A characterization of the eigenvalues of Schrödinger operators with Dirichlet and Neumann boundary conditions

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The eigenvalues of the selfadjoint Schrödinger operators on a bounded domain with Dirichlet and Neumann boundary conditions are characterized by the singularities of an associated Dirichlet-to-Neumann map and its inverse, respectively.

1 Introduction

A classical result from Sturm-Liouville theory states that the spectra of ordinary selfadjoint differential operators with certain boundary conditions can be completely described by the structure of singularities of an associated analytic object, the Titchmarsh-Weyl coefficient, see, e.g., [4,5,13]. For example, the isolated eigenvalues of the selfadjoint realization of the expression $-f'' + Vf$ in $L^2(0,\infty)$ with a Dirichlet boundary condition at 0, where the potential V is real-valued and bounded, are exactly the poles of the mapping $m(\cdot)$, where $m(\lambda)f_\lambda(0) = f'_\lambda(0)$ holds for all solutions $f_\lambda \in L^2(0,\infty)$ of the equation $-f''_\lambda + Vf_\lambda = \lambda f_\lambda$ and all nonreal λ. The aim of this note is to show that this result can be translated into Schrödinger operators on a bounded domain $\Omega \subseteq \mathbb{R}^n, n \geq 2$. The spectra (which in this case are discrete) of the selfadjoint realizations of the differential expression $-\Delta u + Vu$ with Dirichlet and Neumann boundary conditions will be characterized by an appropriate analogue of the Titchmarsh-Weyl coefficient, the Dirichlet-to-Neumann map, and its inverse, respectively. The result can be extended to larger classes of selfadjoint Schrödinger operators with different, nonlocal boundary conditions. Other recent approaches to spectral problems for partial differential equations using methods from operator and extension theory can be found in, e.g., [1,3,8,11].

2 A characterization of the eigenvalues of the Dirichlet and Neumann operators

Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$, $n \geq 2$. Denote by $H^s(\Omega)$ the Sobolev space of order $s > 0$ on Ω, by $u|_{\partial\Omega}$ the trace of a Sobolev function u at the boundary $\partial\Omega$ of Ω, and by $\partial_\nu u|_{\partial\Omega}$ the Neumann trace of u, that is, the derivative of u in the direction of the outer normal vector ν at $\partial\Omega$. Consider the Schrödinger differential expression $\mathcal{L} = -\Delta + V$ in Ω, where $V \in L^\infty(\Omega)$ is a real-valued, bounded potential. It is well known that the *Dirichlet operator*

$$A_D u = \mathcal{L}u, \quad \operatorname{dom} A_D = \big\{u \in H^2(\Omega) : u|_{\partial\Omega} = 0\big\}$$

and the *Neumann operator*

$$A_N u = \mathcal{L}u, \quad \operatorname{dom} A_N = \big\{u \in H^2(\Omega) : \partial_\nu u|_{\partial\Omega} = 0\big\}$$

associated with $\mathcal{L}$ in the Hilbert space $L^2(\Omega)$ are selfadjoint and have discrete spectra, see, e.g., [7,10].

The central objects which will be used to characterize the spectra of the operators A_D and A_N will be introduced in the following definition. Here for each fixed $\lambda \in \mathbb{C}$, $\mathcal{N}_\lambda$ denotes the space of solutions $u_\lambda \in H^2(\Omega)$ of the equation $\mathcal{L}u_\lambda = \lambda u_\lambda$.

Definition 2.1 For $\lambda \in \rho(A_D)$ the linear mapping

$$M_D(\lambda) : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \quad u_\lambda|_{\partial\Omega} \mapsto \partial_\nu u_\lambda|_{\partial\Omega}, \quad u_\lambda \in \mathcal{N}_\lambda,$$

is called *Dirichlet-to-Neumann map* associated with $\mathcal{L}$. For $\lambda \in \rho(A_N)$ the linear mapping

$$M_N(\lambda) : H^{1/2}(\partial\Omega) \to H^{3/2}(\partial\Omega), \quad \partial_\nu u_\lambda|_{\partial\Omega} \mapsto u_\lambda|_{\partial\Omega}, \quad u_\lambda \in \mathcal{N}_\lambda,$$

is called *Neumann-to-Dirichlet map* associated with $\mathcal{L}$.

It follows from classical trace theorems that the mappings $M_D(\lambda)$ and $M_N(\lambda)$ are well-defined. Moreover, $M_D(\cdot)$ and $M_N(\cdot)$ are strongly holomorphic, see [2], that is, $M_D(\cdot)\varphi$ and $M_N(\cdot)\psi$ are holomorphic on $\rho(A_D)$ and $\rho(A_N)$, respectively, for all $\varphi \in H^{3/2}(\partial\Omega)$ and $\psi \in H^{1/2}(\partial\Omega)$, respectively. For the following we agree to say that the mapping $M_D(\cdot)$ has a pole in $\lambda \in \mathbb{R}$ if there exists $\varphi \in H^{3/2}(\partial\Omega)$ such that $M_D(\cdot)\varphi$ has a pole in λ. Analogously for $M_N(\cdot)$.

It is obvious that each pole of $M_D(\cdot)$ is an eigenvalue of A_D and that each pole of $M_N(\cdot)$ is an eigenvalue of A_N. The main objective of the present note is to show that also the reverse inclusions hold.

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Theorem 2.2 *The eigenvalues of the Dirichlet operator A_D coincide with the poles of the Dirichlet-to-Neumann map $\rho(A_D) \ni \lambda \mapsto M_D(\lambda)$. The eigenvalues of the Neumann operator A_N coincide with the poles of the Neumann-to-Dirichlet map $\rho(A_N) \ni \lambda \mapsto M_N(\lambda)$.*

Proof. **Step 1.** We will first show an identity which connects the Dirichlet-to-Neumann map with the resolvent of A_D. In the following we will make use of the Poisson operator $\gamma_D(\lambda) : H^{3/2}(\partial\Omega) \to \mathcal{N}_\lambda, u_\lambda|_{\partial\Omega} \mapsto u_\lambda, u_\lambda \in \mathcal{N}_\lambda$, which is well-defined for all $\lambda \in \rho(A_D)$. Let $\lambda, \mu \in \rho(A_D)$. Take $g_\lambda \in \mathcal{N}_\lambda$ with $g_\lambda|_{\partial\Omega} = \varphi$ and $h_\mu \in \mathcal{N}_\mu$ with $h_\mu|_{\partial\Omega} = \psi$. By *Green's identity*

$$(\mathcal{L}f, g)_{L^2(\Omega)} - (f, \mathcal{L}g)_{L^2(\Omega)} = (f|_{\partial\Omega}, \partial_\nu g|_{\partial\Omega})_{L^2(\partial\Omega)} - (\partial_\nu f|_{\partial\Omega}, g|_{\partial\Omega})_{L^2(\partial\Omega)}, \quad f, g \in H^2(\Omega),$$

see, e.g., [10], we obtain

$$(\lambda - \bar{\mu})(\gamma_D(\lambda)\varphi, \gamma_D(\mu)\psi)_{L^2(\Omega)} = (\varphi, M_D(\mu)\psi)_{L^2(\partial\Omega)} - (M_D(\lambda)\varphi, \psi)_{L^2(\partial\Omega)},$$

which implies $(M_D(\bar{\mu})\varphi, \psi)_{L^2(\partial\Omega)} = (\varphi, M_D(\mu)\psi)_{L^2(\partial\Omega)}$ and leads to the equation

$$(\lambda - \bar{\mu})\gamma_D(\mu)^*\gamma_D(\lambda)\varphi = M_D(\bar{\mu})\varphi - M_D(\lambda)\varphi, \quad \lambda, \mu \in \rho(A_D), \varphi \in H^{3/2}(\partial\Omega).$$

Here $\gamma_D(\mu)^*$ has to be understood as the adjoint of $\gamma_D(\mu)$ regarded as a densely defined operator from $L^2(\partial\Omega)$ to $L^2(\Omega)$. Using this identity and the equation

$$\gamma_D(\lambda)\varphi = \left(I + (\lambda - \eta)(A_D - \lambda)^{-1}\right)\gamma_D(\eta)\varphi, \quad \lambda, \eta \in \rho(A_D), \varphi \in H^{3/2}(\partial\Omega),$$

which can be verified easily by Green's identity, we compute for $\lambda, \mu, \eta \in \rho(A_D)$, $\eta \neq \bar{\mu}$, $\lambda \neq \eta$, $\lambda \neq \bar{\mu}$, and $\varphi \in H^{3/2}(\partial\Omega)$

$$\begin{aligned}\gamma_D(\mu)^*(A_D - \lambda)^{-1}\gamma_D(\eta)\varphi &= \frac{1}{\lambda - \eta}\frac{M_D(\bar{\mu})\varphi - M_D(\lambda)\varphi}{\lambda - \bar{\mu}} - \frac{1}{\lambda - \eta}\frac{M_D(\bar{\mu})\varphi - M_D(\eta)\varphi}{\eta - \bar{\mu}} \\ &= \frac{M_D(\lambda)\varphi}{(\eta - \lambda)(\lambda - \bar{\mu})} - \frac{M_D(\bar{\mu})\varphi}{(\lambda - \bar{\mu})(\eta - \bar{\mu})} + \frac{M_D(\eta)\varphi}{(\lambda - \eta)(\eta - \bar{\mu})} \qquad (1)\end{aligned}$$

in the same way as in [6].

Step 2. In this step we will show that the span of the spaces $\mathcal{N}_\lambda = \operatorname{ran}\gamma_D(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is dense in $L^2(\Omega)$. For this consider the closed, symmetric restriction S of A_D defined by $S = \mathcal{L} \upharpoonright \{u \in H^2(\Omega) : u|_{\partial\Omega} = \partial_\nu u|_{\partial\Omega} = 0\}$. As is known, the adjoint of S is given by the maximal operator $S^* = \mathcal{L} \upharpoonright \{u \in L^2(\Omega) : \mathcal{L}u \in L^2(\Omega)\}$. Since $\mathcal{N}_\lambda$ is dense in the defect space $\ker(S^* - \lambda)$, it suffices to show that $\operatorname{span}\{\ker(S^* - \lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$ is dense in $L^2(\Omega)$. By [9] and the fact that A_D is a selfadjoint extension of S with discrete spectrum, it is sufficient to show that the operator S has no eigenvalues. Assume there exists an eigenvalue $\lambda \in \mathbb{R}$ of S with corresponding eigenfunction $u \in \operatorname{dom} S$. Extend u by zero to $\widetilde{u} \in H^2(\mathbb{R}^n)$ and extend V to a real-valued function $\widetilde{V} \in L^\infty(\mathbb{R}^n)$. Then $-\Delta\widetilde{u} + \widetilde{V}\widetilde{u} = \lambda\widetilde{u}$ implies the existence of a constant M such that $|\Delta\widetilde{u}| \leq M|\widetilde{u}|$ holds. By known unique continuation theorems, see, e.g., [12, Theorem XIII.63], it follows that $\widetilde{u}$ is trivial, which is a contradiction to the choice of u as an eigenfunction.

Step 3. In this last step we come back to the statement of the theorem. For this, choose μ, η in (1) as non-real numbers. Assume that λ_0 is an eigenvalue of A_D. Then there exist $g, h \in L^2(\Omega)$ such that $((A_D - \cdot)^{-1}g, h)_{L^2(\Omega)}$ has a pole at λ_0. Since by step 2 $\mathcal{D} = \operatorname{span}\{\operatorname{ran}\gamma_D(\mu) : \mu \in \mathbb{C} \setminus \mathbb{R}\}$ is dense in $L^2(\Omega)$, we can assume $g, h \in \mathcal{D}$. In particular, there exist $\mu, \eta \in \mathbb{C} \setminus \mathbb{R}$ and $\varphi, \psi \in H^{3/2}(\partial\Omega)$ such that $((A_D - \cdot)^{-1}\gamma_D(\eta)\varphi, \gamma_D(\mu)\psi)_{L^2(\Omega)}$ has a pole at λ_0. Then (1) implies that $(M_D(\cdot)\varphi, \psi)_{L^2(\partial\Omega)}$ and hence also $M_D(\cdot)$ has a pole at λ_0. This completes the proof of the statement on the eigenvalues of the Dirichlet operator.

The statement on the eigenvalues of the Neumann operator follows in the same way with an analogue of (1), where $\varphi \in H^{3/2}(\partial\Omega)$, $\gamma_D(\cdot)$, A_D, and $M_D(\cdot)$ are replaced by $\varphi \in H^{1/2}(\partial\Omega)$, $\gamma_N(\cdot)$, A_N, and $M_N(\cdot)$, respectively, and $\gamma_N(\cdot)$ is defined by $\gamma_N(\lambda) : H^{1/2}(\partial\Omega) \to \mathcal{N}_\lambda, \partial_\nu u_\lambda|_{\partial\Omega} \mapsto u_\lambda, \lambda \in \rho(A_N)$. □

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