PROPERTIES OF THE SPECTRUM OF TYPE π_+ AND TYPE π_- OF SELF-ADJOINT OPERATORS IN KREIN SPACES

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Abstract. We investigate spectral points of type π_+ and type π_- for self-adjoint operators in Krein spaces. In particular a sharp lower bound for the codimension of the linear manifold $\mathcal{H}_0$ occuring in the definition of spectral points of type π_+ and type π_- is determined. Furthermore, we describe the structure of the spectrum in a small neighbourhood of such points and we construct a finite dimensional perturbation which turns a real spectral point of type π_+ (type π_-) into a point of positive (resp. negative) type. As an application we study a singular Sturm-Liouville operator with an indefinite weight.

1. Introduction

Let A be a self-adjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. A real point λ_0 of the spectrum $\sigma(A)$ of A is called a spectral point of positive (negative) type, if for every approximative eigensequence (x_n) corresponding to λ_0 all accumulation points of the sequence $([x_n, x_n])$ are positive (resp. negative) (see Definition 2.1).

For bounded self-adjoint operators these spectral points were introduced by P. Lancaster, A. Markus and V. Matsaev in [10]. Thereafter H. Langer, A. Markus and V. Matsaev proved in [12] the existence of a local spectral function for a closed interval Δ containing only spectral points of positive (negative) type of A or points of the resolvent set $\rho(A)$. It turns out that the inner product $[\cdot,\cdot]$ is positive (negative) on the spectral subspaces corresponding to subintervals of Δ and therefore A locally has the same spectral properties as a self-adjoint operator in a Hilbert space.

In this paper the object of investigation are points of the approximative point spectrum of A which are defined in almost the same way as the points of positive and negative type but we require the positivity (negativity) of the accumulation points of $([x_n, x_n])$ only for approximative eigensequences (x_n) belonging to some linear manifold $\mathcal{H}_0$ of finite codimension, cf. Definition 2.2. These points are called of type π_+ and type π_-, respectively, and were recently introduced by T. Azizov, P. Jonas and C. Trunk in [2]. Under compact perturbations of the operator A spectral points of positive (negative) type turn into spectral points of type π_+ (resp. type π_-) or become points of $\rho(A)$. If each point of a closed interval Δ is an accumulation point of $\rho(A)$ and all spectral points in Δ are of type π_+ (type π_-), then there exists a local spectral function, the spectral subspaces corresponding to subintervals of Δ are Pontryagin spaces with finite rank of negativity (resp. positivity) and it follows that A is a so-called locally definitizable operator (see [2], [7]).

In this paper we continue the investigation of spectral points of type π_+ and type π_- of self-adjoint (in general unbounded) operators started in [2]. In Theorem 3.3 we determine a sharp lower bound for the codimension of the linear manifold $\mathcal{H}_0$ occuring in the definition of a spectral point λ_0 of type π_+ (type π_-) and in the case of a locally

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definitizable operator we show that this number is smaller or equal to the negativity (resp. positivity) index of the spectral subspaces corresponding to small intervals containing λ_0, cf. Theorem 3.6.

In Theorem 4.1 we describe the structure of the spectrum in a neighbourhood of a spectral point λ_0 of type π_+ or type π_-. In contrast to [2] we allow λ_0 to be an inner point of the spectrum of A. Using a Fredholm argument we show that there is an open neighbourhood $\mathcal{U}$ of λ_0 such that for all $\lambda \in \mathcal{U}\backslash\{\lambda_0\}$ the eigenspaces are nonnegative (nonpositive) and the dimension of their isotropic parts is constant. In the special case that the latter constant is zero and that λ_0 is real we retrieve a result from [2], namely, in this case $\mathcal{U}\backslash\{\lambda_0\}$ consists only of spectral points of positive (resp. negative) type and of points from $\rho(A)$.

In Section 5 we construct a special finite dimensional perturbation which turns a real point of type π_+ (type π_-) into a point of positive (resp. negative) type. Finally, as an example for spectral points of type π_+ and type π_- we consider a singular Sturm-Liouville operator with the indefinite weight sgn x in Section 6.

2. Preliminaries

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. In the following all topological notions are understood with respect to some Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent.

Let A be a closed operator in $\mathcal{H}$. We define the extended spectrum $\sigma_e(A)$ of A by $\sigma_e(A) := \sigma(A)$ if A is bounded and $\sigma_e(A) := \sigma(A)\cup\{\infty\}$ if A is unbounded. The resolvent set of A is denoted by $\rho(A)$ and the extended resolvent set is defined by $\rho_e(A) := \overline{\mathbb{C}}\backslash\sigma_e(A)$. A point $\lambda_0 \in \mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{ap}(A)$ of A if there exists a sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \dots$, and $\|(A - \lambda_0)x_n\| \to 0$ if $n \to \infty$. For a self-adjoint operator A in $\mathcal{H}$ all real spectral points of A belong to $\sigma_{ap}(A)$ (see e.g. [3, Corollary VI.6.2]).

First we recall the notions of spectral points of positive and negative type and of type π_+ and type π_-. The following definition was given in [10] and [12] for bounded self-adjoint operators.

Definition 2.1. For a self-adjoint operator A in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of *positive* (*negative*) *type of* A if $\lambda_0 \in \sigma_{ap}(A)$ and for every sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad \bigl(\textit{resp.} \ \limsup_{n\to\infty} [x_n, x_n] < 0\bigr).$$

The point ∞ is said to be of *positive* (*negative*) *type of* A if A is unbounded and for every sequence $(x_n) \subset \mathcal{D}(A)$ with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|Ax_n\| = 1$ we have

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad \bigl(\textit{resp.} \ \limsup_{n\to\infty} [Ax_n, Ax_n] < 0\bigr).$$

We denote the set of all points of $\sigma_e(A)$ of positive (negative) type by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$).

It is not difficult to see that the sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\overline{\mathbb{R}}$. Moreover the non-real spectrum of A cannot accumulate to $\sigma_{++}(A) \cup \sigma_{--}(A)$.

In a similar way as above we define some subsets $\sigma_{\pi_+}(A)$ and $\sigma_{\pi_-}(A)$ of $\sigma_e(A)$ containing $\sigma_{++}(A)$ and $\sigma_{--}(A)$, respectively (cf. [2, Definition 5]).

Definition 2.2. For a self-adjoint operator A in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of *type* π_+ (*type* π_-) *of* A if $\lambda_0 \in \sigma_{ap}(A)$ and if there exists a linear submanifold $\mathcal{H}_0 \subset \mathcal{H}$ with $\operatorname{codim} \mathcal{H}_0 < \infty$ such that for every sequence $(x_n) \subset \mathcal{H}_0 \cap \mathcal{D}(A)$ with

$\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad \big(\textit{resp.} \limsup_{n\to\infty} [x_n, x_n] < 0\big). \tag{2.1}$$

The point ∞ is said to be of *type* π_+ (*type* π_-) *of* A if A is unbounded and if there exists a linear submanifold $\mathcal{H}_0 \subset \mathcal{H}$ with $\operatorname{codim} \mathcal{H}_0 < \infty$ such that for every sequence $(x_n) \subset \mathcal{H}_0 \cap \mathcal{D}(A)$ with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|Ax_n\| = 1$ we have

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad \big(\textit{resp.} \limsup_{n\to\infty} [Ax_n, Ax_n] < 0\big).$$

We denote the set of all points of $\sigma_e(A)$ of type π_+ (type π_-) of A by $\sigma_{\pi_+}(A)$ (resp. $\sigma_{\pi_-}(A)$).

By [2, Proposition 6] a point $\lambda_0 \in \sigma_{ap}(A)$ is of type π_+ (type π_-) if and only if there exists a linear manifold $\mathcal{D}_0 \subset \mathcal{D}(A)$ with finite codimension in $\mathcal{D}(A)$ such that for every sequence $(x_n) \subset \mathcal{D}_0$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$ the property (2.1) holds. An analogous statement holds for $\lambda_0 = \infty$. We note that if ∞ is a spectral point of type π_+ (type π_-) of the self-adjoint operator A, then it follows that ∞ is of positive (resp. negative) type, cf. [2, Lemma 10]. Since our investigations in this paper mainly concern spectral points of type π_+ or type π_- which are not of positive or negative type, respectively, no special attention is paid to ∞ in the next sections.

We recall a criterion for a spectral point of A not belonging to $\sigma_{\pi_+}(A)$ ($\sigma_{\pi_-}(A)$) which will be used frequently in this paper, see [2, Theorem 14].

Theorem 2.3. *Let $\lambda_0 \in \sigma_{ap}(A)$. Then $\lambda_0 \notin \sigma_{\pi_+}(A)$ ($\lambda_0 \notin \sigma_{\pi_-}(A)$) if and only if there exists a sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$, which converges weakly to zero such that*

$$\liminf_{n\to\infty} [x_n, x_n] \le 0 \quad \big(\textit{resp.} \limsup_{n\to\infty} [x_n, x_n] \ge 0\big).$$

Recall, that a self-adjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called definitizable if the resolvent set $\rho(A)$ is nonempty and there exists a polynomial $p \neq 0$ such that $[p(A)x, x] \ge 0$ for all $x \in \mathcal{D}(p(A))$. For a detailed study of the spectral properties of definitizable operators we refer to the fundamental paper [11] of H. Langer. Here we note only that a definitizable operator possesses a spectral function and that the non-real spectrum consists of no more than a finite number of eigenvalues.

In the next definition we recall the notion of locally definitizable operators, see e.g. [7, Definition 4.4]. We emphasize that a self-adjoint operator is definitizable if and only if it is definitizable over $\overline{\mathbb{C}}$, cf. [7]. As usual we denote the open half planes by $\mathbb{C}^\pm := \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$.

Definition 2.4. Let Ω be a domain in $\overline{\mathbb{C}}$ which is symmetric with respect to $\mathbb{R}$ such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ are simply connected. Let A be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum of A. The operator A is called *definitizable over* Ω, if the following holds.

(i) For every closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighbourhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \ge 1$, $M > 0$ such that

$$\|(A - \lambda)^{-1}\| \le M \, \frac{(1 + |\lambda|)^{2m-2}}{|\operatorname{Im} \lambda|^m}$$

holds for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$.

(ii) Every point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_λ in $\overline{\mathbb{R}}$ such that the spectral points in each component of $I_\lambda \setminus \{\lambda\}$ are either all of positive type or of negative type with respect to A.

Let A be a self-adjoint operator in a Krein space and let Δ be a closed connected subset of $\overline{\mathbb{R}}$ with

$$\Delta \cap \sigma_e(A) \subset \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A)$$

and assume, in addition, that each point of Δ is an accumulation point of $\rho(A)$. Then, by [2, Theorem 23] there exists a domain Ω with the properties as in Definition 2.4 such that the operator A is definitizable over Ω.

If Ω is a domain as in Definition 2.4 and A is definitizable over Ω, then A possesses a local spectral function E. For the construction and the properties of this spectral function we refer to [7] (see also [6]). We mention only that $E(\Delta)$ is defined and is a self-adjoint projection in $(\mathcal{H}, [\cdot,\cdot])$ for every union Δ of a finite number of connected subsets Δ_i, $i = 1, \ldots, n$, of $\Omega \cap \overline{\mathbb{R}}$, $\overline{\Delta_i} \subset \Omega \cap \overline{\mathbb{R}}$, such that the endpoints of Δ_i belong to $\sigma_{++}(A) \cup \sigma_{--}(A) \cup \rho_e(A)$.

If A is self-adjoint and definitizable over Ω then the real spectral points of type π_+ (type π_-) of A in Ω can be characterized with the help of the local spectral function of A, see [2, Theorem 26].

Theorem 2.5. *Let A be definitizable over Ω and let E be the local spectral function of A. A real point $\lambda \in \sigma(A) \cap \Omega$ belongs to $\sigma_{\pi_+}(A)$ $(\sigma_{\pi_-}(A))$ if and only if there exists a bounded open interval Δ, $\lambda \in \Delta$, such that $E(\Delta)$ is defined and $(E(\Delta)\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity* (*resp. finite rank of positivity*).

3. A lower bound for the codimension of $\mathcal{H}_0$

In this section we investigate the properties of the linear submanifolds $\mathcal{H}_0$ in Definition 2.2 corresponding to a spectral point $\lambda_0 \in \sigma_{\pi_+}(A)$ (or $\lambda_0 \in \sigma_{\pi_-}(A)$). In particular we will establish a sharp lower bound for the codimension of $\mathcal{H}_0$ with the help of a fundamental decomposition of $\ker(A - \lambda_0)$ (see Theorem 3.3) and in the case of a locally definitizable operator A we show that this minimal codimension is smaller or equal to the rank of negativity of the spectral subspaces corresponding to small open intervals containing λ_0, cf. Theorem 3.6.

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. For an arbitrary subset $\mathcal{L}$ of $\mathcal{H}$ we denote the orthogonal companion by $\mathcal{L}^{[\perp]}$,

$$\mathcal{L}^{[\perp]} = \big\{x \in \mathcal{H} : [x,y] = 0 \text{ for all } y \in \mathcal{L}\big\}.$$

It is clear from the definition that $\mathcal{L}^{[\perp]}$ is a subspace. Throughout this paper a subspace is a closed linear manifold. If a subspace $\mathcal{M} \subset \mathcal{H}$ is the direct sum of two subspaces $\mathcal{L}$, $\mathcal{N} \subset \mathcal{H}$ such that $[x,y] = 0$ holds for all $x \in \mathcal{L}, y \in \mathcal{N}$, then we write

$$\mathcal{M} = \mathcal{L}[\dot{+}]\mathcal{N}.$$

Recall that a subspace $\mathcal{N}$ of a Krein space always admits a fundamental decomposition

$$\mathcal{N} = \mathcal{N}_0[\dot{+}]\mathcal{N}_+[\dot{+}]\mathcal{N}_-,$$

where $\mathcal{N}_0 = \mathcal{N} \cap \mathcal{N}^{[\perp]}$, $\mathcal{N}_+$ is a positive subspace of $\mathcal{H}$ and $\mathcal{N}_-$ is a negative subspace of $\mathcal{H}$. Here and in the following the linear manifolds $\mathcal{N}_+$ and $\mathcal{N}_-$ in a fundamental decomposition are always assumed to be closed. Note that if $\mathcal{N} = \mathcal{N}_0'[\dot{+}]\mathcal{N}_+'[\dot{+}]\mathcal{N}_-'$ is another fundamental decomposition of $\mathcal{N}$, then

$$\mathcal{N}_0 = \mathcal{N}_0', \quad \dim \mathcal{N}_+' = \dim \mathcal{N}_+ \quad \text{and} \quad \dim \mathcal{N}_+' = \dim \mathcal{N}_+ \tag{3.1}$$

holds (see [1]).

Lemma 3.1. *Let A be a self-adjoint operator in $\mathcal{H}$, let $\mathcal{K}_0 \subset \mathcal{H}$ be a linear manifold and let $\lambda_0 \in \sigma_{\pi_+}(A) \setminus \{\infty\}$ $(\lambda_0 \in \sigma_{\pi_-}(A) \setminus \{\infty\})$. Let $\mathcal{K}_0^A$ be the closure of $\mathcal{K}_0 \cap \mathcal{D}(A)$ in $\mathcal{D}(A)$ with respect to the graph norm $\|\cdot\|_A$ of A,*

$$\mathcal{K}_0^A := \overline{\mathcal{K}_0 \cap \mathcal{D}(A)}^{\|\cdot\|_A}.$$

Then the following assertions are equivalent.

(i) *All non-zero elements of $\mathcal{K}_0^A \cap \ker(A - \lambda_0)$ are positive (resp. negative) in the Krein space $\mathcal{H}$.*

(ii) *For every sequence $(x_n) \subset \mathcal{K}_0 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$ we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0). \tag{3.2}$$

If, in addition, $\mathcal{K}_0$ is closed, then assertions (i) *and* (ii) *are equivalent to*

(iii) *All non-zero elements of $\mathcal{K}_0 \cap \ker(A - \lambda_0)$ are positive (resp. negative) in the Krein space $\mathcal{H}$.*

Remark 3.2. A linear manifold $\mathcal{H}_0$ from Definition 2.2 satisfies (ii) (and hence (i) and, if it is a subspace also (iii)) in Lemma 3.1.

Proof of Lemma 3.1. We will prove this lemma only for $\lambda_0 \in \sigma_{\pi_+}(A) \setminus \{\infty\}$. Note first, that the equivalence of (i) and (iii) is evident if $\mathcal{K}_0$ is closed.

We show that (ii) implies (i). Let $y \in \mathcal{K}_0^A \cap \ker(A - \lambda_0)$, $y \neq 0$. Then there exists a sequence $(y_n) \subset \mathcal{K}_0 \cap \mathcal{D}(A)$ with $y_n \to y$ and $(A - \lambda_0)y_n \to (A - \lambda_0)y = 0$ as $n \to \infty$. Hence if (ii) holds then it follows from (3.2) that y is a positive vector, i.e. (i) is valid.

In order to show that (i) implies (ii) we verify first that $\mathcal{K}_0^A \cap \ker(A - \lambda_0)$ is uniformly positive. Assume the contrary. Then there exists a sequence (y_n) in $\mathcal{K}_0^A \cap \ker(A - \lambda_0)$, $\|y_n\| = 1$, $n \in \mathbb{N}$, which converges weakly to some y_0 and satisfies

$$[y_n, y_n] \leq \frac{1}{n}, \quad n \in \mathbb{N}.$$

On the space $\ker(A - \lambda_0)$ the norm of $\mathcal{H}$ and the graph norm $\|\cdot\|_A$ are equivalent. Therefore $\mathcal{K}_0^A \cap \ker(A - \lambda_0)$ is closed in $\mathcal{H}$ and $y_0 \in \mathcal{K}_0^A \cap \ker(A - \lambda_0)$. For $n \in \mathbb{N}$ we have

$$|[y_0, y_0]| \leq |[y_0 - y_n, y_0]| + [y_n, y_n]^{\frac{1}{2}} [y_0, y_0]^{\frac{1}{2}}$$

and $y_0 = 0$ follows, which is a contradiction to $\lambda_0 \in \sigma_{\pi_+}(A)$ (cf. Theorem 2.3). Hence $\mathcal{K}_0^A \cap \ker(A - \lambda_0)$ is a uniformly positive subspace of $\mathcal{H}$ and there exists a Krein subspace $\mathcal{G}_0 \subset \mathcal{H}$ with

$$\mathcal{H} = \left(\mathcal{K}_0^A \cap \ker(A - \lambda_0)\right) [\dot{+}] \mathcal{G}_0. \tag{3.3}$$

Assume now that assertion (ii) is not true. Then there exists a sequence (x_n) in $\mathcal{K}_0 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} [x_n, x_n] \leq 0. \tag{3.4}$$

It is no restriction to assume that (x_n) converges weakly to some x_0. By the closedness of the operator A, it follows that $x_0 \in \ker(A - \lambda_0)$. Moreover, (x_n) converges weakly to x_0 in the Hilbert space $(\mathcal{D}(A), \|\cdot\|_A)$, therefore

$$x_0 \in \mathcal{K}_0^A \cap \ker(A - \lambda_0). \tag{3.5}$$

According to (3.3) we write x_n, $n \in \mathbb{N}$, in the form

$$x_n = u_n + v_n$$

for some $u_n \in \mathcal{K}_0^A \cap \ker(A-\lambda_0)$ and some $v_n \in \mathcal{G}_0$. By (3.4) and the positivity of $\mathcal{K}_0^A \cap \ker(A-\lambda_0)$ we obtain

$$0 \geq \lim_{n\to\infty}[x_n,x_n] = \lim_{n\to\infty}\left([u_n,u_n]+[v_n,v_n]\right) \geq \lim_{n\to\infty}[v_n,v_n]. \tag{3.6}$$

Assume that there exits a subsequence (v_{n_k}) of (v_n) with $\lim_{k\to\infty}\|v_{n_k}\| = 0$. Then, by $\|x_n\| = 1$, we have $\lim_{k\to\infty}\|u_{n_k}\| = 1$ and, by (3.6), $0 = \lim_{k\to\infty}[u_{n_k},u_{n_k}]$, which is a contradiction to the fact that $\mathcal{K}_0^A \cap \ker(A-\lambda_0)$ is uniformly positive. Therefore, $\liminf_{n\to\infty}\|v_n\| > 0$ and for $n \in \mathbb{N}$ we have

$$(A-\lambda_0)v_n = (A-\lambda_0)(x_n-u_n) = (A-\lambda_0)x_n,$$

which implies $(A-\lambda_0)v_n \to 0$ as $n\to\infty$. Moreover, if P denotes the self-adjoint projector onto $\mathcal{G}_0$, then $v_n = Px_n$ and, by (3.5), (v_n) converges weakly to zero. Hence together with (3.6) and Theorem 2.3 this is a contradiction to $\lambda_0 \in \sigma_{\pi_+}(A)$. This completes the proof of Lemma 3.1. ☐

With the help of Lemma 3.1 we will determine the minimal possible codimension of a linear manifold $\mathcal{H}_0$ occuring in Definition 2.2. We do not exclude the case of spectral points of positive or negative type. The orthogonal complement and orthogonal sum with respect to the inner product corresponding to the Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ will be denoted by $\perp$ and $\oplus$, respectively.

Theorem 3.3. *Let A be a self-adjoint operator in $\mathcal{H}$ and let $\lambda_0 \in \sigma_{\pi_+}(A)\setminus\{\infty\}$ ($\lambda_0 \in \sigma_{\pi_-}(A)\setminus\{\infty\}$). Let $\mathcal{N}_0[\dot{+}]\mathcal{N}_+[\dot{+}]\mathcal{N}_-$ be a fundamental decomposition of* $\ker(A-\lambda_0)$*. Then the following holds.*

(i) *The finite nonnegative number*

$$\dim\mathcal{N}_0 + \dim\mathcal{N}_- \quad \big(\textit{resp. } \dim\mathcal{N}_0 + \dim\mathcal{N}_+\big)$$

is a lower bound for the codimension of every linear manifold $\mathcal{H}_0$ satisfying the conditions from Definition 2.2 *and does not depend on the fundamental decomposition of* $\ker(A-\lambda_0)$*, that is, for every linear manifold $\mathcal{H}_0 \subset \mathcal{H}$ from Definition* 2.2 *we have*

$$\begin{gathered}\operatorname{codim}\mathcal{H}_0 \geq \dim\mathcal{N}_0 + \dim\mathcal{N}_- \\ \big(\textit{resp. } \operatorname{codim}\mathcal{H}_0 \geq \dim\mathcal{N}_0 + \dim\mathcal{N}_+\big).\end{gathered} \tag{3.7}$$

(ii) *The subspace*

$$\mathcal{H}_0' = \mathcal{N}_+ \oplus \ker(A-\lambda_0)^\perp \quad \big(\textit{resp. } \mathcal{H}_0' = \mathcal{N}_- \oplus \ker(A-\lambda_0)^\perp\big) \tag{3.8}$$

has the properties required in Definition 2.2 *and we have*

$$\begin{gathered}\operatorname{codim}\mathcal{H}_0' = \dim\mathcal{N}_0 + \dim\mathcal{N}_- \\ \big(\textit{resp. } \operatorname{codim}\mathcal{H}_0' = \dim\mathcal{N}_0 + \dim\mathcal{N}_+\big).\end{gathered} \tag{3.9}$$

Proof. We will prove this theorem only for $\lambda_0 \in \sigma_{\pi_+}(A)\setminus\{\infty\}$. Note that by Theorem 2.3 we have

$$\dim\mathcal{N}_0 + \dim\mathcal{N}_- < \infty \tag{3.10}$$

and, by (3.1), this number does not depend on the fundamental decomposition of $\ker(A-\lambda_0)$. Since $\operatorname{codim}\mathcal{H}_0 < \infty$ and $\mathcal{H}_0 \cap (\mathcal{N}_0[\dot{+}]\mathcal{N}_-) = \{0\}$ inequality (3.7) is evident and assertion (i) holds. Obviously $\mathcal{H}_0'$ in (3.8) is closed and (3.9) holds. We have

$$\mathcal{H}_0' \cap \ker(A-\lambda_0) = \mathcal{N}_+,$$

and therefore (iii) from Lemma 3.1 is satisfied. From Lemma 3.1 (ii) we conclude that $\mathcal{H}_0'$ has the properties required in Definition 2.2. ☐

Corollary 3.4. *Let* $\lambda_0 \in \sigma_{\pi_+}(A)\setminus\{\infty\}$ $(\lambda_0 \in \sigma_{\pi_-}(A)\setminus\{\infty\})$. *Then* $\lambda_0 \in \sigma_{++}(A)$ (*resp.* $\lambda_0 \in \sigma_{--}(A)$) *if and only if* $\mathcal{N}_0 = \mathcal{N}_- = \{0\}$ (*resp.* $\mathcal{N}_0 = \mathcal{N}_+ = \{0\}$).

The following statement was proved in [2]. Here it follows immediately from Theorem 3.3.

Corollary 3.5. *If* $\lambda_0 \in \sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$ $(\lambda_0 \in \sigma_{\pi_-}(A) \setminus \sigma_{--}(A))$ *then* λ_0 *is an eigenvalue of* A *with a corresponding nonpositive* (*resp. nonnegative*) *eigenvector. In particular, we have*

$$\sigma_{\pi_+}(A) \setminus \overline{\mathbb{R}} \subset \sigma_p(A) \quad \textit{and} \quad \sigma_{\pi_-}(A) \setminus \overline{\mathbb{R}} \subset \sigma_p(A).$$

In the next theorem we consider the case that λ_0 is a real spectral point of type π_+ or type π_- of a locally definitizable operator. We note that if $\lambda_0 \in \sigma_{\pi_+}(A) \cap \mathbb{R}$ is an accumulation point of the resolvent set of A, then there exists a domain Ω with the properties as in Definition 2.4 such that $\lambda_0 \in \Omega$ and A is definitizable over Ω, cf. [2, Theorem 23] and Section 2. The algebraic eigenspace of A corresponding to λ_0 will be denoted by $\mathcal{L}_{\lambda_0}(A)$.

Theorem 3.6. *Let* A *be a self-adjoint operator in* $\mathcal{H}$ *and assume that* A *is definitizable over* Ω. *Let* $\lambda_0 \in \sigma_{\pi_+}(A)$ $(\lambda_0 \in \sigma_{\pi_-}(A))$ *belong to* $\Omega \cap \mathbb{R}$ *and let* $\Delta \subset \Omega \cap \mathbb{R}$ *be a closed interval such that* λ_0 *is an inner point of* Δ *and*

$$\Delta \setminus \{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A) \quad \big(\textit{resp. } \Delta \setminus \{\lambda_0\} \subset \sigma_{--}(A) \cup \rho(A)\big)$$

holds. Then the following assertions (i)-(iii) *are true.*

(i) *The spectral projection* $E(\Delta)$ *is defined and* $(E(\Delta)\mathcal{H}, [\cdot,\cdot])$ *is a Pontryagin space with finite rank of negativity* κ_- (*resp. finite rank of positivity* κ_+).

(ii) *If* $\mathcal{N}_0[\dot+]\mathcal{N}_+[\dot+]\mathcal{N}_-$ *and* $\mathcal{L}_0[\dot+]\mathcal{L}_+[\dot+]\mathcal{L}_-$ *are fundamental decompositions of* $\ker(A - \lambda_0)$ *and* $\mathcal{L}_{\lambda_0}(A)$, *respectively, then we have*

$$\begin{aligned} &\dim \mathcal{N}_0 + \dim \mathcal{N}_- \le \dim \mathcal{L}_0 + \dim \mathcal{L}_- = \kappa_- \\ &\big(\textit{resp. } \dim \mathcal{N}_0 + \dim \mathcal{N}_+ \le \dim \mathcal{L}_0 + \dim \mathcal{L}_+ = \kappa_+\big). \end{aligned} \tag{3.11}$$

(iii) *If* $\mathcal{H}_0 \subseteq \mathcal{H}$ *is a subspace as in Definition* 2.2 *such that* $\operatorname{codim}\mathcal{H}_0$ *is minimal, then* $\operatorname{codim}\mathcal{H}_0 = \kappa_-$ $(\operatorname{codim}\mathcal{H}_0 = \kappa_+)$ *if and only if*

$$\dim \mathcal{N}_0 + \dim \mathcal{N}_- = \dim \mathcal{L}_0 + \dim \mathcal{L}_-$$
$$\big(\textit{resp. } \dim \mathcal{N}_0 + \dim \mathcal{N}_+ = \dim \mathcal{L}_0 + \dim \mathcal{L}_+\big).$$

Proof. We prove the theorem in the case $\lambda_0 \in \sigma_{\pi_+}(A)$. Assertion (i) was already proved in [2]. Let us show (ii). According to [3, Theorem IX.2.5] we find subspaces $\mathcal{P}, \mathcal{M} \subseteq E(\Delta)\mathcal{H}$ such that $\mathcal{P}$ is neutral, skewly linked to $\mathcal{L}_0$ and

$$E(\Delta)\mathcal{H} = \mathcal{L}_+[\dot+]\mathcal{L}_-[\dot+](\mathcal{L}_0 \dotplus \mathcal{P})[\dot+]\mathcal{M} \;\text{ and }\; \mathcal{L}_{\lambda_0}(A)^{[\perp]} = \mathcal{L}_0[\dot+]\mathcal{M}$$

hold, where $\mathcal{L}_{\lambda_0}(A)^{[\perp]}$ denotes the orthogonal companion of $\mathcal{L}_{\lambda_0}(A)$ in $E(\Delta)\mathcal{H}$. Since $A|E(\Delta)\mathcal{H}$ is a definitizable operator in $E(\Delta)\mathcal{H}$ and $\mathcal{L}_{\lambda_0}(A) = \mathcal{L}_{\lambda_0}(A|E(\Delta)\mathcal{H})$ we conclude from [11, Propositions II.5.1 and II.5.2] that the subspace $\mathcal{L}_{\lambda_0}(A)^{[\perp]}$ is nonnegative and therefore $\mathcal{M}$ is positive. In order to verify $\dim \mathcal{L}_0 + \dim \mathcal{L}_- = \kappa_-$ we show that $\mathcal{L}_0[\dot+]\mathcal{L}_-$ is a maximal nonpositive subspace in $E(\Delta)\mathcal{H}$. Assume that this is not true. Then there exists a vector

$$e := \ell_+ + m + p, \qquad \ell_+ \in \mathcal{L}_+,\; m \in \mathcal{M},\; p \in \mathcal{P},$$

such that $\mathcal{L}_0 \dotplus \mathcal{L}_- \dotplus \operatorname{span}\{e\}$ is nonpositive. From $[e,e] \le 0$ we obtain $e = p$. Since $\mathcal{L}_0$ and $\mathcal{P}$ are skewly linked we find $\ell_0 \in \mathcal{L}_0$ with $[p,\ell_0] > 0$. But then

$$p + \ell_0 \in \mathcal{L}_0 \dotplus \mathcal{L}_- \dotplus \operatorname{span}\{e\}$$

is a positive vector, which is a contradiction, i.e. $\dim \mathcal{L}_0 + \dim \mathcal{L}_- = \kappa_-$. The inequality in (3.11) follows from $\mathcal{N}_0[\dot{+}]\mathcal{N}_- \subseteq E(\Delta)\mathcal{H}$. Finally, assertion (iii) is an immediate consequence of (ii) and Theorem 3.3. $\square$

The following simple example shows that in general the number $\dim \mathcal{N}_0 + \dim \mathcal{N}_-$ in Theorem 3.6 (ii) does not coincide with the negativity index κ_- of the corresponding spectral subspace.

Example. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then A is self-adjoint in the Pontryagin space $\Pi_2 := (\mathbb{C}^4, (J\cdot,\cdot))$. The rank of negativity of Π_2 is 2 and the rank of positivity is 2. It is obvious that

$$\ker(A-1) = \operatorname{span}\,(1,0,0,0)^{\mathrm{T}}$$

is a neutral subspace and the algebraic eigenspace $\mathcal{L}_1(A)$ coincides with Π_2. Thus the numbers κ_+ and κ_- in Theorem 3.6 are both equal to two but $\dim \mathcal{N}_0 + \dim \mathcal{N}_-$ is one. Here the subspace $\mathcal{H}_0$ from Definition 2.2 can be chosen as the $(\cdot,\cdot)$-orthogonal complement of $\ker(A-1)$, cf. Theorem 3.3 (ii).

4. Structure of the spectrum in a neighbourhood of a spectral point of type π_+ or type π_-

Let A be a self-adjoint operator in the Krein space $\mathcal{H}$, let $\Delta \subset \mathbb{R}$ be a closed bounded interval and assume that

$$\Delta \cap \sigma(A) \subset \sigma_{\pi_+}(A)$$

holds. If $\rho(A) \neq \emptyset$ and if each point of Δ is an accumulation point of $\rho(A)$, then by [2, Theorem 18] there exists an open neighbourhood $\mathcal{U}$ in $\mathbb{C}$ of Δ such that $\mathcal{U} \setminus \mathbb{R} \subset \rho(A)$ and there are at most finitely many points $\lambda_1, \dots, \lambda_n$ in $\mathcal{U} \cap \mathbb{R}$ which belong to $\sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$. In the following theorem we give a more complete description.

Theorem 4.1. *Let A be a self-adjoint operator in $\mathcal{H}$ and let Δ be a closed bounded interval with*

$$\Delta \cap \sigma(A) \subset \sigma_{\pi_+}(A) \quad \big(\Delta \cap \sigma(A) \subset \sigma_{\pi_-}(A)\big).$$

For $\lambda \in \mathbb{C}$ denote by $\mathcal{N}_0(A-\lambda)[\dot{+}]\mathcal{N}_+(A-\lambda)[\dot{+}]\mathcal{N}_-(A-\lambda)$ a fundamental decomposition of $\ker(A-\lambda)$. *Then there exist an open neighbourhood $\mathcal{U}$ in $\mathbb{C}$ of Δ, a finite nonnegative number α and at most finitely many points $\lambda_1, \dots, \lambda_n$ in $\sigma_{\pi_+}(A)\setminus\sigma_{++}(A)$ $(\sigma_{\pi_-}(A)\setminus\sigma_{--}(A))$ such that for all $\lambda \in \mathcal{U} \setminus \{\lambda_1, \dots, \lambda_n\}$ we have*

$$\alpha = \dim \mathcal{N}_0(A-\lambda) \leq \min_{j=1,\dots,n} \dim \mathcal{N}_0(A-\lambda_j) \tag{4.1}$$

and

$$\dim \mathcal{N}_-(A-\lambda) = 0 \quad \big(\textit{resp.}\ \dim \mathcal{N}_+(A-\lambda) = 0\big). \tag{4.2}$$

Moreover, in the case $\alpha = 0$ we have

$$\mathcal{U} \setminus \mathbb{R} \subset \rho(A) \quad \textit{and} \quad (\mathcal{U} \cap \sigma(A) \cap \mathbb{R}) \setminus \{\lambda_1, \dots, \lambda_n\} \subset \sigma_{++}(A)$$
$$\big(\textit{resp.}\ \mathcal{U} \setminus \mathbb{R} \subset \rho(A) \quad \textit{and} \quad (\mathcal{U} \cap \sigma(A) \cap \mathbb{R}) \setminus \{\lambda_1, \dots, \lambda_n\} \subset \sigma_{--}(A)\big)$$

and in the case $\alpha > 0$

$$\mathcal{U} \subset \sigma_{\pi_+}(A) \setminus \sigma_{++}(A) \quad \big(\textit{resp.}\ \mathcal{U} \subset \sigma_{\pi_+}(A) \setminus \sigma_{--}(A)\big)$$

holds, in particular $\mathcal{U} \subset \sigma_p(A)$.

Proof. We prove the theorem only for $\Delta \cap \sigma(A) \subset \sigma_{\pi_+}(A)$. Let $\lambda_0 \in \Delta$. We will show that there exist an open neighbourhood $\mathcal{U}_{\lambda_0}$ of λ_0 and a finite nonnegative number α_{λ_0} such that

$$\alpha_{\lambda_0} = \dim \mathcal{N}_0(A-\lambda) \le \dim \mathcal{N}_0(A-\lambda_0) \quad \text{and} \quad \dim \mathcal{N}_-(A-\lambda) = 0 \tag{4.3}$$

holds for all $\lambda \in \mathcal{U}_{\lambda_0}\backslash\{\lambda_0\}$.

If $\lambda_0 \notin \sigma_{ap}(A)$, then $\lambda_0 \in \rho(A)$ and there exists an open neighbourhood $\mathcal{U}_{\lambda_0} \subset \rho(A)$ of λ_0 such that relation (4.3) holds with $\alpha_{\lambda_0} = 0$.

If $\lambda_0 \in \sigma_{ap}(A)$, there exists (see [2, Lemma 12]) an open neighbourhood $\mathcal{V}_{\lambda_0}$ of λ_0 with $\mathcal{V}_{\lambda_0} \cap \sigma_{ap}(A) \subset \sigma_{\pi_+}(A)$. We set

$$\widetilde{\mathcal{N}}_0 := \operatorname{span}\left\{\mathcal{N}_0(A-\lambda) : \lambda \in \mathcal{V}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R}),\ \ \lambda \neq \lambda_0\right\}$$

and

$$\widetilde{\mathcal{N}}_- := \operatorname{span}\left\{\mathcal{N}_-(A-\lambda) : \lambda \in \mathcal{V}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R}),\ \ \lambda \neq \lambda_0\right\}.$$

Then $\widetilde{\mathcal{N}}_0$ is neutral, $\widetilde{\mathcal{N}}_-$ is negative and $[x,y] = 0$ for all $x \in \widetilde{\mathcal{N}}_0$, $y \in \widetilde{\mathcal{N}}_-$. Denote by $\mathcal{L}_0$ the closure of $\widetilde{\mathcal{N}}_0$ and by $\mathcal{L}$ the closure of $\widetilde{\mathcal{N}}_0[\dot{+}]\widetilde{\mathcal{N}}_-$ in $\mathcal{H}$. Then $\mathcal{L}_0$ is a neutral subspace and $\mathcal{L}$ is a nonpositive subspace of $\mathcal{H}$. Denote by $A_{\mathcal{L}_0}$ ($A_{\mathcal{L}}$) the closure of $A|\widetilde{\mathcal{N}}_0$ (resp. $A|\widetilde{\mathcal{N}}_0[\dot{+}]\widetilde{\mathcal{N}}_-$) in $\mathcal{L}_0$ (resp. $\mathcal{L}$). Obviously $A_{\mathcal{L}_0} - \lambda_0$ (resp. $A_{\mathcal{L}} - \lambda_0$) maps $\widetilde{\mathcal{N}}_0$ (resp. $\widetilde{\mathcal{N}}_0[\dot{+}]\widetilde{\mathcal{N}}_-$) onto itself, hence it has a dense range.

Assume that the range of $A_{\mathcal{L}_0} - \lambda_0$ (resp. of $A_{\mathcal{L}} - \lambda_0$) is not closed. Then for any $\epsilon > 0$ and every subspace $\mathcal{M}$ of $\mathcal{L}_0$ (resp. $\mathcal{L}$) with finite codimension in $\mathcal{L}_0$ (resp. $\mathcal{L}$) there exists an $f \in \mathcal{M} \cap \mathcal{D}(A_{\mathcal{L}_0})$ (resp. $f \in \mathcal{M} \cap \mathcal{D}(A_{\mathcal{L}})$) such that $\|f\| = 1$ and $\|(A_{\mathcal{L}_0} - \lambda_0)f\| < \epsilon$ (resp. $\|(A_{\mathcal{L}} - \lambda_0)f\| < \epsilon$). Hence there exists an orthonormal sequence $(f_n) \subset \mathcal{D}(A_{\mathcal{L}_0})$ (resp. $(f_n) \subset \mathcal{D}(A_{\mathcal{L}})$) such that $(A_{\mathcal{L}_0} - \lambda_0)f_n \to 0$ (resp. $(A_{\mathcal{L}} - \lambda_0)f_n \to 0$) as $n \to \infty$. The sequence (f_n) converges weakly to zero. Since $f_n \in \mathcal{L}_0$ (resp. $f_n \in \mathcal{L}$) we have for $n \in \mathbb{N}$

$$[f_n, f_n] = 0 \quad (\text{resp. } [f_n, f_n] \le 0).$$

From $A_{\mathcal{L}_0}, A_{\mathcal{L}} \subset A$ and Theorem 2.3 we find that this contradicts $\lambda_0 \in \sigma_{\pi_+}(A)$, hence the ranges of $A_{\mathcal{L}_0} - \lambda_0$ and $A_{\mathcal{L}} - \lambda_0$ are closed. In particular $A_{\mathcal{L}_0} - \lambda_0$ and $A_{\mathcal{L}} - \lambda_0$ are surjective operators in $\mathcal{L}_0$ and $\mathcal{L}$, respectively. Thus, they are semi-Fredholm.

For some $\lambda \in \mathcal{V}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R})$, $\lambda \neq \lambda_0$, let $x \in \ker(A_{\mathcal{L}_0} - \lambda)$. Then $x \in \ker(A-\lambda)$ and we write $x = x_0 + x_+ + x_-$ with $x_0 \in \mathcal{N}_0(A-\lambda)$, $x_+ \in \mathcal{N}_+(A-\lambda)$ and $x_- \in \mathcal{N}_-(A-\lambda)$. There exists a sequence (x_n) in $\widetilde{\mathcal{N}}_0$ with $x_n \to x$ as $n \to \infty$. By $[x_+, x_n] = [x_-, x_n] = 0$ for $n \in \mathbb{N}$, we have

$$[x_+, x_+] = [x_+, x] = \lim_{n\to\infty}[x_+, x_n] = 0, \quad [x_-, x_-] = [x_-, x] = \lim_{n\to\infty}[x_-, x_n] = 0.$$

Therefore $x_+ = x_- = 0$ and we obtain $x \in \mathcal{N}_0(A-\lambda)$. Together with $\mathcal{N}_0(A-\lambda) \subset \ker(A_{\mathcal{L}_0} - \lambda)$ we conclude for $\lambda \in \mathcal{V}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R})$, $\lambda \neq \lambda_0$,

$$\ker(A_{\mathcal{L}_0} - \lambda) = \mathcal{N}_0(A-\lambda) \tag{4.4}$$

and, using similar arguments as above,

$$\ker(A_{\mathcal{L}} - \lambda) = \mathcal{N}_0(A-\lambda)[\dot{+}]\mathcal{N}_-(A-\lambda). \tag{4.5}$$

As $A_{\mathcal{L}_0} - \lambda_0$ and $A_{\mathcal{L}} - \lambda_0$ are surjective semi-Fredholm operators by [9, IV 5.31], (3.10), (4.4) and (4.5) there exist finite nonnegative numbers α_{λ_0} and β_{λ_0} and an open neighbourhood $\widetilde{\mathcal{U}}_{\lambda_0}$ of λ_0, $\widetilde{\mathcal{U}}_{\lambda_0} \subset \mathcal{V}_{\lambda_0}$, such that for all $\lambda \in \widetilde{\mathcal{U}}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R})$, $\lambda \neq \lambda_0$, we have

$$\alpha_{\lambda_0} = \dim\ker(A_{\mathcal{L}_0} - \lambda) = \dim \mathcal{N}_0(A-\lambda) = \dim\ker(A_{\mathcal{L}_0} - \lambda_0) \tag{4.6}$$

and

$$\beta_{\lambda_0} = \dim\ker(A_{\mathcal{L}} - \lambda) = \dim\big(\mathcal{N}_0(A-\lambda)[\dot{+}]\mathcal{N}_-(A-\lambda)\big). \tag{4.7}$$

Since $\mathcal{N}_-(A-\lambda) = 0$ for $\lambda \in \mathbb{C}^+$ and $\dim \mathcal{N}_0(A-\lambda) = \alpha_{\lambda_0}$ for all $\lambda \in \widetilde{\mathcal{U}}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R})$, $\lambda \neq \lambda_0$, relations (4.6) and (4.7) imply $\mathcal{N}_-(A-\lambda) = 0$ also for $\lambda \in \widetilde{\mathcal{U}}_{\lambda_0} \cap \mathbb{R}$, $\lambda \neq \lambda_0$. Hence we have

$$\alpha_{\lambda_0} = \dim \mathcal{N}_0(A-\lambda) = \dim\ker(A_{\mathcal{L}_0} - \lambda_0) \quad \text{and} \quad \dim \mathcal{N}_-(A-\lambda) = 0 \tag{4.8}$$

for all $\lambda \in \widetilde{\mathcal{U}}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R})$, $\lambda \neq \lambda_0$.

Let $x \in \ker(A_{\mathcal{L}_0} - \lambda_0)$. Then there exists a sequence (x_n) in $\widetilde{\mathcal{N}}_0$ with $x_n \to x$ as $n \to \infty$. For $y \in \ker(A-\lambda_0)$ we have $[x_n, y] = 0$, $n \in \mathbb{N}$, hence $[x,y] = 0$. Thus, $x \in \ker(A-\lambda_0) \cap (\ker(A-\lambda_0))^{[\perp]} = \mathcal{N}_0(A-\lambda_0)$. This and (4.8) imply that (4.3) holds for all $\lambda \in \widetilde{\mathcal{U}}_{\lambda_0} \cap (\mathbb{C}^+ \cup \mathbb{R})$, $\lambda \neq \lambda_0$. It is easily seen that the above reasoning still holds true if $\mathbb{C}^+$ is replaced by $\mathbb{C}^-$. Therefore there exists an open neighbourhood $\mathcal{U}_{\lambda_0}$ of λ_0 such that (4.3) holds for all $\lambda \in \mathcal{U}_{\lambda_0} \setminus \{\lambda_0\}$.

The compactness of the interval Δ implies the existence of $\mathcal{U}$ and of points $\lambda_1, \dots, \lambda_n$ with the properties mentioned in Theorem 4.1 such that (4.1) and (4.2) hold.

In the case $\alpha = 0$ we have $\ker(A-\lambda) = \{0\}$ for $\lambda \in \mathcal{U}\backslash\mathbb{R}$. If $\lambda \in \sigma_{ap}(A)$, then $\lambda_0 \in \sigma_{\pi_+}(A)$ and Corollary 3.4 implies $\lambda \in \sigma_{++}(A)$, which is impossible since $\sigma_{++}(A) \subset \overline{\mathbb{R}}$. Hence for each $\lambda \in \mathcal{U}\backslash\mathbb{R}$ we obtain that ran $(A-\lambda)$ is closed and it is not difficult to see that $\mathcal{U}\backslash\mathbb{R} \subset \rho(A)$ holds. A similar argument shows that with the exception of finitely many points $\mathcal{U} \cap \sigma(A) \cap \mathbb{R}$ belongs to $\sigma_{++}(A)$. The remaining assertions of Theorem 4.1 follow from Corollary 3.5. □

For a self-adjoint operator A in $\mathcal{H}$ and a non-real point λ_0 belonging to $\sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A)$ we have a similar situation, see Lemma 4.2 and Theorem 4.3 below.

Lemma 4.2. *For a non-real $\lambda_0 \in \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A)$ the operator $A - \lambda_0$ is semi-Fredholm and* $\dim\ker(A-\lambda_0) < \infty$.

Proof. Let $\lambda_0 \in \sigma_{\pi_+}(A)$. Let $\mathcal{H}_0 \subset \mathcal{H}$ be a linear manifold with the properties from Definition 2.2 and assume that $\mathcal{H}_0$ is closed, choose e.g. $\mathcal{H}_0 = \mathcal{N}_+ \oplus \ker(A-\lambda_0)^\perp$, cf. Theorem 3.3. Assume that there exists no $\epsilon > 0$ with

$$\|(A-\lambda_0)x\| \geq \epsilon\|x\| \quad \text{for all} \ \ x \in \mathcal{H}_0 \cap \mathcal{D}(A). \tag{4.9}$$

Then there exists a sequence $(x_n) \subset \mathcal{H}_0 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A-\lambda_0)x_n\| \to 0$ as $n \to \infty$. Therefore, by

$$(-\operatorname{Im}\lambda_0) \liminf_{n\to\infty} [x_n, x_n] = \liminf_{n\to\infty} \operatorname{Im}[(A-\lambda_0)x_n, x_n] = 0,$$

$\liminf_{n\to\infty}[x_n, x_n] = 0$ follows, a contradiction to $\lambda_0 \in \sigma_{\pi_+}(A)$. Thus there exists $\epsilon > 0$ such that (4.9) holds and $A - \lambda_0$ is semi-Fredholm with dim ker $(A-\lambda_0) < \infty$. A similar proof holds for points from $\sigma_{\pi_-}(A)$. □

The following theorem is a direct consequence of Theorem 4.1 and Lemma 4.2

Theorem 4.3. *Let K be a connected compact set in $\mathbb{C}$ such that $K \cap \sigma_{ap}(A)$ belongs to $\sigma_{\pi_+}(A)$ ($\sigma_{\pi_-}(A)$). For $\lambda \in \mathbb{C}$ denote by*

$$\mathcal{N}_0(A-\lambda)[\dot{+}]\mathcal{N}_+(A-\lambda)[\dot{+}]\mathcal{N}_-(A-\lambda)$$

a fundamental decomposition of $\ker(A-\lambda)$. *Then there exist an open neighbourhood $\mathcal{U}$ in $\mathbb{C}$ of K, a finite nonnegative number α and at most finitely many points $\lambda_1, \dots, \lambda_n \in K$ which belong to $\sigma_{\pi_+}(A)\backslash\sigma_{++}(A)$ (resp. $\sigma_{\pi_-}(A)\backslash\sigma_{--}(A)$) such that*

$$\alpha = \dim \mathcal{N}_0(A-\lambda) \leq \min_{j=1,\dots,n} \dim \mathcal{N}_0(A-\lambda_j)$$

and

$$\dim \mathcal{N}_-(A-\lambda) = 0 \qquad \big(\textit{resp. } \dim \mathcal{N}_+(A-\lambda) = 0\big)$$

hold for all $\lambda \in \mathcal{U} \setminus \{\lambda_1, \dots, \lambda_n\}$.

We will give an example where the number α from Theorem 4.1 and Theorem 4.3 is larger than zero.

Example. Let $l^2(\mathbb{N})$ denote the Hilbert space of all square summable sequences equipped with the usual inner product. Denote by T the right shift operator and by S the left shift operator in $l^2(\mathbb{N})$. On $\mathcal{H} := l^2(\mathbb{N}) \times l^2(\mathbb{N})$ we introduce an indefinite inner product $[\cdot,\cdot]$ by

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] := (x_2, y_1) + (x_1, y_2), \quad x_1, x_2, y_1, y_2 \in l^2(\mathbb{N}).$$

Then $(\mathcal{H}, [\cdot,\cdot])$ is a Krein space and the operator A, defined by

$$A := \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$$

is self-adjoint. For λ with $|\lambda| < 1$ the operator $A - \lambda$ is Fredholm, we have

$$\dim \ker (A - \lambda) = 1$$

and all eigenvectors corresponding to λ are neutral in $\mathcal{H}$. Hence, the open unit disc belongs to $\sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$ and also to $\sigma_{\pi_-}(A) \setminus \sigma_{--}(A)$.

5. Finite dimensional perturbations

In this section we construct a special finite dimensional perturbation which turns a real point of type π_+ (type π_-) into a point of positive (resp. negative) type. In the case of a definitizable operator this was shown in [8]. For locally definitizable operators it follows from Theorem 2.5 combined with [8].

Theorem 5.1. *Let A be a self-adjoint operator in $\mathcal{H}$ and let $\lambda_0 \in \sigma_{\pi_+}(A) \cap \mathbb{R}$ ($\lambda_0 \in \sigma_{\pi_-}(A) \cap \mathbb{R}$). Let $\mathcal{N}_0[\dot+]\mathcal{N}_+[\dot+]\mathcal{N}_-$ be a fundamental decomposition of* $\ker(A-\lambda_0)$*. Then there exists a finite dimensional bounded self-adjoint operator F with*

$$\dim \operatorname{ran} F = \dim \mathcal{N}_0 + \dim \mathcal{N}_-$$

and

$$\lambda_0 \in \sigma_{++}(A+F) \cup \rho(A+F) \quad \big(\textit{resp. } \lambda_0 \in \sigma_{--}(A+F) \cup \rho(A+F)\big).$$

Moreover, there exists an open neighbourhood $\mathcal{U}$ in $\mathbb{C}$ of λ_0 such that

$$\mathcal{U} \cap \sigma(A+F) \cap \mathbb{R} \subset \sigma_{++}(A+F) \quad \big(\textit{resp. } \mathcal{U} \cap \sigma(A+F) \cap \mathbb{R} \subset \sigma_{--}(A+F)\big)$$

and $\mathcal{U} \setminus \mathbb{R} \subset \rho(A+F)$ holds.

Proof. We will prove this theorem only for $\lambda_0 \in \sigma_{\pi_+}(A) \cap \mathbb{R}$.

The subspace $\mathcal{N}_+$ is uniformly positive. Otherwise there exists a sequence $(y_n) \subset \mathcal{N}_+$, $\|y_n\| = 1$, $n \in \mathbb{N}$, which converges weakly to some $y_0 \in \mathcal{N}_+$ and satisfies $\lim_{n\to\infty}[y_n, y_n] \leq 0$. As in the proof of Lemma 3.1 we conclude from

$$|[y_0, y_0]| \leq |[y_0 - y_n, y_0]| + [y_n, y_n]^{\frac{1}{2}}[y_0, y_0]^{\frac{1}{2}}$$

that $y_0 = 0$, which is a contradiction to $\lambda_0 \in \sigma_{\pi_+}(A)$ (cf. Theorem 2.3).

As $\mathcal{N}_-$ is finite dimensional (see Theorem 3.3), the space $\mathcal{N}_+[\dot+]\mathcal{N}_-$ is a Pontryagin space. Hence there exists a Krein subspace $\mathcal{K}$ of $\mathcal{H}$ with

$$\mathcal{H} = \mathcal{N}_+[\dot+]\mathcal{N}_-[\dot+]\mathcal{K}.$$

Then $\mathcal{D}(A) \cap \mathcal{K}$ is dense in $\mathcal{K}$ and we have $\mathcal{N}_0 \cap (\mathcal{D}(A) \cap \mathcal{K})^{[\perp]} = \{0\}$. $\mathcal{N}_0$ is finite dimensional (see Theorem 3.3) and by [3, Lemma I.10.4] we find a basis $e_1, \dots, e_n$ of $\mathcal{N}_0$ and $f_1, \dots, f_n \in \mathcal{D}(A) \cap \mathcal{K}$ such that $[e_j, f_k] = \delta_{jk}$, $j,k = 1, \dots, n$, holds. Set

$$\mathcal{G} := \mathcal{N}_0 \dot{+} \operatorname{span}\{f_1 \dots, f_n\}.$$

Then $(\mathcal{G}, [\cdot,\cdot])$ is a Krein space. Denote by $J_{\mathcal{G}}$ a fundamental symmetry in $\mathcal{G}$. Then

$$\mathcal{P} := J_{\mathcal{G}} \mathcal{N}_0$$

is a neutral subspace with $\mathcal{G} = \mathcal{N}_0 \dot{+} \mathcal{P}$. There exists a Krein subspace $\mathcal{M}$ of $\mathcal{H}$ with

$$\mathcal{H} = \mathcal{N}_+ [\dot{+}] \mathcal{N}_- [\dot{+}] \mathcal{G} [\dot{+}] \mathcal{M}$$

and

$$\mathcal{H} = \mathcal{N}_+ \dot{+} \mathcal{N}_- \dot{+} \mathcal{N}_0 \dot{+} \mathcal{P} \dot{+} \mathcal{M}.$$

Denote by P_- the self-adjoint projection in the Krein space $\mathcal{H}$ onto $\mathcal{N}_-$ and by Q_0 and Q_1 the bounded projections onto $\mathcal{N}_0$ and $\mathcal{P} \dot{+} \mathcal{M}$, respectively. For $x, y \in \mathcal{H}$ we have that $J_{\mathcal{G}} Q_0 x$ and $J_{\mathcal{G}} Q_0 y$ belong to $(\mathcal{P} \dot{+} \mathcal{M})^{[\perp]}$ and, hence,

$$\begin{aligned} [J_{\mathcal{G}} Q_0 x, y] &= [J_{\mathcal{G}} Q_0 x, Q_0 y + Q_1 y] = [J_{\mathcal{G}} Q_0 x, Q_0 y] \\ &= [Q_0 x, J_{\mathcal{G}} Q_0 y] = [x, J_{\mathcal{G}} Q_0 y]. \end{aligned}$$

Therefore, the operator $J_{\mathcal{G}} Q_0$, considered as an operator in $\mathcal{H}$, is self-adjoint. We define the operator F by

$$F := P_- + J_{\mathcal{G}} Q_0.$$

Then the operator $A + F$ is also self-adjoint. Hence, the real point λ_0 belongs either to $\sigma_{ap}(A+F)$ or to $\rho(A+F)$. Assume $\lambda_0 \in \sigma_{ap}(A+F)$. The space

$$\mathcal{H}_0 := \mathcal{N}_+ [\dot{+}] \mathcal{P} [\dot{+}] \mathcal{M}$$

is closed, has finite codimension and from Lemma 3.1 we obtain that every sequence (x_n) in $\mathcal{H}_0 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0) x_n\| \to 0$ as $n \to \infty$ fulfils

$$\liminf_{n\to\infty} [x_n, x_n] > 0.$$

From $A|\mathcal{H}_0 = (A+F)|\mathcal{H}_0$ we conclude $\lambda_0 \in \sigma_{\pi_+}(A+F)$. Moreover the inclusion $\mathcal{N}_+ \subset \ker(A + F - \lambda_0)$ holds. For $x \in \ker(A + F - \lambda_0)$ it follows

$$\begin{aligned} 0 &= [(A + F - \lambda_0)x, P_- x] = [(A - \lambda_0)x + P_- x + J_{\mathcal{G}} Q_0 x, P_- x] \\ &= [P_- x, P_- x] \end{aligned} \tag{5.1}$$

and

$$0 = [(A + F - \lambda_0)x, Q_0 x] = [P_- x + J_{\mathcal{G}} Q_0 x, Q_0 x] = [J_{\mathcal{G}} Q_0 x, Q_0 x]. \tag{5.2}$$

From (5.1) and (5.2) we conclude $P_- x = Q_0 x = 0$, hence $Fx = 0$. This implies $x \in \ker(A - \lambda_0)$, thus $x \in \mathcal{N}_+$. Therefore

$$\mathcal{N}_+ = \ker(A + F - \lambda_0)$$

and with Corollary 3.4 we have

$$\lambda_0 \in \sigma_{++}(A + F).$$

The remaining assertions of Theorem 5.1 follow from the fact that $\sigma_{++}(A+F)$ is open in $\sigma_e(A+F)$ and that the non-real spectrum of $A+F$ does not accumulate to $\sigma_{++}(A+F)$ (see [2, Lemma 2 and Proposition 4] and [12]). $\square$

6. An example: spectral points of type π_+ and type π_- of indefinite Sturm-Liouville operators

In this section we consider the singular Sturm-Liouville differential expression

$$(\operatorname{sgn} x)\big(-f''(x)+q(x)f(x)\big), \qquad x \in \mathbb{R},$$

with the signum function as indefinite weight and a real potential $q \in L^1_{loc}(\mathbb{R})$, where it is assumed that q is continuous in $\mathbb{R}\backslash[-\eta,\eta]$ for some positive η and the limits

$$q_\infty^+ := \lim_{x\to+\infty} q(x) \quad \text{and} \quad q_\infty^- := \lim_{x\to-\infty} q(x) \tag{6.1}$$

exist.

Let in the following $L^2(\mathbb{R},\operatorname{sgn})$ be the Krein space $(L^2(\mathbb{R}),[\cdot,\cdot])$, where

$$[f,g] = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\operatorname{sgn} x\,dx, \qquad f,g \in L^2(\mathbb{R}),$$

and denote by J the fundamental symmetry of $L^2(\mathbb{R},\operatorname{sgn})$ defined by

$$(Jf)(x) := (\operatorname{sgn} x)f(x), \quad x \in \mathbb{R}.$$

Then $(\cdot,\cdot) := [J\cdot,\cdot]$ is the usual Hilbert scalar product of $L^2(\mathbb{R})$.

Proposition 6.1. *Let $q \in L^1_{loc}(\mathbb{R})$ be a real valued function as above such that the limits q_∞^+ and q_∞^- in* (6.1) *exist. Then the operator*

$$\begin{aligned} (Af)(x) &= (\operatorname{sgn} x)\big(-f''(x)+q(x)f(x)\big),\\ \mathcal{D}(A) &= \big\{f \in L^2(\mathbb{R}) \,|\, f,f' \in W^{1,2}(\mathbb{R}),\ -f''+qf \in L^2(\mathbb{R})\big\}, \end{aligned} \tag{6.2}$$

is self-adjoint in the Krein space $L^2(\mathbb{R},\operatorname{sgn})$ and the interval $(-q_\infty^-,\infty)$ is of type π_+ and the interval $(-\infty,q_\infty^+)$ is of type π_- with respect to A.

Proof. The differential expression $-\frac{d^2}{dx^2}+q$ is in the limit point case at both singular endpoints ∞ and $-\infty$ since the limits $\lim_{x\to+\infty} q(x)$ and $\lim_{x\to-\infty} q(x)$ exist (see e.g. [13]). Hence

$$(JAf)(x) = -f''(x)+q(x)f(x), \quad \mathcal{D}(JA) = \mathcal{D}(A),$$

is a self-adjoint operator in the Hilbert space $(L^2(\mathbb{R}),(\cdot,\cdot))$ and therefore A is self-adjoint in the Krein space $L^2(\mathbb{R},\operatorname{sgn})$.

In the following the elements f of $L^2(\mathbb{R})$ will be identified with the elements $\{f_+,f_-\}$, $f_+ := f|\mathbb{R}^+$, $f_- := f|\mathbb{R}^-$, of $L^2(\mathbb{R}^+)\times L^2(\mathbb{R}^-)$; similarly for $q=\{q_+,q_-\}$. Further we denote the set of all $f_\pm \in L^2(\mathbb{R}^\pm)$ such that $f_\pm, f_\pm' \in W^{1,2}(\mathbb{R}^\pm)$ and $\mp f_\pm'' \pm q_\pm f_\pm \in L^2(\mathbb{R}^\pm)$ by $\mathcal{D}_{\max,\pm}$. Let

$$\mathcal{D}_0 := \big\{f \in \mathcal{D}(A) \,|\, f(0)=f'(0)=0\big\}.$$

Then the codimension of $\mathcal{D}_0$ in $\mathcal{D}(A)$ is two and the operator $S := A|\mathcal{D}_0$ is the direct sum of the closed symmetric operator

$$\begin{aligned} S_+f_+ &= -f_+''+q_+f_+,\\ \mathcal{D}(S_+) &= \big\{f_+ \in \mathcal{D}_{\max,+} \,|\, f_+(0)=f_+'(0)=0\big\}, \end{aligned}$$

in the Hilbert space $(L^2(\mathbb{R}^+),(\cdot,\cdot))$ and the closed symmetric operator

$$\begin{aligned} S_-f_- &= f_-''-q_-f_-,\\ \mathcal{D}(S_-) &= \big\{f_- \in \mathcal{D}_{\max,-} \,|\, f_-(0)=f_-'(0)=0\big\}, \end{aligned}$$

in $(L^2(\mathbb{R}^-), (\cdot,\cdot))$. The deficiency indices of both S_+ and S_- are $(1,1)$. It is well known (see e.g. [5], [13]) that the spectrum of the self-adjoint extension

$$\begin{aligned} T_+ f_+ &= -f_+'' + q_+ f_+, \\ \mathcal{D}(T_+) &= \big\{ f_+ \in \mathcal{D}_{\max,+} \,|\, f_+(0) = 0 \big\}, \end{aligned} \tag{6.3}$$

of S_+ in $L^2(\mathbb{R}^+)$ is semibounded from below, $\sigma(T_+) \cap (-\infty, q_\infty^+)$ consists of eigenvalues of multiplicity one and the essential spectrum of T_+ coincides with $[q_\infty^+, \infty)$. Analogously the spectrum of the operator

$$\begin{aligned} T_- f_- &= f_-'' - q_- f_-, \\ \mathcal{D}(T_-) &= \big\{ f_- \in \mathcal{D}_{\max,-} \,|\, f_-(0) = 0 \big\}, \end{aligned} \tag{6.4}$$

in $L^2(\mathbb{R}^-)$ is semibounded from above, $\sigma(T_-) \cap (-q_\infty^-, \infty)$ consists of eigenvalues of multiplicity one and the essential spectrum of T_- coincides with $(-\infty, -q_\infty^-]$.

Let us show that each point in $\sigma(A) \cap (-q_\infty^-, \infty)$ belongs to $\sigma_{\pi_+}(A)$. Assume that $\lambda \in \sigma(A)$, $\lambda > -q_\infty^-$, and let $(f_n) \subset \mathcal{D}_0$ be a sequence with $\|f_n\| = 1$ and $\|(A-\lambda)f_n\| \to 0$ as $n \to \infty$. Then we have $\|(S-\lambda)f_n\| \to 0$ for $n \to \infty$ and as S is the direct sum of S_+ and S_- we obtain

$$\lim_{n\to\infty} \|(S_+ - \lambda) f_{n,+}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(S_- - \lambda) f_{n,-}\| = 0.$$

Since $\lambda > -q_\infty^-$ we conclude that λ belongs to $\rho(T_-)$ or λ is an eigenvalue of T_- of multiplicity one. In both cases ran $(T_- - \lambda)$ is closed in $L^2(\mathbb{R}^-)$ and therefore ran $(S_- - \lambda)$ is also closed and moreover $(S_- - \lambda)$ is injective. This implies $f_{n,-} \to 0$ as $n \to \infty$ and we obtain

$$\liminf_{n\to\infty} \, [f_n, f_n] = \liminf_{n\to\infty} \, [f_{n,+}, f_{n,+}] = 1,$$

that is, by the remark below Definition 2.2 (cf. [2, Proposition 6]) we have $\lambda \in \sigma_{\pi_+}(A)$. The same argument shows $\sigma(A) \cap (-\infty, q_\infty^+) \subset \sigma_{\pi_-}(A)$. □

Proposition 6.2. *Let q and A be as in Proposition* 6.1 *and assume that $-q_\infty^- < q_\infty^+$. Then the operator A is definitizable.*

Proof. As in the proof of Proposition 6.1 we identify the elements $f \in L^2(\mathbb{R})$ with the elements $\{f_+, f_-\} \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^-)$. We consider the operator

$$\begin{aligned} B\{f_+, f_-\} &= \big\{ -f_+'' + q_+ f_+, f_-'' - q_- f_- \big\}, \\ \mathcal{D}(B) &= \big\{ \{f_+, f_-\} \in \mathcal{D}(T_+) \times \mathcal{D}(T_-) \big\}, \end{aligned}$$

in $L^2(\mathbb{R}, \mathrm{sgn})$ which is the direct sum of the operators T_+ and T_- from (6.3) and (6.4), respectively. Since T_+ is self-adjoint in $(L^2(\mathbb{R}^+), (\cdot,\cdot))$ and T_- is self-adjoint in $(L^2(\mathbb{R}^-), -(\cdot,\cdot))$ the operator B is self-adjoint in $L^2(\mathbb{R}, \mathrm{sgn})$ and B is a so-called fundamentally reducible operator. Here we have $\sigma_{++}(B) = \sigma(T_+)$ and $\sigma_{--}(B) = \sigma(T_-)$.

We fix some point $\mu_0 \in (-q_\infty^-, q_\infty^+)$ which belongs to $\rho(T_+) \cap \rho(T_-)$ and consider the self-adjoint operator $B - \mu_0$ in $L^2(\mathbb{R}, \mathrm{sgn})$. We have $0 \in \rho(B - \mu_0)$ and the assumption $-q_\infty^- < q_\infty^+$ implies that with the exception of at most finitely many positive (negative) eigenvalues with onedimensional negative (resp. positive) eigenspaces all positive (negative) spectral points belong to $\sigma_{++}(B - \mu_0)$ (resp. $\sigma_{--}(B - \mu_0)$). Hence $B - \mu_0$ is an operator with finitely many negative squares (see [11], [4]). If S denotes the symmetric restriction of B as in the proof of Proposition 6.1 then $S - \mu_0$ has also finitely many negative squares and it follows from [4, Proposition 1.1] that $A - \mu_0$ has finitely many negative squares and a nonempty resolvent set. Hence $A - \mu_0$ and A are definitizable (see [11]). □

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