An L^2 model for selfadjoint elliptic differential operators with constant coefficients on bounded domains

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The selfadjoint realization of a second order elliptic differential expression with Dirichlet boundary conditions is shown to be unitarily equivalent to the maximal multiplication operator with the independent variable in an explicit L^2 model space.

1 Introduction

It is well known that every selfadjoint operator in a Hilbert space is unitarily equivalent to a multiplication operator in an abstract L^2 space. For the case of a selfadjoint Sturm–Liouville differential operator on $(0, \infty)$, where, e.g., ∞ is in the limit point case and 0 is a regular endpoint, the integral representation of the classical Titchmarsh–Weyl m-function gives rise to a multiplication operator model in a more explicit L^2 space; cf. [4, 10, 13, 14]. The main objective of the present note is to construct an L^2 model space in a similar way for the Dirichlet realization A of a second order elliptic differential expression with constant coefficients on a bounded domain $\Omega \subset \mathbb{R}^n$, $n > 1$. It will be shown that the maximal multiplication operator in this model space is unitarily equivalent to A. L^2 models for other selfadjoint realizations can be constructed analogously.

2 An L^2 model for a selfadjoint elliptic operator with Dirichlet boundary conditions

Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded domain with a smooth boundary $\partial\Omega$ and denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$, $s \in \mathbb{R}$, the Sobolev spaces of order s on Ω and $\partial\Omega$, respectively. The trace of $u \in H^s(\Omega)$, $s > 1/2$, on $\partial\Omega$ is denoted by $u|_{\partial\Omega}$ and belongs to the space $H^{s-1/2}(\partial\Omega)$. The inner product $(\cdot, \cdot)$ on $L^2(\partial\Omega)$ can be extended by continuity to $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. Let $\iota_\pm$ be isomorphisms from $H^{\pm 1/2}(\partial\Omega)$ onto $L^2(\partial\Omega)$ with $(x, y)_{1/2 \times -1/2} = (\iota_+ x, \iota_- y)$ for all $x \in H^{1/2}(\partial\Omega)$ and $y \in H^{-1/2}(\partial\Omega)$. If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{H}, \mathcal{K})$, or $\mathcal{L}(\mathcal{H})$ if $\mathcal{H} = \mathcal{K}$.

Let $a_{jk} \in \mathbb{C}$, $j, k = 1, \ldots, n$, suppose that the $n \times n$-matrix $(a_{jk})_{j,k=1}^n$ is positive and let $c > 0$. In the following we consider the elliptic differential expression $\Lambda = -\sum_{j,k=1}^n a_{jk}\partial_j\partial_k + c$. It is well known that the operator

$$Au = \Lambda u = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + cu, \quad \operatorname{dom} A = \big\{u \in H^2(\Omega) : u|_{\partial\Omega} = 0\big\}, \tag{1}$$

is a positive selfadjoint operator in $L^2(\Omega)$ with compact resolvent, see, e.g., [6]. Besides the selfadjoint operator A we shall make use of the so-called minimal operator $A_{\min} u = \Lambda u$, $\operatorname{dom} A_{\min} = \{u \in H^2(\Omega) : u|_{\partial\Omega} = \partial_\nu^\Lambda u|_{\partial\Omega} = 0\}$, where $\partial_\nu^\Lambda u|_{\partial\Omega}$ denotes the conormal derivative of u, $\partial_\nu^\Lambda u = \sum_{j,k=1}^n a_{jk}\nu_j\partial_k u$ and $\nu = (\nu_1, \ldots, \nu_n)$ is the normal vector pointing outwards. Clearly, the minimal operator is a restriction of A and hence symmetric. Furthermore, $\operatorname{dom} A_{\min}$ is dense in $L^2(\Omega)$ and $A_{\min}$ is a closed operator with infinite deficiency indices. The adjoint $A_{\min}^*$ is the maximal operator $A_{\max}$ associated to Λ which is defined on $\operatorname{dom} A_{\max} = \{u \in L^2(\Omega) : \Lambda u \in L^2(\Omega)\}$. According to [9, Theorem 2.1] the trace map $u \mapsto u|_{\partial\Omega}$, $u \in H^s(\Omega)$, $s > 1/2$, can be extended by continuity to a surjective mapping from $\operatorname{dom} A_{\max}$ onto $H^{-1/2}(\partial\Omega)$, where $\operatorname{dom} A_{\max}$ is equipped with the graph norm. As A is positive and $\operatorname{dom} A_{\max} = \operatorname{dom} A \dotplus \ker A_{\max}$ holds, it follows that for $y \in L^2(\partial\Omega)$ there is a unique function $u_0(y) \in \ker A_{\max}$ such that $y = \iota_- u_0(y)|_{\partial\Omega}$.

Theorem 2.1 *For λ from the resolvent set $\rho(A)$ of A and $y \in L^2(\partial\Omega)$ we define*

$$M(\lambda)y := -\lambda\, \iota_+\big(\partial_\nu^\Lambda (A - \lambda)^{-1} u_0(y)\big)|_{\partial\Omega}.$$

Then $M(\lambda)$ is a bounded operator in $L^2(\partial\Omega)$, and the function $M : \rho(A) \to \mathcal{L}(L^2(\partial\Omega))$, $\lambda \mapsto M(\lambda)$ is an operator-valued Nevanlinna function, which admits an integral representation

$$M(\lambda) = \alpha + \int_{\mathbb{R}} \Big(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\Big) \mathrm{d}\Sigma(t), \tag{2}$$

where $\alpha \in \mathcal{L}(L^2(\partial\Omega))$ is a selfadjoint operator and $\Sigma : \mathbb{R} \to \mathcal{L}(L^2(\partial\Omega))$ is a nondecreasing operator function which satisfies $\int_{\mathbb{R}} (1 + t^2)^{-1} d\Sigma(t) \in \mathcal{L}(L^2(\partial\Omega))$.

The proof of Theorem 2.1 will be published elsewhere. It makes use of the notion of boundary triplets and Weyl functions

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associated to symmetric operators from [5, 7], see also [1, 3, 8] for the elliptic case.

Let $\Sigma : \mathbb{R} \to \mathcal{L}(L^2(\partial\Omega))$ be the nondecreasing operator function from the integral representation (2). The space $L^2_\Sigma(L^2(\partial\Omega))$ is defined as in [2, 7, 12]. Very roughly speaking it consists of $L^2(\partial\Omega)$-valued functions on $\mathbb{R}$ which are square-integrable with respect to the measure $\mathrm{d}\Sigma$. The next theorem is the main result in this note.

Theorem 2.2 *The Dirichlet operator A in* (1) *is unitarily equivalent to the maximal multiplication operator with the independent variable in $L^2_\Sigma(L^2(\partial\Omega))$.*

Proof. The proof of Theorem 2.2 consists of two steps. In the first step it will be shown that the span of the defect spaces of the minimal operator $A_{\min}$ is dense in $L^2(\Omega)$. In the second step a unitary operator $U : L^2(\Omega) \to L^2_\Sigma(L^2(\partial\Omega))$ will be constructed, which fulfills $A = U^* A_\Sigma U$, where A_Σ is the maximal multiplication operator with the independent variable in the model space $L^2_\Sigma(L^2(\partial\Omega))$.

Step 1. We claim that $A_{\min}$ has no eigenvalues. In fact, assume that $u \in \operatorname{dom} A_{\min}$ is a solution of $A_{\min} u = \lambda u$ for some $\lambda \in \mathbb{R}$ and define the function $\widetilde{u}$ to be the extension of u by 0 on $\mathbb{R}^n \backslash \Omega$. Then $u|_{\partial\Omega} = \partial_\nu^\Lambda u|_{\partial\Omega} = 0$ and the equivalence of the graph norm induced by $A_{\min}$ to the H^2 norm imply $\widetilde{u} \in H^2(\mathbb{R}^n)$. It follows that $\widetilde{u}$ satisfies the equation $\Lambda\widetilde{u} = \lambda\widetilde{u}$ on $\mathbb{R}^n$. Hence $\widetilde{u}$ is an eigenfunction of the selfadjoint operator $\widetilde{A}$ associated to Λ in $L^2(\mathbb{R}^n)$ defined on $\operatorname{dom}\widetilde{A} = H^2(\mathbb{R}^n)$. But $\widetilde{A}$ has no eigenvalues (this can be seen, for example, with the help of the Fourier transform), and therefore $\widetilde{u} = 0$. This implies $u = 0$ and hence $A_{\min}$ has no eigenvalues.

Since the spectrum of the selfadjoint operator A in (1) consists only of eigenvalues it follows that $A_{\min}$ does not contain a nontrivial selfadjoint part, i.e., there is no nontrivial subspace $\mathcal{H} \subset L^2(\Omega)$ which is invariant for the operator $A_{\min}$ such that the restriction $A_{\min} \upharpoonright (\operatorname{dom} A_{\min} \cap \mathcal{H})$ is selfadjoint in $\mathcal{H}$. It is well known (see, e.g., [11]) that this is equivalent to

$$L^2(\Omega) = \overline{\operatorname{span}}\{\ker(A^*_{\min} - \lambda) : \lambda \in \mathbb{C}\backslash\mathbb{R}\} = \overline{\operatorname{span}}\{\ker(A_{\max} - \lambda) : \lambda \in \mathbb{C}\backslash\mathbb{R}\}. \tag{3}$$

Step 2. Let A_Σ be the maximal multiplication operator with the independent variable in $L^2_\Sigma(L^2(\partial\Omega))$ and denote the restriction of A_Σ onto the dense subspace $\{f \in \operatorname{dom} A_\Sigma : \int_\mathbb{R} f d\Sigma = 0\}$ by S_Σ. For further details and the precise definition of $\operatorname{dom} S_\Sigma$ we refer to [12, §7]. For $\lambda \in \mathbb{C}\backslash\mathbb{R}$ we define $\gamma(\lambda) \in \mathcal{L}(L^2(\partial\Omega), L^2(\Omega))$ and $\tilde\gamma(\lambda) \in \mathcal{L}(L^2(\partial\Omega), L^2_\Sigma(L^2(\partial\Omega)))$ by

$$\gamma(\lambda)y = (I + \lambda(A-\lambda)^{-1})u_0(y) \quad \text{and} \quad \tilde\gamma(\lambda)y = (t-\lambda)^{-1}y, \qquad y \in L^2(\partial\Omega),$$

where $u_0(y)$ is the unique function in $\ker A_{\max}$ such that $\iota_- u_0(y)|_{\partial\Omega} = y$. Then we have $\operatorname{ran}\gamma(\lambda) = \ker(A_{\max} - \lambda)$ and $\operatorname{ran}\tilde\gamma(\lambda) = \ker(S^*_\Sigma - \lambda)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Moreover, the equation

$$\gamma(\mu)^*\gamma(\lambda) = \frac{M(\lambda) - M(\mu)^*}{\lambda - \overline{\mu}} = \tilde\gamma(\mu)^*\tilde\gamma(\lambda), \qquad \lambda, \mu \in \mathbb{C}\backslash\mathbb{R}, \tag{4}$$

holds, and $\gamma(\lambda) = (I + (\lambda - i)(A-\lambda)^{-1})\gamma(i)$ and $\tilde\gamma(\lambda) = (I + (\lambda - i)(A_\Sigma - \lambda)^{-1})\tilde\gamma(i)$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. It follows from (3) and (4) that

$$V\left(\sum_{j=0}^{l} \gamma(\lambda_j)y_j\right) = \sum_{j=0}^{l} \tilde\gamma(\lambda_j)y_j, \quad \operatorname{dom} V = \left\{\sum_{j=0}^{l} \gamma(\lambda_j)y_j : \lambda_j \in \mathbb{C}\backslash\mathbb{R}, y_j \in L^2(\partial\Omega), j = 0, \ldots, l, l \in \mathbb{N}\right\},$$

is a well-defined isometric operator with dense domain in $L^2(\Omega)$. As a consequence of [12, Proposition 7.9 (i)] $\operatorname{ran} V$ is dense in $L^2_\Sigma(L^2(\partial\Omega))$ and hence V admits a unique unitary extension $U : L^2(\Omega) \to L^2_\Sigma(L^2(\partial\Omega))$. For $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the equation $U\gamma(\lambda) = \tilde\gamma(\lambda)$ holds by definition of U and for $\lambda \neq i$ we obtain

$$U(A-\lambda)^{-1}\gamma(i) = U\frac{1}{\lambda - i}\big(\gamma(\lambda) - \gamma(i)\big) = \frac{1}{\lambda - i}\big(\tilde\gamma(\lambda) - \tilde\gamma(i)\big) = (A_\Sigma - \lambda)^{-1}\tilde\gamma(i) = (A_\Sigma - \lambda)^{-1}U\gamma(i).$$

This implies $A_\Sigma U u = UAu$ for all $u \in \operatorname{dom} A$, that is, A and A_Σ are unitarily equivalent. □

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