in order to obtain the spectral bounds for $A_{[\alpha]}$ that are contained in the next theorem. We remark that [75] contains further bounds in space dimension two and in the special case when Ω_+ is a convex domain, which could be combined with our theorems; however, we do not include this in the next theorem.

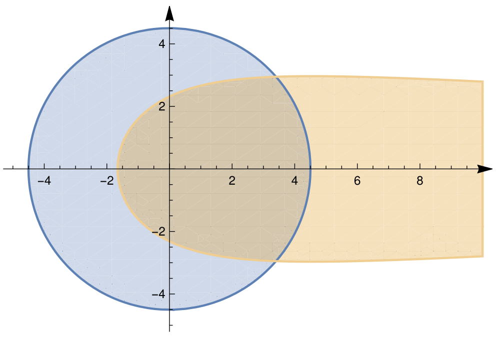
Theorem 8.6. *Let Σ be compact and let $\alpha \in L^\infty(\Sigma)$ be a complex-valued function which satisfies (8.8). Then there exist constants $C_1, C_2 > 0$, which are independent of α, such that the spectrum of $A_{[\alpha]}$ satisfies*

$$\sigma(A_{[\alpha]}) \setminus \mathbb{R}_+ \subset \mathbb{V}_{\alpha,C_1} \cap \mathbb{W}_{\alpha,C_2}, \tag{8.11}$$

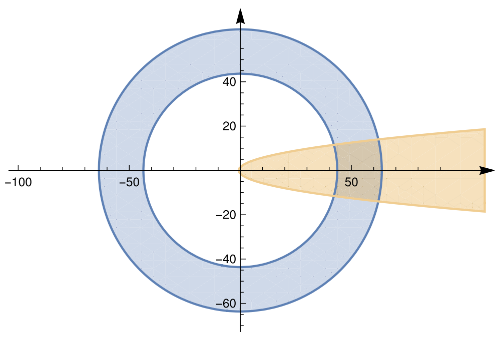
where (see Fig. 4)

$$\mathbb{V}_{\alpha,C_1} := \begin{cases} \left\{ z \in \mathbb{C} \setminus \{0\} : C_1\|\alpha\|_\infty \big(2 + |z|\big)^{-\frac{1}{4}} \ln\big(2 + |z|^{-1}\big) \geq 1 \right\}, & n = 2, \\ \left\{ z \in \mathbb{C} : C_1\|\alpha\|_\infty \big(2 + |z|\big)^{-\frac{1}{4}} \ln\big(2 + |z|\big) \geq 1 \right\}, & n \geq 3, \end{cases}$$

$$\mathbb{W}_{\alpha,C_2} := \begin{cases} \left\{ z \in \mathbb{C} \setminus \{0\} : C_2\|\alpha\|_\infty \big(2 + |\operatorname{Im}\sqrt{z}|^2\big)^{-\frac{1}{2}} \ln\big(2 + |z|^{-1}\big) \geq 1 \right\}, & n = 2, \\ \left\{ z \in \mathbb{C} : C_2\|\alpha\|_\infty \big(2 + |\operatorname{Im}\sqrt{z}|^2\big)^{-\frac{1}{2}} \geq 1 \right\}, & n \geq 3. \end{cases}$$



(a) $n = 2$



(b) $n \geq 3$

Fig. 4. The sets $\mathbb{V}_{\alpha,C_1}$ (blue) and $\mathbb{W}_{\alpha,C_2}$ (yellow) in Theorem 8.6 for (a) $n = 2$ and (b) $n \geq 3$, respectively, where $C_1\|\alpha\|_\infty = C_2\|\alpha\|_\infty = 0.5$ in (a), and $C_1\|\alpha\|_\infty = 1.47$ and $C_2\|\alpha\|_\infty = 0.6$ in (b). (For interpretation of the colours in the figures, the reader is referred to the web version of this article.)

Proof. By [75, Theorems 1.2 and 1.3] there exist constants $C_1, C_2 > 0$ (the constants here differ from the ones in [75] by a factor $\frac{1}{2}$) such that

$$\|\alpha\overline{M(\lambda)}\| \leq \begin{cases} C_1\|\alpha\|_\infty\big(2+|\lambda|\big)^{-\frac{1}{4}}\ln\big(2+|\lambda|^{-1}\big), & n=2, \\ C_1\|\alpha\|_\infty\big(2+|\lambda|\big)^{-\frac{1}{4}}\ln\big(2+|\lambda|\big), & n\geq 3, \end{cases}$$

$$\|\alpha\overline{M(\lambda)}\| \leq \begin{cases} C_2\|\alpha\|_\infty\big(2+|\operatorname{Im}\sqrt{\lambda}|^2\big)^{-\frac{1}{2}}\ln\big(2+|\lambda|^{-1}\big), & n=2, \\ C_2\|\alpha\|_\infty\big(2+|\operatorname{Im}\sqrt{\lambda}|^2\big)^{-\frac{1}{2}}, & n\geq 3, \end{cases}$$

hold for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. Thanks to condition (8.8) we can view the multiplication with α as an operator in $L^2(\Sigma)$ with domain $H^1(\Sigma)$ and range contained in $H^{1/2}(\Sigma)$. Hence, by Theorem 4.1, any point $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ for which at least one of the above two upper bounds on $\|\alpha\overline{M(\lambda)}\|$ is strictly less than one belongs to the resolvent set of $A_{[\alpha]}$. Thus, the enclosure in (8.11) follows. □

Furthermore, we obtain certain Schatten–von Neumann estimates for the difference of the resolvents of $A_{[\alpha]}$ and the free Laplacian. They are analogues of the first and the third estimates in Theorem 7.11, and the proofs are analogous, where one uses the relations

$$\gamma(\lambda)^* \in \mathfrak{S}_p\big(L^2(\mathbb{R}^n), L^2(\Sigma)\big) \qquad \text{for all } p > \frac{2(n-1)}{3}$$

and

$$M(\lambda)^{-1}\gamma(\overline{\lambda})^* \in \mathfrak{S}_q\big(L^2(\mathbb{R}^n), L^2(\Sigma)\big) \qquad \text{for all } q > 2(n-1).$$

Theorem 8.7. *Let all assumptions of Theorem 8.4 be satisfied. Moreover, assume that Σ is compact. Then*

$$(A_{[\alpha]} - \lambda)^{-1} - (-\Delta_{\mathbb{R}^n} - \lambda)^{-1} \in \mathfrak{S}_p\big(L^2(\mathbb{R}^n)\big) \qquad \text{for all } p > \frac{2(n-1)}{3} \tag{8.12}$$

and all $\lambda \in \rho(A_{[\alpha]})$. If, in addition, α is bounded, then

$$(A_{[\alpha]} - \lambda)^{-1} - (-\Delta_{\mathbb{R}^n} - \lambda)^{-1} \in \mathfrak{S}_p\big(L^2(\mathbb{R}^n)\big) \qquad \text{for all } p > \frac{n-1}{3} \tag{8.13}$$

and all $\lambda \in \rho(A_{[\alpha]})$.

Remark 8.8. In the same way as in Remark 7.12, we can reformulate Theorem 8.7 for weak Schatten–von Neumann classes. In this setting the endpoints for the intervals of admissible values of p can be included in both (8.12) and (8.13); cf. [26, Section 4.2].

Remark 8.9. In the case of a real, bounded coefficient α, in space dimensions 2 and 3 the previous theorem can be used in order to derive existence and completeness of wave operators for the scattering pair $\{A_{[\alpha]}, -\Delta_{\mathbb{R}^n}\}$. In space dimension 2, the same is true for certain unbounded α; cf. Example 7.9. Let us also mention [118] where Schatten–von Neumann properties were proved for certain δ-interactions with unbounded real-valued coefficients.

Finally, in the last theorem of this section we show that in two space dimensions for $\|\alpha\|_\infty$ small enough the spectrum of $A_{[\alpha]}$ outside $[0, \infty)$ is contained in a disc with radius that converges to 0 exponentially as $\|\alpha\|_\infty \to 0$ and that in higher dimensions $A_{[\alpha]}$ has no spectrum outside $[0, \infty)$ if $\|\alpha\|_\infty$ is small enough. The result in two dimensions agrees well with the asymptotic expansion in [99] in the self-adjoint setting. Related conditions for absence of non-real eigenvalues in higher dimensions for Schrödinger operators with complex-valued regular potentials can be found in [70,71]. In the self-adjoint setting absence of negative eigenvalues for $\|\alpha\|_\infty$ small enough is also a consequence of the Birman–Schwinger bounds in [37]; see also [63].

Theorem 8.10. *Let Σ be compact and let $\alpha \in L^\infty(\Sigma)$ be a complex-valued function that satisfies (8.8). Then $\sigma_{\mathrm{ess}}(A_{[\alpha]}) = [0, \infty)$, and the following statements hold.*

(i) *Let $n = 2$ and let $C_1 > 0$ be as in Theorem 8.6. If $0 < \|\alpha\|_\infty \le \frac{1}{2C_1 \ln 2}$, then*

$$\sigma(A_{[\alpha]}) \setminus \mathbb{R}_+ \subset \left\{ z \in \mathbb{C} : |z| \le 2 \exp\left(-\frac{1}{C_1 \|\alpha\|_\infty} \right) \right\}.$$

(ii) *Let $n \ge 3$. There exists $\varepsilon = \varepsilon(\Sigma) > 0$ such that $\sigma(A_{[\alpha]}) = \sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty)$ if $\|\alpha\|_\infty < \varepsilon$.*

Proof. The statement about the essential spectrum follows directly from Theorem 8.7.

(i) Assume that $0 < \|\alpha\|_\infty \le \frac{1}{2C_1 \ln 2}$ and let $z \in \sigma(A_{[\alpha]}) \setminus [0,\infty)$. It follows from Theorem 8.6 that $z \in \mathbb{V}_{\alpha,C_1}$ and hence

$$C_1\|\alpha\|_\infty \ln\bigl(2+|z|^{-1}\bigr) \ge C_1\|\alpha\|_\infty \bigl(2+|z|\bigr)^{-1/4} \ln\bigl(2+|z|^{-1}\bigr) \ge 1,$$

which implies that

$$|z| \le \frac{1}{\exp\bigl(\frac{1}{C_1\|\alpha\|_\infty}\bigr)-2} = \frac{\exp\bigl(-\frac{1}{C_1\|\alpha\|_\infty}\bigr)}{1-2\exp\bigl(-\frac{1}{C_1\|\alpha\|_\infty}\bigr)} \le 2\exp\left(-\frac{1}{C_1\|\alpha\|_\infty}\right).$$

(ii) Since the maximum of the function $g(t)=t^{-1/4}\ln t$, $t\in[2,\infty)$ is $\frac{4}{e}$ (attained at $t=e^4$), it follows that

$$\bigl(2+|z|\bigr)^{-1/4}\ln\bigl(2+|z|\bigr) \le \frac{4}{e} \qquad \text{for all } z\in\mathbb{C}.$$

If

$$\|\alpha\|_\infty < \varepsilon := \frac{e}{4C_1},$$

then

$$C_1\|\alpha\|_\infty\bigl(2+|z|\bigr)^{-1/4}\ln\bigl(2+|z|\bigr) < 1$$

for every $z\in\mathbb{C}$, and Theorem 8.6 implies that $\sigma(A_{[\alpha]})\setminus[0,\infty)=\emptyset$. Together with the relation $\sigma_{\mathrm{ess}}(A_{[\alpha]})=[0,\infty)$ this shows that $\sigma(A_{[\alpha]})=[0,\infty)$. □

9. Infinitely many point interactions on the real line

In this section we provide applications of the results in Section 5 to Hamiltonians with non-local, non-Hermitian interactions supported on a discrete set of points $X=\{x_n : n\in\mathbb{Z}\}$, where (x_n) is a strictly increasing sequence of real numbers. The investigation of such Hamiltonians has been initiated almost a century ago in [105] for periodically distributed, local, Hermitian point δ-interactions. Classical results are summarized in the monograph [7]; see also the references therein and [97,102]. More recently, non-Hermitian interactions attracted attention (see [6,9]) and also non-local interactions were studied; see [9,107].

Throughout this section we make the assumption

$$d := \inf_{n\in\mathbb{Z}}(x_{n+1}-x_n) > 0; \tag{9.1}$$

in particular, the sequence (x_n) does not have a finite accumulation point. We remark that this assumption can be avoided by using the methods of [8,101], but we do not focus on this here.

For each interval $I_n := (x_n, x_{n+1})$ we denote by $H^2(I_n)$ the usual Sobolev space on I_n of second order. Moreover, we set $f_n := f|_{I_n}$ for $f \in L^2(\mathbb{R})$ and introduce

$$H^2(\mathbb{R} \setminus X) := \left\{ f \in L^2(\mathbb{R}) : f_n \in H^2(I_n) \text{ for all } n \in \mathbb{Z}, \sum_{n\in\mathbb{Z}} \|f_n\|^2_{H^2(I_n)} < \infty \right\},$$

equipped with the norm

$$\|f\|^2_{H^2(\mathbb{R}\setminus X)} := \sum_{n\in\mathbb{Z}} \|f_n\|^2_{H^2(I_n)}, \qquad f \in H^2(\mathbb{R} \setminus X). \tag{9.2}$$

In order to construct a boundary triple which is suitable for the parameterization of Hamiltonians with interactions supported on X, we define operators S and T in $L^2(\mathbb{R})$ by

$$Sf = -f'' \quad \text{on } \mathbb{R} \setminus X, \quad \operatorname{dom} S = \{ f \in H^2(\mathbb{R}) : f(x_n) = 0 \text{ for all } n \in \mathbb{Z} \}, \tag{9.3}$$

and

$$Tf = -f'' \quad \text{on } \mathbb{R} \setminus X, \qquad \operatorname{dom} T = H^2(\mathbb{R} \setminus X) \cap H^1(\mathbb{R}), \tag{9.4}$$

that is, $\operatorname{dom} T$ consists of all $f \in H^2(\mathbb{R} \setminus X)$ such that $f_{n-1}(x_n) = f_n(x_n)$ for all $n \in \mathbb{Z}$. Moreover, for $f \in \operatorname{dom} T$ we define

$$\Gamma_0 f = \big(f_n'(x_n) - f_{n-1}'(x_n) \big)_{n\in\mathbb{Z}} \qquad \text{and} \qquad \Gamma_1 f = \big(-f(x_n) \big)_{n\in\mathbb{Z}}. \tag{9.5}$$

In fact, Γ_0 and Γ_1 are boundary mappings for an ordinary boundary triple, as the following proposition shows; see also [100, Proposition 7 (i)] where a very similar boundary triple was constructed.

Proposition 9.1. *The operator S in (9.3) is closed, symmetric and densely defined with $S^* = T$ for T in (9.4), and the triple $\{\ell^2(\mathbb{Z}), \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for S^* with the following properties.*

(i) $A_0 = S^* \upharpoonright \ker \Gamma_0$ *is given by*

$$A_0 f = -f'', \quad \operatorname{dom} A_0 = H^2(\mathbb{R}), \tag{9.6}$$

and $A_1 = S^* \upharpoonright \ker \Gamma_1$ *is given by*

$$\begin{aligned} A_1 f &= -f'' \quad \textit{on } \mathbb{R} \setminus X, \\ \operatorname{dom} A_1 &= \{ f \in H^2(\mathbb{R} \setminus X) \cap H^1(\mathbb{R}) : f(x_n) = 0 \ \textit{for all } n \in \mathbb{Z} \}. \end{aligned} \tag{9.7}$$

(ii) *For* $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ *the associated* γ*-field acts as*

$$\big(\gamma(\lambda)\xi\big)(x) = \frac{-i}{2\sqrt{\lambda}} \sum_{n \in \mathbb{Z}} e^{i\sqrt{\lambda}|x_n - x|} \xi_n, \qquad x \in \mathbb{R}, \quad \xi = (\xi_n) \in \ell^2(\mathbb{Z}), \tag{9.8}$$

and the associated Weyl function satisfies

$$M(\lambda)\xi = \left(\frac{i}{2\sqrt{\lambda}} \sum_{n \in \mathbb{Z}} e^{i\sqrt{\lambda}|x_n - x_m|} \xi_n \right)_{m \in \mathbb{Z}}, \qquad \xi = (\xi_n) \in \ell^2(\mathbb{Z}). \tag{9.9}$$

Proof. Let us first check that Γ_0 and Γ_1 are well-defined mappings from $\operatorname{dom} T$ to $\ell^2(\mathbb{Z})$. For this we make use of the following estimate, which can be found in, e.g. [106, Lemma 8]: if $[a,b]$ is a compact interval then for each $l \in (0, b-a]$ one has

$$|f(a)|^2 \leq \frac{2}{l} \|f\|^2_{L^2(a,b)} + l \|f'\|^2_{L^2(a,b)} \quad \text{for all } \ f \in H^1(a,b). \tag{9.10}$$

The same estimate holds for $|f(a)|^2$ replaced by $|f(b)|^2$. From (9.1) we obtain that $d \in (0, x_{n+1} - x_n]$ for each $n \in \mathbb{Z}$, and (9.10) yields

$$\sum_{n \in \mathbb{Z}} |f(x_n)|^2 \leq \frac{2}{d} \sum_{n \in \mathbb{Z}} \|f_n\|^2_{L^2(I_n)} + d \sum_{n \in \mathbb{Z}} \|f_n'\|^2_{L^2(I_n)} < \infty$$

for all $f \in \operatorname{dom} T \subset H^1(\mathbb{R})$. Hence $\Gamma_1 f \in \ell^2(\mathbb{Z})$ for all $f \in \operatorname{dom} T$. Similarly, using (9.10) for f replaced by f' we get $\Gamma_0 f \in \ell^2(\mathbb{Z})$ for all $f \in \operatorname{dom} T$.

To show that $\{\ell^2(\mathbb{Z}), \Gamma_0, \Gamma_1\}$ is a boundary triple for S^*, let us verify the conditions of Theorem 2.3. In fact, it is clear that $T \upharpoonright \ker \Gamma_0$ is given by the operator A_0 in (9.6), which is self-adjoint. Moreover, for all $f, g \in \operatorname{dom} T$ we have

$$\begin{aligned}
(Tf, g)_{L^2(\mathbb{R})} - (f, Tg)_{L^2(\mathbb{R})} &= \sum_{n \in \mathbb{Z}} \Big((-f_n'', g_n)_{L^2(I_n)} - (f_n, -g_n'')_{L^2(I_n)} \Big) \\
&= \sum_{n \in \mathbb{Z}} \Big(f_n'(x_n)\overline{g(x_n)} - f_n'(x_{n+1})\overline{g(x_{n+1})} \Big) \\
&\quad - \sum_{n \in \mathbb{Z}} \Big(f(x_n)\overline{g_n'(x_n)} - f(x_{n+1})\overline{g_n'(x_{n+1})} \Big) \\
&= \sum_{n \in \mathbb{Z}} \big(f_n'(x_n) - f_{n-1}'(x_n) \big) \overline{g(x_n)} - \sum_{n \in \mathbb{Z}} f(x_n) \overline{\big(g_n'(x_n) - g_{n-1}'(x_n) \big)} \\
&= (\Gamma_1 f, \Gamma_0 g)_{\ell^2(\mathbb{Z})} - (\Gamma_0 f, \Gamma_1 g)_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

Furthermore, the pair of mappings $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ has a dense range since it can be checked easily that all pairs of unit sequences $\{e_j, e_k\}$, $j, k \in \mathbb{Z}$, belong to the range. It follows from Theorem 2.3 that S is closed with $S^* = \overline{T}$ and that $\{\ell^2(\mathbb{Z}), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for S^*.

In order to conclude that $\{\ell^2(\mathbb{Z}), \Gamma_0, \Gamma_1\}$ is even an ordinary boundary triple, let us verify that the operator T is closed. To this end define a mapping

$$K : H^2(\mathbb{R} \setminus X) \to \ell^2(\mathbb{Z}), \qquad f \mapsto \big(f_n(x_n) - f_{n-1}(x_n)\big)_{n\in\mathbb{Z}}.$$

For all $f \in H^2(\mathbb{R} \setminus X)$ we have

$$\begin{aligned}\|Kf\|^2_{\ell^2(\mathbb{Z})} &\le 2 \sum_{n\in\mathbb{Z}} \big(|f_n(x_n)|^2 + |f_{n-1}(x_n)|^2\big) \\ &\le 2 \sum_{n\in\mathbb{Z}} \bigg(\frac{2}{d}\|f_n\|^2_{L^2(I_n)} + d\|f_n'\|^2_{L^2(I_n)} + \frac{2}{d}\|f_{n-1}\|^2_{L^2(I_{n-1})} + d\|f_{n-1}'\|^2_{L^2(I_{n-1})}\bigg) \\ &\le 2 \max\bigg\{\frac{4}{d}, 2d\bigg\} \sum_{n\in\mathbb{Z}} \|f_n\|^2_{H^2(I_n)},\end{aligned}$$

where we have used (9.10) with $l = d$. Therefore K is a bounded operator and, hence, its kernel, which equals $\operatorname{dom} T$, is closed in $H^2(\mathbb{R} \setminus X)$. Equivalently, $\operatorname{dom} T$ equipped with the norm of $H^2(\mathbb{R} \setminus X)$ is complete. It follows from [129, Satz 6.24], its proof and (9.1) that for each $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for all $n \in \mathbb{Z}$ one has

$$\|f_n'\|^2_{L^2(I_n)} \le \varepsilon\|f_n''\|^2_{L^2(I_n)} + C(\varepsilon)\|f_n\|^2_{L^2(I_n)}, \qquad f_n \in H^2(I_n).$$

This implies that $\operatorname{dom} T$ is also complete when equipped with the graph norm of T, that is, T is a closed operator. Hence $\{\ell^2(\mathbb{Z}), \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for S^*.

The remaining assertion (9.7) in (i) is obvious. For the assertions in (ii) let $\lambda \in \mathbb{C} \setminus [0, \infty)$. According to [129, Satz 11.26] or [127, page 190] we have

$$\big((A_0 - \lambda)^{-1} f\big)(y) = \frac{i}{2\sqrt{\lambda}} \int_{\mathbb{R}} e^{i\sqrt{\lambda}|y-x|} f(x)\, \mathrm{d}x, \qquad y \in \mathbb{R},\ f \in L^2(\mathbb{R}).$$

Hence for each compactly supported $f \in L^2(\mathbb{R})$ and each $\xi = \{\xi_n\}_n \in \ell^2(\mathbb{Z})$ we obtain from (2.3) and the definition of Γ_1 that

$$\begin{aligned}\big(f, \gamma(\lambda)\xi\big)_{L^2(\mathbb{R})} &= \big(\gamma(\lambda)^* f, \xi\big)_{\ell^2(\mathbb{Z})} = \big(\Gamma_1(A_0 - \overline{\lambda})^{-1} f, \xi\big)_{\ell^2(\mathbb{Z})} \\ &= \sum_{n\in\mathbb{Z}} \bigg(-\frac{i}{2\sqrt{\overline{\lambda}}} \int_{\mathbb{R}} e^{i\sqrt{\overline{\lambda}}|x_n-x|} f(x)\, \mathrm{d}x\bigg)\overline{\xi_n} \\ &= \int_{\mathbb{R}} f(x) \overline{\bigg(\frac{-i}{2\sqrt{\lambda}} \sum_{n\in\mathbb{Z}} e^{i\sqrt{\lambda}|x_n-x|}\xi_n\bigg)}\, \mathrm{d}x,\end{aligned}$$

where we have used that $\overline{i\sqrt{\lambda}} = i\sqrt{\overline{\lambda}}$. This proves (9.8). With the definition of Γ_1 also relation (9.9) follows. $\square$

Next we use the representation of the Weyl function in (9.9) to estimate its norm.

Lemma 9.2. *The Weyl function associated with the boundary triple in Proposition 9.1 satisfies*

$$\|M(\lambda)\| \leq \frac{\coth\left(\frac{d}{2}\,\mathrm{Im}\,\sqrt{\lambda}\right)}{2\sqrt{|\lambda|}} \tag{9.11}$$

for all $\lambda \in \mathbb{C} \setminus [0, \infty)$. In particular, the following estimates hold.

(i) *For each $\mu < 0$,*

$$\|M(\lambda)\| \leq \frac{\coth\left(\frac{d}{2}\sqrt{-\mu}\right)}{2(\mu - \lambda)^{1/2}} \qquad \text{for all } \lambda < \mu.$$

(ii) *For each $w_0 < 0$ and each $\nu \in (0, \pi)$ we have*

$$\|M(\lambda)\| \leq \frac{\coth(J_0)}{2\sqrt{|\lambda|}} \qquad \text{for all } \lambda \in \mathbb{U}_{w_0,\nu}, \tag{9.12}$$

where $J_0 = J_0(w_0, \nu) := \frac{d}{2}\sqrt{|w_0| \sin \nu}\, \sin\left(\frac{\nu}{2}\right) > 0$ and $\mathbb{U}_{w_0,\nu}$ is defined in (6.2).

Proof. Recall that for $\lambda \in \mathbb{C} \setminus [0, \infty)$ the operator $M(\lambda)$ has the explicit representation (9.9). In order to estimate its norm, we make use of the Schur test; see, e.g. [129, Korollar 6.7]. For this note that $|x_n - x_m| \geq |n - m|d$ holds for all $n, m \in \mathbb{Z}$ and, thus,

$$\begin{aligned}
\sup_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \left| e^{i\sqrt{\lambda}|x_n - x_m|} \right| &= \sup_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} e^{-\,\mathrm{Im}\,\sqrt{\lambda}|x_n - x_m|} \leq \sup_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} e^{-\,\mathrm{Im}\,\sqrt{\lambda}|n-m|d} \\
&= \sum_{n\in\mathbb{Z}} e^{-\,\mathrm{Im}\,\sqrt{\lambda}|n|d} = \frac{1 + e^{-\,\mathrm{Im}\,\sqrt{\lambda}d}}{1 - e^{-\,\mathrm{Im}\,\sqrt{\lambda}d}} = \coth\left(\frac{d}{2}\,\mathrm{Im}\,\sqrt{\lambda}\right).
\end{aligned}$$

Since the last term is finite and the same estimate holds by symmetry when the roles of m and n are interchanged, the Schur test can be applied and yields (9.11).

The statement (i) is a direct consequence of the estimate in (9.11) and the monotonicity properties of the function coth. For the remaining statement (ii) we calculate

$$J_0 = J_0(w_0, \nu) := \frac{d}{2} \min\left\{\mathrm{Im}\,\sqrt{\lambda} : \lambda \in \mathbb{U}_{w_0,\nu}\right\}. \tag{9.13}$$

By symmetry it is clear that it suffices to consider $\lambda \in \mathbb{U}_{w_0,\nu}$ with $\mathrm{Im}\,\lambda \geq 0$. Since the function $\mathbb{C} \setminus \{0\} \ni \lambda \mapsto \mathrm{Im}\,\sqrt{\lambda}$ has no local extremum, the minimum in (9.13) will be attained on the boundary of $\mathbb{U}_{w_0,\nu}$. Let us first consider the case when $\nu \in (0, \pi/2)$. Writing $\lambda = x + iy$ with $x, y \in \mathbb{R}$, for $\lambda \in \partial\mathbb{U}_{w_0,\nu}$ with $\operatorname{Im}\lambda \geq 0$ we have

$$\begin{aligned}\operatorname{Im}\sqrt{\lambda} = \operatorname{Im}\sqrt{x+iy} &\geq \sqrt[4]{x^2+y^2}\,\sin\Big(\frac{\nu}{2}\Big)\\ &= \sqrt[4]{x^2+\tan^2\nu\cdot(x-w_0)^2}\,\sin\Big(\frac{\nu}{2}\Big),\end{aligned} \tag{9.14}$$

and the right-hand side will be minimal if and only if $x^2 + \tan^2\nu \cdot (x - w_0)^2$ is minimal. The latter happens for $x = (w_0 \tan^2\nu)/(1 + \tan^2\nu)$. Plugging this into (9.14) and using elementary trigonometric identities we obtain the claimed expression for J_0. The case $\nu \in (\pi/2, \pi)$ can be treated analogously with $\tan\nu$ replaced by $\tan(\pi - \nu)$, and for $\nu = \pi/2$ we have

$$\operatorname{Im}\sqrt{\lambda} \geq \sqrt[4]{w_0^2+y^2}\sin\Big(\frac{\pi}{4}\Big) \geq \sqrt{|w_0|}\sin\Big(\frac{\pi}{4}\Big). \qquad \square$$

We are now able to formulate consequences of the results in Section 5. The assertions of the next theorem follow directly from Lemma 9.2 in combination with Corollary 5.7, Proposition 5.9 (a), [56, Proposition 1.4 (i)] and the fact that $\{\ell^2(\mathbb{Z}), \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple.

Theorem 9.3. *Let B be a closed operator in $\ell^2(\mathbb{Z})$. Then the operator $A_{[B]}$*

$$\begin{aligned}A_{[B]}f &= -f'' \quad \text{on } \mathbb{R}\setminus X,\\ \operatorname{dom} A_{[B]} &= \Big\{f \in H^2(\mathbb{R}\setminus X)\cap H^1(\mathbb{R}) : \Gamma_0 f = B\Gamma_1 f\Big\},\end{aligned} \tag{9.15}$$

in $L^2(\mathbb{R})$ is closed, the resolvent formula

$$(A_{[B]} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1}B\gamma(\overline{\lambda})^*$$

holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$ and the following assertions are true.

(i) *If B is self-adjoint, then $A_{[B]}$ is self-adjoint. If B is maximal dissipative (maximal accumulative, respectively), then $A_{[B]}$ is maximal accumulative (maximal dissipative, respectively).*
(ii) $A_{[B^*]} = A^*_{[B]}$.

Assume, additionally, that $B \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and let $b \in \mathbb{R}$ be such that

$$\operatorname{Re}(B\zeta,\zeta)_{\ell^2(\mathbb{Z})} \leq b\|\zeta\|^2_{\ell^2(\mathbb{Z})} \qquad \text{for all } \zeta \in \ell^2(\mathbb{Z}).$$

Then the operator $A_{[B]}$ is m-sectorial; in particular the inclusion $\sigma(A_{[B]}) \subset \overline{W(A_{[B]})}$ holds, and for any $\mu < 0$ and $C := \frac{1}{2}\coth(\frac{d}{2}\sqrt{-\mu})$ the following assertions are true.

(a) *If $b > 0$, then for every $\xi < \mu - (Cb)^2$,*

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq \mu - (Cb)^2, \ |\operatorname{Im} z| \leq K_\xi (\operatorname{Re} z - \xi)^{1/2} \right\},$$

where

$$K_\xi = \frac{2C \big\| \operatorname{Im} B \big\|}{1 - \frac{Cb}{(\mu - \xi)^{1/2}}}.$$

(b) *If $b \leq 0$, then*

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq 0, \ |\operatorname{Im} z| \leq \frac{2C \big\| \operatorname{Im} B \big\| (\operatorname{Re} z - \mu)}{(\operatorname{Re} z - \mu)^{1/2} - Cb} \right\}.$$

(c) *For any $w_0 < 0$ and each $\nu \in (0, \pi)$*

$$\sigma(A_{[B]}) \cap \mathbb{U}_{w_0,\nu} \subset \left\{ z \in \mathbb{U}_{w_0,\nu} : \operatorname{dist}(z, \mathbb{R}_+) \leq \frac{1}{4} \coth^2(J_0) \|B\|^2 \right\},$$

where $\mathbb{U}_{w_0,\nu}$ is defined in (6.2) and $J_0 = \frac{d}{2} \sqrt{|w_0| \sin \nu} \, \sin(\frac{\nu}{2})$.

Finally, we remark that the class of Hamiltonians under consideration in this section includes Schrödinger operators in $L^2(\mathbb{R})$ with local point δ-interactions supported on the set X, with possibly non-real coupling constants. Such operators are obtained by choosing $B = \operatorname{diag}(\alpha_n)$ with $\alpha_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. The constant α_n can be viewed as intensity (or strength) of the point δ-interaction supported on x_n; cf. [7, Chapter III.2].

10. Quantum graphs with δ-type vertex couplings

In this section we apply the results of the abstract part of this paper to Laplacians on metric graphs. For a survey on this actively developing field and references we refer the reader to the monograph [32] and the survey articles [31,104,106]. In the present section we consider the Laplacian on a finite, not necessarily compact metric graph, equipped with δ or more general non-self-adjoint vertex couplings; for further recent work on non-self-adjoint quantum graphs see [89,90,126]. Furthermore, for the treatment of quantum graphs via boundary triples and similar techniques we refer to, e.g. [46,59, 61,109,120,123].

Let G be a finite graph consisting of a finite set V of vertices and a finite set E of edges, where we allow infinite edges, i.e. edges 'connecting a vertex to a point ∞'. Without loss of generality we assume that there are no vertices of degree 0, i.e. each vertex belongs to at least one edge, and that G does not contain loops, i.e. no edge connects a vertex to itself; this can always be achieved by introducing additional vertices to the graph. We equip each finite edge $e \in E$ with a length $L(e) > 0$ and identify it with the interval $[0, L(e)]$. Moreover, we identify each infinite edge with the interval $[0, \infty)$. This identification gives rise to a natural metric on G and to a natural L^2 space $L^2(G)$ on G. For a vertex $v \in V$ and an edge $e \in E$ we write $v = o(e)$ or $v = t(e)$ if e originates or terminates, respectively, at v, and we occasionally simply write $v \sim e$ if one of these two properties holds. For each vertex v we denote by $\deg(v)$ the vertex degree, that is, the number of edges which originate from or terminate at v.

In $\mathcal{H} = L^2(G)$ we consider the Laplace differential expression

$$(-\Delta f)_e = -f_e'', \qquad e \in E,$$

where f_e denotes the restriction of f to the edge $e \in E$. In the following we write $\widetilde{H}^k(G) := \bigoplus_{e\in E} H^k(0, L(e))$, $k = 1, 2, \ldots$, for the orthogonal sum of the usual Sobolev spaces on the edges of G. We say that a function $f \in \widetilde{H}^k(G)$ is continuous at a vertex v whenever $v \sim e$ and $v \sim \hat{e}$ imply that the values of f_e and $f_{\hat{e}}$ at v coincide. We define

$$H^1(G) := \bigl\{ f \in \widetilde{H}^1(G) : f \text{ is continuous at each } v \in V \bigr\}.$$

Note that for $f \in H^1(G)$ we can just write $f(v)$ for the evaluation of f at a vertex v. For $f \in \widetilde{H}^2(G)$ and a vertex v we write

$$\partial_\nu f(v) := \sum_{t(e)=v} f_e'\bigl(L(e)\bigr) - \sum_{o(e)=v} f_e'(0).$$

In order to construct an ordinary boundary triple let us consider the operators

$$\begin{aligned} Sf &= -\Delta f, \\ \operatorname{dom} S &= \bigl\{ f \in H^1(G) \cap \widetilde{H}^2(G) : f(v) = \partial_\nu f(v) = 0 \text{ for all } v \in V \bigr\}, \end{aligned} \tag{10.1}$$

and

$$Tf = -\Delta f, \qquad \operatorname{dom} T = H^1(G) \cap \widetilde{H}^2(G), \tag{10.2}$$

in $L^2(G)$. Moreover, we choose an enumeration $V = \{v_1, \ldots, v_{|V|}\}$ of the vertex set V and define mappings $\Gamma_0, \Gamma_1 : H^1(G) \cap \widetilde{H}^2(G) \to \mathbb{C}^{|V|}$ by

$$\begin{aligned} (\Gamma_0 f)_j &= \partial_\nu f(v_j), \\ (\Gamma_1 f)_j &= f(v_j), \end{aligned} \qquad j = 1, \ldots, |V|,\ f \in \operatorname{dom} T.$$

The mappings Γ_0 and Γ_1 give rise to an ordinary boundary triple with finite-dimensional boundary space. The following proposition is a consequence of Theorem 2.3 and some elementary calculations. It can also be derived from [60, Lemma 2.14 and Theorem 2.16]. For the convenience of the reader we provide its proof below.

Proposition 10.1. *The operator S in (10.1) is closed, symmetric and densely defined with $S^* = T$ for T in (10.2), and the triple $\{\mathbb{C}^{|V|}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for S^* with the following properties.*

(i) *$A_0 := S^* \upharpoonright \ker \Gamma_0$ coincides with the standard (or Kirchhoff) Laplacian*

$$\begin{aligned} -\Delta_G f &= -\Delta f, \\ \operatorname{dom}(-\Delta_G) &= \Big\{ f \in H^1(G) \cap \widetilde{H}^2(G) : \partial_\nu f(v) = 0 \text{ for all } v \in V \Big\}, \end{aligned} \tag{10.3}$$

and $A_1 := S^* \upharpoonright \ker \Gamma_1$ coincides with the Dirichlet Laplacian

$$\begin{aligned} -\Delta_{\mathrm{D}} f &= -\Delta f, \\ \operatorname{dom}(-\Delta_{\mathrm{D}}) &= \Big\{ f \in H^1(G) \cap \widetilde{H}^2(G) : f(v) = 0 \text{ for all } v \in V \Big\}. \end{aligned}$$

In particular, A_0 and A_1 are both self-adjoint and non-negative operators in $L^2(G)$.

(ii) *For $\lambda \in \mathbb{C} \setminus \sigma(-\Delta_G)$, the corresponding γ-field is given by*

$$\gamma(\lambda) \begin{pmatrix} \partial_\nu f(v_1) \\ \vdots \\ \partial_\nu f(v_{|V|}) \end{pmatrix} = f, \tag{10.4}$$

where $f \in H^1(G) \cap \widetilde{H}^2(G)$ is any function that satisfies $-\Delta f = \lambda f$, and the corresponding Weyl function is given by

$$M(\lambda) \begin{pmatrix} \partial_\nu f(v_1) \\ \vdots \\ \partial_\nu f(v_{|V|}) \end{pmatrix} = \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_{|V|}) \end{pmatrix}. \tag{10.5}$$

For each $\lambda \in \mathbb{C} \setminus \big(\sigma(-\Delta_G) \cup \sigma(-\Delta_{\mathrm{D}})\big)$ we have

$$\big(M(\lambda)^{-1}\big)_{jk} = \begin{cases} \sqrt{\lambda} \sum\limits_{\substack{e \sim v_j \\ L(e) < \infty}} \cot\big(\sqrt{\lambda} L(e)\big) \\ \qquad\qquad -i\sqrt{\lambda} \, \big|\{e : o(e) = v_j, L(e) = \infty\}\big|, & j = k, \\ \sum\limits_{\substack{e \sim v_j, \\ e \sim v_k}} \dfrac{-\sqrt{\lambda}}{\sin\big(\sqrt{\lambda} L(e)\big)}, & j \neq k. \end{cases} \tag{10.6}$$

Proof. Let us verify the conditions of Theorem 2.3. Note first that $T \upharpoonright \ker \Gamma_0$ clearly equals the standard Laplacian (10.3), which is self-adjoint in $L^2(G)$. Moreover, it can easily be seen by explicit construction that the pair $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathbb{C}^{|V|} \times \mathbb{C}^{|V|}$ is surjective. Finally, let us verify the abstract Green identity. For $f, g \in \operatorname{dom} T$ integration by parts yields

$$
\begin{aligned}
(Tf, g)_{L^2(G)} - (f, Tg)_{L^2(G)} &= \sum_{e \in E} \left(\int_0^{L(e)} \left(-f_e''(x)\right) \overline{g_e(x)} \, \mathrm{d}x - \int_0^{L(e)} f_e(x) \left(\overline{-g_e''(x)}\right) \mathrm{d}x \right) \\
&= \sum_{e \in E} \Bigg(\int_0^{L(e)} f_e'(x) \overline{g_e'(x)} \, \mathrm{d}x - \int_0^{L(e)} f_e'(x) \overline{g_e'(x)} \, \mathrm{d}x \\
&\quad + f_e'(0)\overline{g_e(0)} - f_e'\big(L(e)\big)\overline{g_e\big(L(e)\big)} - f_e(0)\overline{g_e'(0)} + f_e\big(L(e)\big)\overline{g_e'\big(L(e)\big)} \Bigg) \\
&= \sum_{j=1}^{|V|} f(v_j) \overline{\left(\sum_{t(e)=v_j} g_e'\big(L(e)\big) - \sum_{o(e)=v_j} g_e'(0) \right)} \\
&\quad - \sum_{j=1}^{|V|} \left(\sum_{t(e)=v_j} f_e'\big(L(e)\big) - \sum_{o(e)=v_j} f_e'(0) \right) \overline{g(v_j)} \\
&= (\Gamma_1 f, \Gamma_0 g)_{\mathbb{C}^{|V|}} - (\Gamma_0 f, \Gamma_1 g)_{\mathbb{C}^{|V|}}.
\end{aligned}
$$

From Theorem 2.3 it follows that S is closed, densely defined and symmetric with $S^* = T$ and that $\{\mathbb{C}^{|V|}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for $T = S^*$. Assertion (i) and the identities (10.4), (10.5) are obvious from the definition of the mappings Γ_0, Γ_1.

It remains to verify the representation of $M(\lambda)^{-1}$ in (10.6). To this end fix $\lambda \in \mathbb{C} \setminus (\sigma(-\Delta_G) \cup \sigma(-\Delta_{\mathrm{D}}))$ and denote by $m^e(\lambda)$ the Dirichlet-to-Neumann map corresponding to the equation $-f'' = \lambda f$ on the interval $[0, L(e)]$; if e is finite then $m^e(\lambda)$ is the matrix satisfying

$$
\begin{aligned}
\begin{pmatrix} f'(0) \\ -f'\big(L(e)\big) \end{pmatrix} &= \begin{pmatrix} m_{11}^e(\lambda) & m_{12}^e(\lambda) \\ m_{21}^e(\lambda) & m_{22}^e(\lambda) \end{pmatrix} \begin{pmatrix} f(0) \\ f\big(L(e)\big) \end{pmatrix} \\
&= \begin{pmatrix} m_{11}^e(\lambda) f(0) + m_{12}^e(\lambda) f\big(L(e)\big) \\ m_{21}^e(\lambda) f(0) + m_{22}^e(\lambda) f\big(L(e)\big) \end{pmatrix}
\end{aligned}
\tag{10.7}
$$

for each $f \in H^2(0, L(e))$ with $-f'' = \lambda f$; if e is infinite then m^e is the scalar function satisfying

$$
f'(0) = m^e(\lambda) f(0) \tag{10.8}
$$

for each $f \in H^2(0, \infty)$ with $-f'' = \lambda f$. Let us define the matrix $\Lambda(\lambda)$ by

$$(\Lambda(\lambda))_{jk} = \begin{cases} \sum\limits_{\substack{o(e)=v_j \\ L(e)<\infty}} m_{11}^e(\lambda) + \sum\limits_{\substack{t(e)=v_j \\ L(e)<\infty}} m_{22}^e(\lambda) + \sum\limits_{\substack{o(e)=v_j \\ L(e)=\infty}} m^e(\lambda), & j = k, \\ \sum\limits_{\substack{o(e)=v_j \\ t(e)=v_k}} m_{12}^e(\lambda) + \sum\limits_{\substack{o(e)=v_k \\ t(e)=v_j}} m_{21}^e(\lambda), & j \neq k. \end{cases} \tag{10.9}$$

We show that $\Lambda(\lambda) = -M(\lambda)^{-1}$. Indeed, let $f \in \ker(T - \lambda)$. Then for $j = 1, \dots, |V|$ we have

$$\begin{aligned}
\big(\Lambda(\lambda)\Gamma_1 f\big)_j &= \sum_{k=1}^{|V|} (\Lambda(\lambda))_{jk} f(v_k) \\
&= \sum_{k\neq j} \Bigg(\sum_{\substack{o(e)=v_j \\ t(e)=v_k}} m_{12}^e(\lambda) + \sum_{\substack{o(e)=v_k \\ t(e)=v_j}} m_{21}^e(\lambda) \Bigg) f(v_k) \\
&\quad + \Bigg(\sum_{\substack{o(e)=v_j \\ L(e)<\infty}} m_{11}^e(\lambda) + \sum_{\substack{t(e)=v_j \\ L(e)<\infty}} m_{22}^e(\lambda) + \sum_{\substack{o(e)=v_j \\ L(e)=\infty}} m^e(\lambda) \Bigg) f(v_j) \\
&= \sum_{k\neq j} \Bigg(\sum_{\substack{o(e)=v_j \\ t(e)=v_k}} \Big(m_{11}^e(\lambda) f(v_j) + m_{12}^e(\lambda) f(v_k) \Big) \\
&\quad + \sum_{\substack{o(e)=v_k \\ t(e)=v_j}} \Big(m_{21}^e(\lambda) f(v_k) + m_{22}^e(\lambda) f(v_j) \Big) \Bigg) + \sum_{\substack{o(e)=v_j \\ L(e)=\infty}} m^e(\lambda) f(v_j),
\end{aligned}$$

where we have used that G does not contain loops. Taking (10.7) and (10.8) into account we obtain that

$$\begin{aligned}
\big(\Lambda(\lambda)\Gamma_1 f\big)_j &= \sum_{k\neq j} \Bigg(\sum_{\substack{o(e)=v_j \\ t(e)=v_k}} f_e'(0) - \sum_{\substack{o(e)=v_k \\ t(e)=v_j}} f_e'\big(L(e)\big) \Bigg) + \sum_{\substack{o(e)=v_j \\ L(e)=\infty}} f_e'(0) \\
&= \sum_{o(e)=v_j} f_e'(0) - \sum_{t(e)=v_j} f_e'\big(L(e)\big) = -(\Gamma_0 f)_j,
\end{aligned}$$

which implies that $\Lambda(\lambda) = -M(\lambda)^{-1}$. Note that m^e can be calculated explicitly and is given by the expressions

$$m^e(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}}{\sin\big(\sqrt{\lambda}L(e)\big)} \begin{pmatrix} -\cos\big(\sqrt{\lambda}L(e)\big) & 1 \\ 1 & -\cos\big(\sqrt{\lambda}L(e)\big) \end{pmatrix} & \text{if } L(e) < \infty, \\ i\sqrt{\lambda} & \text{if } L(e) = \infty. \end{cases}$$

Plugging these representations into (10.9) we arrive at (10.6). $\square$

The next lemma provides a decay property of the Weyl function.

Lemma 10.2. *Let M be the Weyl function corresponding to the boundary triple in Proposition 10.1. Then for each $w_0 < 0$ and $\nu \in (0, \pi)$ there exists $C = C(w_0, \nu) > 0$ such that*

$$\|M(\lambda)\| \leq \frac{C}{\sqrt{|\lambda|}} \qquad \text{for all } \lambda \in \mathbb{U}_{w_0,\nu}, \tag{10.10}$$

where $\mathbb{U}_{w_0,\nu}$ is defined in (6.2).

Proof. Let $w_0 < 0$ and $\nu \in (0, \pi)$. If $|\lambda| \to \infty$ for $\lambda \in \mathbb{U}_{w_0,\nu}$, then $\sqrt{\lambda} \to \infty$ within the sector $\{re^{i\varphi} : r > 0, \varphi \in (\nu/2, \pi - \nu/2)\}$. In particular, $\operatorname{Im} \sqrt{\lambda}$ tends to $+\infty$, and thus

$$-\cot\big(\sqrt{\lambda} L(e)\big) \to i \qquad \text{and} \qquad \frac{1}{\sin\big(\sqrt{\lambda} L(e)\big)} \to 0$$

for all e as $|\lambda| \to \infty$, and the convergence is uniform in $\mathbb{U}_{w_0,\nu}$. Hence it follows from (10.6) that

$$M(\lambda)^{-1} \to -\sqrt{\lambda}\, \operatorname{diag}\big(\deg(v_1)i, \ldots, \deg(v_{|V|})i\big)$$

uniformly as $|\lambda| \to \infty$, $\lambda \in \mathbb{U}_{w_0,\nu}$. It follows that

$$M(\lambda)^{-1}\big(M(\lambda)^{-1}\big)^* \to |\lambda|\, \operatorname{diag}\big(\deg(v_1)^2, \ldots, \deg(v_{|V|})^2\big) \tag{10.11}$$

uniformly as $|\lambda| \to \infty$, $\lambda \in \mathbb{U}_{w_0,\nu}$. Let $C_1 > 1$ be arbitrary. Since the matrix $\operatorname{diag}(\deg(v_1)^2, \ldots, \deg(v_{|V|})^2)$ is positive definite with smallest eigenvalue greater than or equal to 1, it follows from (10.11) that there exists $r_0 > 0$ such that the smallest eigenvalue of $M(\lambda)^{-1}\big(M(\lambda)^{-1}\big)^*$ satisfies

$$\lambda_1\Big(M(\lambda)^{-1}\big(M(\lambda)^{-1}\big)^*\Big) \geq \frac{|\lambda|}{C_1^2}$$

for all $\lambda \in \mathbb{U}_{w_0,\nu}$ with $|\lambda| > r_0$. Thus we obtain that

$$\|M(\lambda)\| = \frac{1}{\sqrt{\lambda_1\big(M(\lambda)^{-1}\big(M(\lambda)^{-1}\big)^*\big)}} \leq \frac{C_1}{\sqrt{|\lambda|}} \tag{10.12}$$

for all $\lambda \in \mathbb{U}_{w_0,\nu}$ with $|\lambda| > r_0$. On the other hand, since $\lambda \mapsto \sqrt{|\lambda|}\|M(\lambda)\|$ is continuous on the compact set

$$\mathbb{U}^0_{w_0,\nu} := \big\{\lambda \in \mathbb{U}_{w_0,\nu} : |\lambda| \leq r_0\big\},$$

there exists $C_2 > 0$ with

$$\|M(\lambda)\| \le \frac{C_2}{\sqrt{|\lambda|}}, \qquad \lambda \in \mathbb{U}^0_{w_0,\nu}. \tag{10.13}$$

With $C := \max\{C_1, C_2\}$ the claim of the lemma follows from the inequalities (10.12) and (10.13). $\square$

The assertions of the following theorem are direct consequences of Proposition 10.1, Lemma 10.2 and Corollary 5.7. For characterizations of self-adjoint vertex conditions for Laplacians on metric graphs we refer the reader to [45,103].

Theorem 10.3. *Let $B \in \mathbb{C}^{|V|\times|V|}$. Then the operator*

$$\begin{aligned} A_{[B]} f &= -\Delta f, \\ \operatorname{dom} A_{[B]} &= \left\{ f \in H^1(G) \cap \widetilde{H}^2(G) : \begin{pmatrix} \partial_\nu f(v_1) \\ \vdots \\ \partial_\nu f(v_{|V|}) \end{pmatrix} = B \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_{|V|}) \end{pmatrix} \right\}, \end{aligned} \tag{10.14}$$

in $L^2(G)$ is m-sectorial, one has $\sigma(A_{[B]}) \subset \overline{W(A_{[B]})}$, the resolvent formula

$$(A_{[B]} - \lambda)^{-1} = (-\Delta_G - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\gamma(\overline{\lambda})^*$$

holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(-\Delta_G)$ and the following assertions are true.

(i) *$A_{[B]}$ is self-adjoint if and only if the matrix B is Hermitian. Moreover, $A_{[B]}$ is maximal dissipative (maximal accumulative, respectively) if and only if B is accumulative (dissipative, respectively).*
(ii) $A_{[B^*]} = A^*_{[B]}$.

Assume in addition that $b \in \mathbb{R}$ is chosen such that

$$\operatorname{Re}(B\xi, \xi) \le b|\xi|^2 \qquad \textit{for all } \xi \in \mathbb{C}^{|V|}.$$

Then the following spectral enclosures hold.

(a) *If $b > 0$ then there exists $C > 0$ such that for each $\xi < -(Cb)^2$*

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \ge \xi,\ |\operatorname{Im} z| \le \frac{2C\|\operatorname{Im} B\|}{1 - \frac{Cb}{(-\xi)^{1/2}}} \big(\operatorname{Re} z - \xi\big)^{1/2} \right\}.$$

(b) *If* $b \leq 0$ *then there exists* $C > 0$ *such that*

$$W(A_{[B]}) \subset \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq 0,\ |\operatorname{Im} z| \leq \frac{2C\|\operatorname{Im} B\|(\operatorname{Re} z)}{(\operatorname{Re} z)^{1/2} - Cb} \right\}.$$

(c) *For each* $w_0 < \min \sigma(A_{\mathrm{N}})$ *and* $\nu \in (0, \pi)$ *there exists* $C > 0$ *such that*

$$\sigma(A_{[B]}) \cap \mathbb{U}_{w_0,\nu} \subset \{ z \in \mathbb{U}_{w_0,\nu} : |z| \leq (C\|B\|)^2 \},$$

where $\mathbb{U}_{w_0,\nu}$ *is defined in* (6.2).

Remark 10.4. Note that the operator $A_{[B]}$ satisfies local matching conditions at all vertices if and only if the matrix B is diagonal, $B = \operatorname{diag}(b_1, \dots b_{|V|})$. In this case $\operatorname{dom} A_{[B]}$ consists of all functions $f \in H^1(G) \cap \widetilde{H}^2(G)$ such that

$$\partial_\nu f(v_j) = b_j f(v_j)$$

holds for $j = 1, \dots, |V|$. These conditions describe δ-couplings of strengths b_j. They have been studied extensively in the literature in the self-adjoint case, i.e. for real $b_1, \dots, b_{|V|}$; see, e.g. [32,60,65,94,106].

Remark 10.5. In more specific situations the spectral estimates in Theorem 10.3 can be made more explicit. Let, for instance, G be combinatorially equal to the complete graph K_n with $n = |V| \geq 2$ vertices, that is, each two vertices are connected by precisely one edge; in particular, $\deg(v_j) = n-1$ for $j = 1, \dots, |V|$. Moreover, let G be equilateral with $L(e) = 1$ for all $e \in E$. It follows from (10.6) that the Weyl function M corresponding to the boundary triple in Proposition 10.1 satisfies

$$\big(M(\lambda)\big)^{-1} = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}} \begin{pmatrix} (n-1)\cos\sqrt{\lambda} & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & -1 \\ -1 & \dots & -1 & (n-1)\cos\sqrt{\lambda} \end{pmatrix}.$$

A straightforward calculation yields that M is given by

$$M(\lambda) = \frac{1}{\alpha(n,\lambda)} \begin{pmatrix} d(n,\lambda) & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 1 \\ 1 & \dots & 1 & d(n,\lambda) \end{pmatrix},$$

where

$$\alpha(n,\lambda) = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}}\left[\left((n-1)\cos\sqrt{\lambda} - \frac{n-2}{2}\right)^2 - \frac{n^2}{4}\right],$$
$$d(n,\lambda) = (n-1)\cos\sqrt{\lambda} - (n-2).$$

Since in this case $M(\lambda)$ is a special case of a circulant matrix, its norm can be calculated and estimated explicitly for $\lambda \in \mathbb{U}_{w_0,\nu}$.

The following example shows that the abstract spectral estimate in Corollary 5.10 cannot be improved in general.

Example 10.6. Let G be a star graph consisting of $|E|$ infinite edges, i.e. each edge of G can be parameterized by the interval $[0,\infty)$ and there exists only one vertex v, which satisfies $o(e) = v$ for all $e \in E$. Then for $B \in \mathbb{C}$ the functions in the domain of the operator $A_{[B]}$ in (10.14) are continuous at v and satisfy the condition

$$-\sum_{e\in E} f_e'(0) = Bf(v).$$

If $B \notin \mathbb{R}$ with $\operatorname{Re} B > 0$ then $A_{[B]}$ has $-B^2/|E|^2$ as its only non-real eigenvalue, as an explicit calculation shows. On the other hand, by Proposition 10.1 (ii) we obtain that $M(\lambda) = i|E|/\sqrt{\lambda}$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and Corollary 5.10 yields that

$$\sigma(A_{[B]}) \cap \big(\mathbb{C}\setminus[0,\infty)\big) \subset \left\{z \in \mathbb{C}\setminus[0,\infty) : |z| \le \frac{|B|^2}{|E|^2}\right\}.$$

This shows that Corollary 5.10 is sharp.

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References

[1] H. Abels, G. Grubb, I. Wood, Extension theory and Kreĭn-type resolvent formulas for nonsmooth boundary value problems, J. Funct. Anal. 266 (2014) 4037–4100.
[2] A.A. Abramov, A. Aslanyan, E.B. Davies, Bounds on complex eigenvalues and resonances, J. Phys. A 34 (2001) 57–72.
[3] M. Adler, M. Bombieri, K.-J. Nagel, Perturbation of analytic semigroups and applications to partial differential equations, J. Evol. Equ. 17 (2017) 1183–1208.
[4] S. Agmon, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, Comm. Pure Appl. Math. 15 (1962) 119–147.
[5] S. Albeverio, J.F. Brasche, M.M. Malamud, H. Neidhardt, Inverse spectral theory for symmetric operators with several gaps: scalar-type Weyl functions, J. Funct. Anal. 228 (2005) 144–188.

[6] S. Albeverio, S.-M. Fei, P. Kurasov, Point interactions: $\mathcal{PT}$-Hermiticity and reality of the spectrum, Lett. Math. Phys. 59 (2002) 227–242.

[7] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, Solvable Models in Quantum Mechanics. With an Appendix by Pavel Exner, AMS Chelsea Publishing, 2005.

[8] S. Albeverio, A. Kostenko, M.M. Malamud, Spectral theory of semibounded Sturm–Liouville operators with local interactions on a discrete set, J. Math. Phys. 51 (2010) 102102, 24 pp.

[9] S. Albeverio, L. Nizhnik, Schrödinger operators with nonlocal point interactions, J. Math. Anal. Appl. 332 (2007) 884–895.

[10] T. Ando, Topics on Operator Inequalities, Division of Applied Mathematics, Research Institute of Applied Electricity, Hokkaido University, Sapporo, 1978.

[11] W. Arendt, A.F.M. ter Elst, The Dirichlet-to-Neumann operator on rough domains, J. Differential Equations 251 (2011) 2100–2124.

[12] W. Arendt, A.F.M. ter Elst, The Dirichlet-to-Neumann operator on exterior domains, Potential Anal. 43 (2015) 313–340.

[13] W. Arendt, A.F.M. ter Elst, J.B. Kennedy, M. Sauter, The Dirichlet-to-Neumann operator via hidden compactness, J. Funct. Anal. 266 (2014) 1757–1786.

[14] Yu. Arlinskii, Abstract boundary conditions for maximal sectorial extensions of sectorial operators, Math. Nachr. 209 (2000) 5–36.

[15] Yu. Arlinskiĭ, Boundary triplets and maximal accretive extensions of sectorial operators, in: Operator Methods for Boundary Value Problems, in: London Math. Soc. Lecture Note Ser., vol. 404, 2012, pp. 35–72.

[16] Yu. Arlinskiĭ, Y. Kovalev, E. Tsekanovskii, Accretive and sectorial extensions of nonnegative symmetric operators, Complex Anal. Oper. Theory 6 (2012) 677–718.

[17] Yu. Arlinskiĭ, A.B. Popov, On m-accretive extensions of a sectorial operator, Mat. Sb. 204 (2013) 3–40 [Russian]; English translation: Sb. Math. 204 (2013) 1085–1121.

[18] Yu. Arlinskiĭ, A. Popov, On m-sectorial extensions of sectorial operators, Zh. Mat. Fiz. Anal. Geom. 13 (2017) 205–241.

[19] W. Bade, R. Freeman, Closed extensions of the Laplace operator determined by a general class of boundary conditions, Pacific J. Math. 12 (1962) 395–410.

[20] R. Beals, Non-local boundary value problems for elliptic operators, Amer. J. Math. 87 (1965) 315–362.

[21] J. Behrndt, G. Grubb, M. Langer, V. Lotoreichik, Spectral asymptotics for resolvent differences of elliptic operators with δ and δ'-interactions on hypersurfaces, J. Spectr. Theory 5 (2015) 697–729.

[22] J. Behrndt, M. Langer, Boundary value problems for elliptic partial differential operators on bounded domains, J. Funct. Anal. 243 (2007) 536–565.

[23] J. Behrndt, M. Langer, On the adjoint of a symmetric operator, J. Lond. Math. Soc. (2) 82 (2010) 563–580.

[24] J. Behrndt, M. Langer, Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples, in: Operator Methods for Boundary Value Problems, in: London Math. Soc. Lecture Note Ser., vol. 404, 2012, pp. 121–160.

[25] J. Behrndt, M. Langer, I. Lobanov, V. Lotoreichik, I.Yu. Popov, A remark on Schatten–von Neumann properties of resolvent differences of generalized Robin Laplacians on bounded domains, J. Math. Anal. Appl. 371 (2010) 750–758.

[26] J. Behrndt, M. Langer, V. Lotoreichik, Schrödinger operators with δ and δ'-potentials supported on hypersurfaces, Ann. Henri Poincaré 14 (2013) 385–423.

[27] J. Behrndt, M. Langer, V. Lotoreichik, Spectral estimates for resolvent differences of self-adjoint elliptic operators, Integral Equations Operator Theory 77 (2013) 1–37.

[28] J. Behrndt, M. Langer, V. Lotoreichik, Trace formulae and singular values of resolvent power differences of self-adjoint elliptic operators, J. Lond. Math. Soc. (2) 88 (2013) 319–337.

[29] J. Behrndt, M. Langer, V. Lotoreichik, J. Rohleder, Quasi boundary triples and semibounded self-adjoint extensions, Proc. Roy. Soc. Edinburgh Sect. A 147 (2017) 895–916.

[30] J. Behrndt, J. Rohleder, Spectral analysis of selfadjoint elliptic differential operators, Dirichlet-to-Neumann maps, and abstract Weyl functions, Adv. Math. 285 (2015) 1301–1338.

[31] G. Berkolaiko, An elementary introduction to quantum graphs, in: Geometric and Computational Spectral Theory, in: Contemp. Math., vol. 700, Amer. Math. Soc., Providence, RI, 2017, pp. 41–72.

[32] G. Berkolaiko, P. Kuchment, Introduction to Quantum Graphs, Mathematical Surveys and Monographs, vol. 186, American Mathematical Society, Providence, RI, 2013.

[33] M.Sh. Birman, M.Z. Solomjak, Asymptotic behavior of the spectrum of variational problems on solutions of elliptic equations in unbounded domains, Funktsional. Anal. i Prilozhen. 14 (1980) 27–35 [Russian]; English translation: Funct. Anal. Appl. 14 (1981) 267–274.

[34] D. Borisov, D. Krejčiřík, $\mathcal{PT}$-symmetric waveguides, Integral Equations Operator Theory 62 (2008) 489–515.
[35] D. Borisov, D. Krejčiřík, The effective Hamiltonian for thin layers with non-Hermitian Robin-type boundary conditions, Asymptot. Anal. 76 (2012) 49–59.
[36] D. Borisov, M. Znojil, On eigenvalues of a $\mathcal{PT}$-symmetric operator in a thin layer, Mat. Sb. 208 (2017) 3–30 [Russian]; English translation: Sb. Math. 208 (2017) 173–199.
[37] J.F. Brasche, P. Exner, Yu.A. Kuperin, P. Šeba, Schrödinger operators with singular interactions, J. Math. Anal. Appl. 184 (1994) 112–139.
[38] F.E. Browder, Estimates and existence theorems for elliptic boundary value problems, Proc. Natl. Acad. Sci. USA 45 (1959) 365–372.
[39] F.E. Browder, On the spectral theory of elliptic differential operators. I, Math. Ann. 142 (1961) 22–130.
[40] B.M. Brown, G. Grubb, I.G. Wood, M-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems, Math. Nachr. 282 (2009) 314–347.
[41] B.M. Brown, M. Marletta, S. Naboko, I.G. Wood, Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices, J. Lond. Math. Soc. (2) 77 (2008) 700–718.
[42] B.M. Brown, M. Marletta, S. Naboko, I. Wood, Inverse problems for boundary triples with applications, Studia Math. 237 (2017) 241–275.
[43] V.M. Bruk, A certain class of boundary value problems with a spectral parameter in the boundary condition, Mat. Sb. (N.S.) 100 (142) (1976) 210–216.
[44] J. Brüning, V. Geyler, K. Pankrashkin, Spectra of self-adjoint extensions and applications to solvable Schrödinger operators, Rev. Math. Phys. 20 (2008) 1–70.
[45] R. Carlson, Adjoint and self-adjoint differential operators on graphs, Electron. J. Differential Equations 1998 (6) (1998), 10 pp.
[46] K.D. Cherednichenko, A.V. Kiselev, L.O. Silva, Functional model for extensions of symmetric operators and applications to scattering theory, Netw. Heterog. Media (2018), in press, arXiv:1703.06220.
[47] J.B. Conway, Functions of One Complex Variable I, Second edition, Graduate Texts in Mathematics, vol. 11, Springer, 1978.
[48] J.-C. Cuenin, Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials, J. Funct. Anal. 272 (2017) 2987–3018.
[49] J.-C. Cuenin, C. Tretter, Non-symmetric perturbations of self-adjoint operators, J. Math. Anal. Appl. 441 (2016) 235–258.
[50] E.B. Davies, Non-self-adjoint differential operators, Bull. Lond. Math. Soc. 34 (2002) 513–532.
[51] M. Demuth, M. Hansmann, G. Katriel, On the discrete spectrum of non-selfadjoint operators, J. Funct. Anal. 257 (2009) 2742–2759.
[52] V.A. Derkach, S. Hassi, M.M. Malamud, H. de Snoo, Boundary relations and their Weyl families, Trans. Amer. Math. Soc. 358 (2006) 5351–5400.
[53] V.A. Derkach, S. Hassi, M.M. Malamud, H. de Snoo, Boundary relations and generalized resolvents of symmetric operators, Russ. J. Math. Phys. 16 (2009) 17–60.
[54] V.A. Derkach, S. Hassi, M.M. Malamud, H. de Snoo, Boundary triplets and Weyl functions. Recent developments, in: Operator Methods for Boundary Value Problems, in: London Math. Soc. Lecture Note Ser., vol. 404, 2012, pp. 161–220.
[55] V.A. Derkach, M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. 95 (1991) 1–95.
[56] V.A. Derkach, M.M. Malamud, The extension theory of Hermitian operators and the moment problem, J. Math. Sci. 73 (1995) 141–242.
[57] D.E. Edmunds, W.D. Evans, Spectral Theory and Differential Operators, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1987.
[58] D.E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers, Differential Operators, Cambridge Tracts in Mathematics, vol. 120, Cambridge University Press, Cambridge, 1996.
[59] Y. Ershova, I. Karpenko, A. Kiselev, Isospectrality for graph Laplacians under the change of coupling at graph vertices, J. Spectr. Theory 6 (2016) 43–66.
[60] Y. Ershova, A. Kiselev, Trace formulae for graph Laplacians with applications to recovering matching conditions, Methods Funct. Anal. Topology 18 (2012) 343–359.
[61] Y. Ershova, A. Kiselev, Trace formulae for Schrödinger operators on metric graphs with applications to recovering matching conditions, Methods Funct. Anal. Topology 20 (2014) 134–148.

[62] P. Exner, Leaky quantum graphs: a review, in: Analysis on Graphs and Its Applications. Selected Papers Based on the Isaac Newton Institute for Mathematical Sciences Programme, Cambridge, UK, 2007, in: Proc. Symp. Pure Math., vol. 77, 2008, pp. 523–564.
[63] P. Exner, M. Fraas, On geometric perturbations of critical Schrödinger operators with a surface interaction, J. Math. Phys. 50 (2009) 112101, 12 pp.
[64] P. Exner, T. Ichinose, Geometrically induced spectrum in curved leaky wires, J. Phys. A 34 (2001) 1439–1450.
[65] P. Exner, M. Jex, On the ground state of quantum graphs with attractive δ-coupling, Phys. Lett. A 376 (2012) 713–717.
[66] P. Exner, S. Kondej, Bound states due to a strong δ-interaction supported by a curved surface, J. Phys. A 36 (2003) 443–457.
[67] P. Exner, H. Kovařík, Quantum Waveguides, Springer, Heidelberg, 2015.
[68] P. Exner, J. Rohleder, Generalized interactions supported on hypersurfaces, J. Math. Phys. 57 (2016) 041507, 23 pp.
[69] P. Exner, K. Yoshitomi, Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop, J. Geom. Phys. 41 (2002) 344–358.
[70] L. Fanelli, D. Krejčiřík, L. Vega, Spectral stability of Schrödinger operators with subordinated complex potentials, J. Spectr. Theory (2018), in press, arXiv:1506.01617.
[71] R.L. Frank, Eigenvalue bounds for Schrödinger operators with complex potentials, Bull. Lond. Math. Soc. 43 (2011) 745–750.
[72] R.L. Frank, Eigenvalues of Schrödinger operators with complex surface potentials, in: Functional Analysis and Operator Theory for Quantum Physics, Europ. Math. Soc., Zürich, 2017, pp. 245–259.
[73] R. Freeman, Closed extensions of the Laplace operator determined by a general class of boundary conditions, for unbounded regions, Pacific J. Math. 12 (1962) 121–135.
[74] R. Freeman, Closed operators and their adjoints associated with elliptic differential operators, Pacific J. Math. 22 (1967) 71–97.
[75] J. Galkowski, H.F. Smith, Restriction bounds for the free resolvent and resonances in lossy scattering, Int. Math. Res. Not. 16 (2015) 7473–7509.
[76] F. Gesztesy, Y. Latushkin, M. Mitrea, M. Zinchenko, Nonselfadjoint operators, infinite determinants, and some applications, Russ. J. Math. Phys. 12 (2005) 443–471.
[77] F. Gesztesy, M. Mitrea, Generalized Robin boundary conditions, Robin-to-Dirichlet maps, Kreĭn-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains, in: Perspectives in Partial Differential Equations, Harmonic Analysis and Applications, in: Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 105–173.
[78] F. Gesztesy, M. Mitrea, A description of all self-adjoint extensions of the Laplacian and Kreĭn-type resolvent formulas on non-smooth domains, J. Anal. Math. 113 (2011) 53–172.
[79] F. Gesztesy, M. Mitrea, M. Zinchenko, Variations on a theme of Jost and Pais, J. Funct. Anal. 253 (2007) 399–448.
[80] F. Gesztesy, M. Mitrea, M. Zinchenko, On Dirichlet-to-Neumann maps and some applications to modified Fredholm determinants, in: Methods of Spectral Analysis in Mathematical Physics, in: Oper. Theory Adv. Appl., vol. 186, Birkhäuser Verlag, Basel, 2009, pp. 191–215.
[81] I.C. Gohberg, M.G. Kreĭn, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI, 1969.
[82] V.I. Gorbachuk, M.L. Gorbachuk, Boundary Value Problems for Operator Differential Equations, Kluwer Academic Publ., Dordrecht, 1991.
[83] A. Grod, S. Kuzhel, Schrödinger operators with non-symmetric zero-range potentials, Methods Funct. Anal. Topology 20 (2014) 34–49.
[84] G. Grubb, A characterization of the non-local boundary value problems associated with an elliptic operator, Ann. Sc. Norm. Super. Pisa (3) 22 (1968) 425–513.
[85] G. Grubb, Remarks on trace estimates for exterior boundary problems, Comm. Partial Differential Equations 9 (1984) 231–270.
[86] G. Grubb, Krein resolvent formulas for elliptic boundary problems in nonsmooth domains, Rend. Semin. Mat. Univ. Politec. Torino 66 (2008) 271–297.
[87] G. Grubb, Distributions and Operators, Graduate Texts in Mathematics, vol. 252, Springer, New York, 2009.
[88] G. Grubb, Spectral asymptotics for Robin problems with a discontinuous coefficient, J. Spectr. Theory 1 (2011) 155–177.
[89] A. Hussein, Maximal quasi-accretive Laplacians on finite metric graphs, J. Evol. Equ. 14 (2014) 477–497.

[90] A. Hussein, D. Krejčiřík, P. Siegl, Non-self-adjoint graphs, Trans. Amer. Math. Soc. 367 (2015) 2921–2957.
[91] A. Ibort, F. Lledó, J.M. Pérez-Pardo, Self-adjoint extensions of the Laplace–Beltrami operator and unitaries at the boundary, J. Funct. Anal. 268 (2015) 634–670.
[92] I.S. Kac, M.G. Krein, On the spectral function of a string, in: F.V. Atkinson (Ed.), Discrete and Continuous Boundary Problems (Russian translation), Addition II, Mir, Moscow, 1968, pp. 648–737; English translation: Amer. Math. Soc. Transl. 2 (103) (1974) 19–102.
[93] I.S. Kac, M.G. Krein, R-functions—analytic functions mapping the upper halfplane into itself, Amer. Math. Soc. Transl. (2) 103 (1974) 1–18.
[94] G. Karreskog, P. Kurasov, I. Trygg Kupersmidt, Schrödinger operators on graphs: symmetrization and Eulerian cycles, Proc. Amer. Math. Soc. 144 (2016) 1197–1207.
[95] T. Kato, Perturbation Theory for Linear Operators, Reprint of the 1980 edition Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[96] A.N. Kochubei, Extensions of symmetric operators and symmetric binary relations, Mat. Zametki 17 (1975) 41–48 [Russian]; English translation: Math. Notes 17 (1975) 25–28.
[97] A.N. Kochubei, One-dimensional point interactions, Ukraïn. Mat. Zh. 41 (1989) 1391–1395 [Russian]; English translation: Ukrainian Math. J. 41 (1989) 1198–1201.
[98] S. Kondej, D. Krejčiřík, Asymptotic spectral analysis in colliding leaky quantum layers, J. Math. Anal. Appl. 446 (2017) 1328–1355.
[99] S. Kondej, V. Lotoreichik, Weakly coupled bound state of 2-D Schrödinger operator with potential-measure, J. Math. Anal. Appl. 420 (2014) 1416–1438.
[100] Yu.G. Kovalev, 1D nonnegative Schrödinger operators with point interactions, Mat. Stud. 39 (2013) 150–163.
[101] A. Kostenko, M. Malamud, 1-D Schrödinger operators with local point interactions on a discrete set, J. Differential Equations 249 (2010) 253–304.
[102] A. Kostenko, M. Malamud, 1-D Schrödinger operators with local point interactions: a review, in: Spectral Analysis, Differential Equations and Mathematical Physics: A Festschrift in Honor of Fritz Gesztesy's 60th Birthday, in: Proc. Symp. Pure Math., vol. 87, Amer. Math. Soc., Providence, RI, 2013, pp. 235–262.
[103] V. Kostrykin, R. Schrader, Kirchhoff's rule for quantum wires, J. Phys. A 32 (1999) 595–630.
[104] V. Kostrykin, R. Schrader, Laplacians on metric graphs: eigenvalues, resolvents and semigroups, in: Quantum Graphs and Their Applications, in: Contemp. Math., vol. 415, Amer. Math. Soc., Providence, RI, 2006, pp. 201–225.
[105] R. de L. Kronig, W.G. Penney, Quantum mechanics of electrons in crystal lattices, Proc. R. Soc. Lond. A 130 (1931) 499–513.
[106] P. Kuchment, Quantum graphs. I. Some basic structures, Waves Random Media 14 (2004) S107–S128.
[107] S. Kuzhel, M. Znojil, Non-self-adjoint Schrödinger operators with nonlocal one-point interactions, Banach J. Math. Anal. 11 (2017) 923–944.
[108] A. Laptev, O. Safronov, Eigenvalue estimates for Schrödinger operators with complex potentials, Comm. Math. Phys. 292 (2009) 29–54.
[109] D. Lenz, C. Schubert, I. Veselić, Unbounded quantum graphs with unbounded boundary conditions, Math. Nachr. 287 (2014) 962–979.
[110] J. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications I, Springer-Verlag, Berlin–Heidelberg–New York, 1972.
[111] V. Lotoreichik, T. Ourmières-Bonafos, On the bound states of Schrödinger operators with δ-interactions on conical surfaces, Comm. Partial Differential Equations 41 (2016) 999–1028.
[112] V. Lotoreichik, J. Rohleder, Schatten–von Neumann estimates for resolvent differences of Robin Laplacians on a half-space, in: Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations, in: Oper. Theory Adv. Appl., vol. 221, Birkhäuser/Springer Basel AG, Basel, 2012, pp. 453–468.
[113] V. Lotoreichik, P. Siegl, Spectra of definite type in waveguide models, Proc. Amer. Math. Soc. 145 (2017) 1231–1246.
[114] M.M. Malamud, Operator holes and extensions of sectorial operators and dual pairs of contractions, Math. Nachr. 279 (2006) 625–655.
[115] M.M. Malamud, Spectral theory of elliptic operators in exterior domains, Russ. J. Math. Phys. 17 (2010) 96–125.
[116] M.M. Malamud, S.M. Malamud, Spectral theory of operator measures in a Hilbert space , Algebra i Analiz 15 (2003) 1–77 [Russian]; English translation: St. Petersburg Math. J. 15 (2004) 323–373.

[117] M.M. Malamud, H. Neidhardt, On the unitary equivalence of absolutely continuous parts of self-adjoint extensions, J. Funct. Anal. 260 (2011) 613–638.
[118] A. Mantile, A. Posilicano, M. Sini, Self-adjoint elliptic operators with boundary conditions on not closed hypersurfaces, J. Differential Equations 261 (2016) 1–55.
[119] V.G. Maz'ya, T.O. Shaposhnikova, Theory of Sobolev Multipliers, Springer-Verlag, Berlin Heidelberg, 2009.
[120] K. Pankrashkin, Spectra of Schrödinger operators on equilateral quantum graphs, Lett. Math. Phys. 77 (2006) 139–154.
[121] A. Pietsch, Eigenvalues and s-Numbers, Cambridge Studies in Advanced Mathematics, vol. 13, Cambridge University Press, Cambridge, 1987.
[122] A. Posilicano, Self-adjoint extensions of restrictions, Oper. Matrices 2 (2008) 1–24.
[123] O. Post, Equilateral quantum graphs and boundary triples, in: Analysis on Graphs and its Applications, in: Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., Providence, RI, 2008, pp. 469–490.
[124] O. Post, Boundary pairs associated with quadratic forms, Math. Nachr. 289 (2016) 1052–1099.
[125] K. Schmüdgen, Unbounded Self-Adjoint Operators on Hilbert Space, Springer, Dordrecht, 2012.
[126] C. Schubert, C. Seifert, J. Voigt, M. Waurick, Boundary systems and (skew-)self-adjoint operators on infinite metric graphs, Math. Nachr. 288 (2015) 1776–1785.
[127] G. Teschl, Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators, American Mathematical Society, Providence, 2009.
[128] M.I. Vishik, On general boundary problems for elliptic differential equations, Tr. Mosk. Mat. Obŝ. 1 (1952) 187–246 [Russian].
[129] J. Weidmann, Lineare Operatoren in Hilberträumen. Teil 1. Grundlagen, Mathematische Leitfäden, B. G. Teubner, Stuttgart, 2000 [German].
[130] H. Winkler, Spectral estimations for canonical systems, Math. Nachr. 220 (2000) 115–141.
[131] C. Wyss, Riesz bases for p-subordinate perturbations of normal operators, J. Funct. Anal. 258 (2010) 208–240.