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Spectral analysis of selfadjoint elliptic differential operators, Dirichlet-to-Neumann maps, and abstract Weyl functions



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ABSTRACT

The spectrum of a selfadjoint second order elliptic differential operator in $L^2(\mathbb{R}^n)$ is described in terms of the limiting behavior of Dirichlet-to-Neumann maps, which arise in a multi-dimensional Glazman decomposition and correspond to an interior and an exterior boundary value problem. This leads to PDE analogs of renowned facts in spectral theory of ODEs. The main results in this paper are first derived in the more abstract context of extension theory of symmetric operators and corresponding Weyl functions, and are applied to the PDE setting afterwards.

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1. Introduction

The Titchmarsh–Weyl function is an indispensable tool in direct and inverse spectral theory of ordinary differential operators and more general systems of ordinary differential equations; see the classical monographs [15,60] and [9,16,28–30,37,41,47,56,57] for a small selection of more recent contributions. For a singular second order Sturm–Liouville

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differential operator of the form $\mathfrak{L}_+ = -\frac{d^2}{dx^2} + q_+$ on \mathbb{R}_+ with a real-valued, bounded potential q_+ the Titchmarsh–Weyl function m_+ can be defined as

$$m_+(\lambda) = \frac{f'_\lambda(0)}{f_\lambda(0)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.1}$$

where f_λ is a square-integrable solution of $\mathfrak{L}_+f = \lambda f$ on \mathbb{R}_+ ; cf. [60,61]. The function $m_+ : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ belongs to the class of Nevanlinna (or Riesz–Herglotz) functions and it is a celebrated fact that it reflects the complete spectral properties of the selfadjoint realizations of \mathfrak{L}_+ in $L^2(\mathbb{R}_+)$. E.g. the eigenvalues of the Dirichlet realization A_D are precisely those $\lambda \in \mathbb{R}$, where $\lim_{\eta \searrow 0} i\eta m_+(\lambda + i\eta) \neq 0$, the isolated eigenvalues among them coincide with the poles of m_+ , and the absolutely continuous spectrum of A_D (roughly speaking) consists of all λ with the property $0 < \text{Im } m_+(\lambda + i0) < +\infty$.

If $\mathfrak{L} = -\frac{d^2}{dx^2} + q$ is a singular Sturm–Liouville expression on \mathbb{R} with q real-valued and bounded, it is most natural to use decomposition methods of Glazman type for the analysis of the corresponding selfadjoint operator in $L^2(\mathbb{R})$; cf. [31]. More precisely, the restriction of \mathfrak{L} to \mathbb{R}_+ gives rise to the Titchmarsh–Weyl function m_+ in (1.1), and similarly a Titchmarsh–Weyl function m_- associated to the restriction of \mathfrak{L} to \mathbb{R}_- is defined. In that case usually the functions

$$m(\lambda) = -(m_+(\lambda) + m_-(\lambda))^{-1} \quad \text{and} \quad \tilde{m}(\lambda) = \begin{pmatrix} -m_+(\lambda) & 1 \\ 1 & m_-(\lambda)^{-1} \end{pmatrix}^{-1} \tag{1.2}$$

are employed for the description of the spectrum. Whereas the scalar function m seems to be more convenient it will in general not contain the complete spectral data, a drawback that is overcome when using the 2×2 -matrix function \tilde{m} . Some of these observations were already made in [39,60], similar ideas can also be found in [36,38,42] for Hamiltonian systems, and more recently in an abstract operator theoretical framework in [17,19], see also [6].

One of the main objectives of this paper is to extend the classical spectral analysis of ordinary differential operators via the Titchmarsh–Weyl functions in (1.2) to the multidimensional setting. For this consider the second order partial differential expression

$$\mathcal{L} = - \sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^n \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \bar{a}_j \right) + a \tag{1.3}$$

with smooth, bounded coefficients $a_{jk}, a_j : \mathbb{R}^n \rightarrow \mathbb{C}$ and $a : \mathbb{R}^n \rightarrow \mathbb{R}$ bounded, and assume that \mathcal{L} is formally symmetric and uniformly elliptic on \mathbb{R}^n . Let A be the selfadjoint operator associated to (1.3) in $L^2(\mathbb{R}^n)$. Our main goal is to describe the spectral data of A , that is, isolated and embedded eigenvalues, continuous, absolutely continuous and singular continuous spectral points, in terms of the limiting behavior of appropriate multidimensional counterparts of the functions in (1.2). Note first that the multidimensional analogue of the Titchmarsh–Weyl function (1.1) is the Dirichlet-to-Neumann map, and

in order to define suitable analogues of the functions in (1.2) we proceed as follows: Split \mathbb{R}^n into a bounded domain Ω_i with smooth boundary Σ and its exterior $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega_i}$. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the Dirichlet-to-Neumann maps for \mathcal{L} in Ω_i and Ω_e , respectively, on the compact interface Σ are given by

$$\Lambda_i(\lambda)u_{\lambda,i}|_{\Sigma} := \frac{\partial u_{\lambda,i}}{\partial \nu_{\mathcal{L}_i}} \Big|_{\Sigma} \quad \text{and} \quad \Lambda_e(\lambda)u_{\lambda,e}|_{\Sigma} := \frac{\partial u_{\lambda,e}}{\partial \nu_{\mathcal{L}_e}} \Big|_{\Sigma}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $u_{\lambda,j} \in H^2(\Omega_j)$ solve $\mathcal{L}u_{\lambda,j} = \lambda u_{\lambda,j}$, $j = i, e$, and $u_{\lambda,j}|_{\Sigma}$ and $\frac{\partial u_{\lambda,j}}{\partial \nu_{\mathcal{L}_j}}|_{\Sigma}$ denote the trace and the conormal derivative, respectively; cf. Section 4.1 for further details. Both $\lambda \mapsto \Lambda_i(\lambda)$ and $\lambda \mapsto \Lambda_e(\lambda)$ are viewed as functions whose values are operators in $L^2(\Sigma)$ defined on the dense subspace $H^{3/2}(\Sigma)$. The multidimensional counterparts of the functions in (1.2) are

$$M(\lambda) = (\Lambda_i(\lambda) + \Lambda_e(\lambda))^{-1} \quad \text{and} \quad \widetilde{M}(\lambda) = \begin{pmatrix} \Lambda_i(\lambda) & 1 \\ 1 & -\Lambda_e(\lambda)^{-1} \end{pmatrix}^{-1} \tag{1.4}$$

(the differences in the signs are due to the definition of the conormal derivative, where the normals of Ω_i and Ω_e point into opposite directions). Observe that, in contrast to the one-dimensional situation described above, \mathbb{R}^n is split into a bounded domain and an unbounded domain. This yields that Λ_i is meromorphic, which in turn essentially allows us to give an almost complete characterization of the spectrum of A with the function M in (1.4) in Theorem 4.1; the only possible spectral points that cannot be detected with M are eigenvalues of A with vanishing traces on Σ , and possible accumulation points of such eigenvalues. A complete picture of the spectrum of A in terms of the limiting behavior of Dirichlet-to-Neumann maps is obtained with help of the 2×2 -block operator matrix function \widetilde{M} in (1.4) in Theorem 4.7.

We mention that in connection with Schrödinger operators in \mathbb{R}^3 the function M in (1.4) was already used in [2] for the extension of a classical convergence property of the Titchmarsh–Weyl function to the three-dimensional case, see also [5,55]. We also remark that for Schrödinger operators on exterior domains with C^2 -boundaries the connection of the spectrum to the limits of the Dirichlet-to-Neumann map was already investigated by the authors in [8]. Furthermore, we refer to [11,13,25–27,33,34,48,52–54] for recent related work on spectral problems for elliptic differential operators.

In this paper our approach to Titchmarsh–Weyl functions and their connection to spectral properties of corresponding selfadjoint differential operators is more abstract and of general nature, based on the concepts of (quasi) boundary triples and their Weyl functions. Recall first that for a symmetric operator S in a Hilbert space \mathcal{H} a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ consists of a “boundary space” \mathcal{G} and two linear mappings $\Gamma_0, \Gamma_1 : \text{dom } S^* \rightarrow \mathcal{G}$, which satisfy an abstract Green identity

$$(S^*f, g)_{\mathcal{H}} - (f, S^*g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}, \quad f, g \in \text{dom } S^*, \tag{1.5}$$

and a maximality condition. The corresponding Weyl function M is defined as

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (1.6)$$

where $f_\lambda \in \mathcal{H}$ solves the equation $S^* f = \lambda f$; the values $M(\lambda)$ of the Weyl function M are bounded operators in the Hilbert space \mathcal{G} . The example of the Sturm–Liouville expression \mathfrak{L}_+ in the beginning of the introduction fits into this scheme: There $\mathcal{H} = L^2(\mathbb{R}_+)$, S is the minimal operator associated with the differential expression \mathfrak{L}_+ in $L^2(\mathbb{R}_+)$, $\mathcal{G} = \mathbb{C}$, and the mappings Γ_0, Γ_1 are given by

$$\Gamma_0 f = f(0) \quad \text{and} \quad \Gamma_1 f = f'(0), \quad f \in \text{dom } S^*,$$

where S^* is the maximal operator associated with \mathfrak{L}_+ in $L^2(\mathbb{R}_+)$. Then the corresponding Weyl function is m_+ in (1.1), the selfadjoint Dirichlet operator A_D coincides with the restriction $S^* \upharpoonright \ker \Gamma_0$, and the spectrum can be described with the help of the limits of the Weyl function. The correspondence between the spectrum of the particular selfadjoint extension

$$A_0 := S^* \upharpoonright \ker \Gamma_0$$

and the limits of the Weyl function is not a special feature of the boundary triple for the above Sturm–Liouville equation. In fact, it holds as soon as the symmetric restriction S (and, thus, the boundary mappings Γ_0 and Γ_1) is chosen properly. More abstract considerations from [20,44–46] yield that the operator A_0 (and hence its spectrum) is determined up to unitary equivalence by the Weyl function if and only if the symmetric operator S is simple or completely non-selfadjoint, that is, there exists no nontrivial subspace of \mathcal{H} which reduces S to a selfadjoint operator. This condition can be reformulated equivalently as

$$\mathcal{H} = \text{clsp}\{\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \mathcal{G}\}, \quad (1.7)$$

where $\gamma(\nu) = (\Gamma_0 \upharpoonright \ker(S^* - \nu))^{-1}$ is the so-called γ -field and clsp denotes the closed linear span; cf. [43]. Under the assumption that S is simple a description of the absolutely continuous and singular continuous spectrum in the framework of boundary triples and their Weyl functions was given in [10]; for more recent related work see also [12,14,35,49,51,58].

The concept of boundary triples and their Weyl functions was extended in [3] in such a way that it is conveniently applicable to PDE problems. For that one defines boundary mappings Γ_0, Γ_1 on a suitable, smaller subset of the domain of the maximal operator and requires Green's identity (1.5) only to hold on this subset; the definition of the Weyl function associated to such a quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is as in (1.6), except that only solutions in the domain of the boundary maps are used; cf. Section 2.1. For the second order elliptic operator \mathcal{L} in (1.3) restricted to the smooth domain $\Omega_i \subset \mathbb{R}^n$ one may choose $\mathcal{G} = L^2(\Sigma)$,

$$\Gamma_0 u = u|_\Sigma \quad \text{and} \quad \Gamma_1 u = -\frac{\partial u}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma, \quad u \in H^2(\Omega_i),$$

in which case the corresponding Weyl function is (minus) the Dirichlet-to-Neumann map $-\Lambda_i$. Based on orthogonal couplings of symmetric operators and extending abstract ideas in [17] also the functions M and \widetilde{M} in (1.4) can be interpreted as Weyl functions associated to properly chosen quasi boundary triples; e.g., M corresponds to the pair of boundary mappings

$$\Gamma_0 u = \frac{\partial u_i}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma + \frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma, \quad \Gamma_1 u = u|_\Sigma, \quad u = u_i \oplus u_e, \quad u_i|_\Sigma = u_e|_\Sigma, \quad (1.8)$$

where $u_j \in H^2(\Omega_j)$, $j = i, e$. Moreover, $\ker \Gamma_0$ is the domain of the unique selfadjoint operator A associated with \mathcal{L} in $L^2(\mathbb{R}^n)$. When trying to link the spectral properties of A to the limiting behavior of the function M it is necessary to extend the known results for boundary triples to the more general notion of quasi boundary triples. Moreover, a subtle difficulty arises: The symmetric operator S corresponding to the boundary mappings in (1.8) may possess eigenvalues and thus in general is not simple.

In the abstract part of the present paper we show how this difficulty can be overcome. In the general setting of quasi boundary triples and their Weyl functions we show that a local simplicity condition on an open interval (or, more generally, a Borel set) $\Delta \subset \mathbb{R}$ suffices to characterize the spectrum of A_0 in Δ . To be more specific, we assume that

$$E(\Delta)\mathcal{H} = \text{clsp} \{E(\Delta)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}, \quad (1.9)$$

where $E(\Delta)$ denotes the spectral projection of $A_0 = S^* \upharpoonright \ker \Gamma_0$ on Δ ; this is a local version of the condition (1.7). Under this assumption we provide characterizations of the isolated and embedded eigenvalues and the corresponding eigenspaces, as well as the continuous, absolutely continuous and singular continuous spectrum of A_0 in Δ in terms of the limits of $M(\lambda)$ when λ approaches the real axis. For instance, we prove that the eigenvalues of A_0 in Δ are those λ , where $\lim_{\eta \searrow 0} i\eta M(\lambda + i\eta)g \neq 0$ for some $g \in \text{ran } \Gamma_0$, and that the absolutely continuous spectrum of A_0 can be characterized by means of the points λ where $0 < \text{Im}(M(\lambda + i0)g, g)_\mathcal{G} < \infty$. Moreover, we prove inclusions and provide conditions for the absence of singular continuous spectrum. Afterwards we apply the obtained results to the selfadjoint elliptic differential operator associated to \mathcal{L} in (1.3) in $L^2(\mathbb{R}^n)$. We prove that, despite the fact that the underlying symmetric operator fails to be simple in general, the whole absolutely continuous spectrum of A_0 can be recovered from the mapping M in (1.4). Moreover, we prove that the eigenvalues of A_0 and the corresponding eigenfunctions can be characterized by limiting properties of M as far as the eigenfunctions do not vanish on the interface Σ . A complete picture of the spectrum of A_0 is obtained when using the function \widetilde{M} in (1.4).

This paper is organized in the following way. In Section 2 we recall the basic facts on quasi boundary triples and corresponding Weyl functions and discuss the local simplicity

property (1.9) in detail. In Section 3 the connection between the spectra of selfadjoint operators and corresponding abstract Weyl functions is investigated. Section 4 contains the application of the abstract results to the mentioned PDE problems.

Finally, let us fix some notation. For a selfadjoint operator A in a Hilbert space \mathcal{H} we denote by $\sigma(A)$ ($\sigma_p(A)$, $\sigma_c(A)$, $\sigma_{ac}(A)$, $\sigma_{sc}(A)$, $\sigma_s(A)$, respectively) the spectrum (set of eigenvalues, continuous, absolutely continuous, singular continuous, singular spectrum, respectively) of A and by $\rho(A) = \mathbb{C} \setminus \sigma(A)$ its resolvent set.

2. Quasi boundary triples, associated Weyl functions, and a local simplicity condition

In this preliminary section we first recall the concepts of quasi boundary triples, their γ -fields and their Weyl functions. Afterwards we discuss a local simplicity property of symmetric operators, which will be assumed to hold in most of the results in Section 3.

2.1. Quasi boundary triples

The notion of quasi boundary triples was introduced in [3] as a generalization of the notions of boundary triples and generalized boundary triples, see [18,20,21,32,40]. The basic definition is the following.

Definition 2.1. Let S be a closed, densely defined, symmetric operator in a separable Hilbert space \mathcal{H} and let $T \subset S^*$ be an operator whose closure coincides with S^* , i.e., $\bar{T} = S^*$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space \mathcal{G} and two linear mappings $\Gamma_0, \Gamma_1 : \text{dom } T \rightarrow \mathcal{G}$ is called a *quasi boundary triple* for S^* if the following conditions are satisfied.

- (i) The range of the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \text{dom } T \rightarrow \mathcal{G} \times \mathcal{G}$ is dense.
- (ii) The identity

$$(Tu, v)_{\mathcal{H}} - (u, Tv)_{\mathcal{H}} = (\Gamma_1 u, \Gamma_0 v)_{\mathcal{G}} - (\Gamma_0 u, \Gamma_1 v)_{\mathcal{G}} \tag{2.1}$$

holds for all $u, v \in \text{dom } T$.

- (iii) The operator $A_0 := T \upharpoonright \ker \Gamma_0$ is selfadjoint in \mathcal{H} .

In the following we suppress the indices in the scalar products and simply write (\cdot, \cdot) , when no confusion can arise.

We recall some facts on quasi boundary triples, which can be found in [3,4]. Let S be a closed, densely defined, symmetric operator in \mathcal{H} . A quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* exists if and only if the defect numbers of S are equal. What we will use frequently is that if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for S^* then $\text{dom } S = \ker \Gamma_0 \cap \ker \Gamma_1$. Recall also that a quasi boundary triple with the additional property $\text{ran}(\Gamma_0, \Gamma_1)^\top = \mathcal{G} \times \mathcal{G}$ becomes an (ordinary) boundary triple and that, in particular, in this case the boundary

mappings Γ_0, Γ_1 are defined on $\text{dom } S^*$ and (2.1) holds with T replaced by S^* . In particular, in the case of finite defect numbers the notions of quasi boundary triples and (ordinary) boundary triples coincide. For more details on quasi boundary triples we refer to [3,4].

In order to prove that a triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for the adjoint S^* of a given symmetric operator S it is not necessary to know S^* explicitly, as the following useful proposition shows; cf. [3, Theorem 2.3] for a proof.

Proposition 2.2. *Let T be a linear operator in a separable Hilbert space \mathcal{H} , let \mathcal{G} be a further Hilbert space, and let $\Gamma_0, \Gamma_1 : \text{dom } T \rightarrow \mathcal{G}$ be linear mappings which satisfy the following conditions.*

- (i) *The range of the map $\Gamma = (\Gamma_0, \Gamma_1)^\top : \text{dom } T \rightarrow \mathcal{G} \times \mathcal{G}$ is dense in $\mathcal{G} \times \mathcal{G}$ and $\ker \Gamma$ is dense in \mathcal{H} .*
- (ii) *The identity (2.1) holds for all $u, v \in \text{dom } T$.*
- (iii) *There exists a selfadjoint restriction A_0 of T in \mathcal{H} with $\text{dom } A_0 \subset \ker \Gamma_0$.*

Then $S := T \upharpoonright \ker \Gamma$ is a closed, densely defined, symmetric operator in \mathcal{H} , $\bar{T} = S^$ holds, and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for S^* with $T \upharpoonright \ker \Gamma_0 = A_0$.*

2.2. Weyl functions and γ -fields

Let S be a closed, densely defined, symmetric operator in \mathcal{H} and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\bar{T} = S^*$ with $A_0 = T \upharpoonright \ker \Gamma_0$. In order to define the γ -field and the Weyl function corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ note that the direct sum decomposition

$$\text{dom } T = \text{dom } A_0 \dot{+} \ker(T - \lambda) = \ker \Gamma_0 \dot{+} \ker(T - \lambda)$$

holds for each $\lambda \in \rho(A_0)$ and that, in particular, the restriction of Γ_0 to $\ker(T - \lambda)$ is injective. The following definition is formally the same as for ordinary and generalized boundary triples.

Definition 2.3. Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\bar{T} = S^*$ and let $A_0 = T \upharpoonright \ker \Gamma_0$. Then the γ -field γ and the Weyl function M associated with $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are given by

$$\gamma(\lambda) = (\Gamma_0 \upharpoonright \ker(T - \lambda))^{-1} \quad \text{and} \quad M(\lambda) = \Gamma_1 \gamma(\lambda), \quad \lambda \in \rho(A_0),$$

respectively.

It follows immediately from the definition that for each $\lambda \in \rho(A_0)$ the operator $M(\lambda)$ satisfies the equality

$$M(\lambda)\Gamma_0 u_\lambda = \Gamma_1 u_\lambda, \quad u_\lambda \in \ker(T - \lambda),$$

and that $\text{ran } \gamma(\lambda) = \ker(T - \lambda)$ holds. We summarize some properties of the γ -field and the Weyl function. For the proofs of items (i)–(iv) in the next lemma we refer to [3, Proposition 2.6], item (v) is a simple consequence of (ii) and (iii).

Lemma 2.4. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\bar{T} = S^*$ with γ -field γ and Weyl function M and let $A_0 = T \upharpoonright \ker \Gamma_0$. Then for $\lambda, \mu, \nu \in \rho(A_0)$ the following assertions hold.*

- (i) $\gamma(\lambda)$ is a bounded operator from \mathcal{G} to \mathcal{H} defined on the dense subspace $\text{ran } \Gamma_0$. The adjoint $\gamma(\lambda)^* : \mathcal{H} \rightarrow \mathcal{G}$ is everywhere defined on \mathcal{H} and is bounded. It is given by

$$\gamma(\lambda)^* = \Gamma_1(A_0 - \bar{\lambda})^{-1}.$$

- (ii) The identity

$$\gamma(\lambda)g = (I + (\lambda - \mu)(A_0 - \lambda)^{-1}) \gamma(\mu)g$$

holds for all $g \in \text{ran } \Gamma_0$.

- (iii) The γ -field and the Weyl function are connected via

$$(\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda)g = M(\lambda)g - M(\mu)^*g, \quad g \in \text{ran } \Gamma_0,$$

and $M(\bar{\lambda}) \subset M(\lambda)^*$ holds.

- (iv) $M(\lambda)$ is an operator in \mathcal{G} defined on the dense subspace $\text{ran } \Gamma_0$ and satisfies

$$M(\lambda)g = \text{Re } M(\mu)g + \gamma(\mu)^* ((\lambda - \text{Re } \mu) + (\lambda - \mu)(\lambda - \bar{\mu})(A_0 - \lambda)^{-1}) \gamma(\mu)g \quad (2.2)$$

for all $g \in \text{ran } \Gamma_0$. In particular, for every $g \in \text{ran } \Gamma_0$ the function $\lambda \mapsto M(\lambda)g$ is holomorphic on $\rho(A_0)$ and each isolated singularity of $\lambda \mapsto M(\lambda)g$ is a pole of first order. Moreover, $\lim_{\eta \searrow 0} i\eta M(\zeta + i\eta)g$ exists for all $g \in \text{ran } \Gamma_0$ and all $\zeta \in \mathbb{R}$.

- (v) The identity

$$\gamma(\mu)^*(A_0 - \lambda)^{-1}\gamma(\nu)g = \frac{M(\lambda)g}{(\lambda - \nu)(\lambda - \bar{\mu})} + \frac{M(\bar{\mu})g}{(\lambda - \bar{\mu})(\nu - \bar{\mu})} + \frac{M(\nu)g}{(\nu - \lambda)(\nu - \bar{\mu})}$$

holds for all $g \in \text{ran } \Gamma_0$ if $\lambda \neq \nu$, $\lambda \neq \bar{\mu}$ and $\nu \neq \bar{\mu}$.

2.3. Simple symmetric operators and local simplicity

Let S be a closed, densely defined, symmetric operator in the separable Hilbert space \mathcal{H} . Recall that S is said to be *simple* or *completely non-selfadjoint* if there is

no nontrivial S -invariant subspace \mathcal{H}_0 of \mathcal{H} which reduces S to a selfadjoint operator in \mathcal{H}_0 , see [1, Chapter VII-81]. According to [43] the simplicity of S is equivalent to the density of the span of the defect spaces of S in \mathcal{H} , i.e., S is simple if and only if

$$\mathcal{H} = \text{clsp}\{\ker(S^* - \nu) : \nu \in \mathbb{C} \setminus \mathbb{R}\} \tag{2.3}$$

holds; here clsp stands for the closed linear span. Assume that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $\bar{T} = S^*$ with $A_0 = T \upharpoonright \ker \Gamma_0$. Then it follows that S is simple if and only if (2.3) holds with $\ker(S^* - \nu)$ replaced by $\ker(T - \nu)$. Moreover, if γ is the γ -field corresponding to the quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ we conclude that S is simple if and only if

$$\mathcal{H} = \text{clsp}\{\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\} \tag{2.4}$$

holds. We also mention that the set $\mathbb{C} \setminus \mathbb{R}$ in (2.4) can be replaced by any set $G \subset \rho(A_0)$ which has an accumulation point in each connected component of $\rho(A_0)$; cf. Lemma 2.5 (v) below.

Our aim is to generalize the notion of simplicity and to replace it by some weaker, local condition, which is satisfied in, e.g., the applications in Section 4. Instead of (2.4) we will assume that

$$E(\Delta)\mathcal{H} = \text{clsp}\{E(\Delta)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\} \tag{2.5}$$

holds on a Borel set (later on usually an open interval) Δ ; here $E(\cdot)$ denotes the spectral measure of A_0 . This condition will be imposed in many of the general results in Section 3. In the next lemma we discuss this condition and some consequences of it.

Lemma 2.5. *Let S be a closed, densely defined, symmetric operator in \mathcal{H} and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\bar{T} = S^*$ with $A_0 = T \upharpoonright \ker \Gamma_0$. Then the following holds.*

- (i) *If S is simple then (2.5) is satisfied for every Borel set $\Delta \subset \mathbb{R}$.*
- (ii) *If (2.5) holds for some Borel set $\Delta \subset \mathbb{R}$ then*

$$E(\Delta')\mathcal{H} = \text{clsp}\{E(\Delta')\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\} \tag{2.6}$$

holds for every Borel set $\Delta' \subset \Delta$.

- (iii) *If $\delta_1, \delta_2, \dots$ are disjoint open intervals such that*

$$E(\delta_j)\mathcal{H} = \text{clsp}\{E(\delta_j)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\} \quad \text{for all } j \tag{2.7}$$

then (2.5) holds for $\Delta = \bigcup_j \delta_j$.

- (iv) *If (2.5) holds for some Borel set $\Delta \subset \mathbb{R}$ then $\Delta \cap \sigma_p(S) = \emptyset$.*

(v) If (2.5) holds and G is a subset of $\rho(A_0)$ which has an accumulation point in each connected component of $\rho(A_0)$ then

$$E(\Delta)\mathcal{H} = \text{clsp}\{E(\Delta)\gamma(\nu)g : \nu \in G, g \in \text{ran } \Gamma_0\}. \tag{2.8}$$

Proof. Assertion (i) is a consequence of item (ii) since (2.5) holds with $\Delta = \mathbb{R}$ when S is simple.

For (ii) note that the inclusion \supset in (2.6) clearly holds. For the converse inclusion let $u \in E(\Delta')\mathcal{H}$. As $\Delta' \subset \Delta$ we have $u \in E(\Delta)\mathcal{H}$ and hence there exists a sequence (v_n) in the linear span of $\{E(\Delta)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}$ which converges to u . Then $(E(\Delta')v_n)$ is a sequence in the linear span of $\{E(\Delta')\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}$ which converges to $E(\Delta')u = u$.

In order to prove (iii) let δ_j be as in the assumptions and let $\Delta = \bigcup_j \delta_j$. The inclusion \supset in (2.5) again is obvious. For the converse inclusion let $u \in E(\Delta)\mathcal{H}$ and define

$$\tilde{\mathcal{H}} := \text{clsp}\{E(\Delta)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}. \tag{2.9}$$

Since

$$u = E(\Delta)u = \sum_j E(\delta_j)u$$

it is sufficient to show $E(\delta_j)u \in \tilde{\mathcal{H}}$ for all j . Note first that by assumption (2.7) we have

$$E(\delta_j)u \in \text{clsp}\{E(\delta_j)\gamma(\mu)h : \mu \in \mathbb{C} \setminus \mathbb{R}, h \in \text{ran } \Gamma_0\}$$

and hence the assertion follows if we verify

$$E(\delta_j)\gamma(\mu)h \in \tilde{\mathcal{H}} \tag{2.10}$$

for all $\mu \in \mathbb{C} \setminus \mathbb{R}$, $h \in \text{ran } \Gamma_0$, and all j . For this purpose consider some fixed $E(\delta_j)\gamma(\mu)h$. According to Lemma 2.4 (ii) we have

$$\gamma(\nu)g = \gamma(\mu)g + (\nu - \mu)(A_0 - \nu)^{-1}\gamma(\mu)g$$

for all $\nu \in \mathbb{C} \setminus \mathbb{R}$ and all $g \in \text{ran } \Gamma_0$, and hence $\tilde{\mathcal{H}}$ in (2.9) can be rewritten in the form

$$\tilde{\mathcal{H}} = \text{clsp}\left\{E(\Delta)\gamma(\mu)g, E(\Delta)(A_0 - \nu)^{-1}\gamma(\mu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\right\}.$$

It follows that for $\eta, \varepsilon > 0$ the element

$$\int_{\alpha_j + \eta}^{\beta_j - \eta} E(\Delta)((A_0 - (\lambda + i\varepsilon))^{-1} - (A_0 - (\lambda - i\varepsilon))^{-1})\gamma(\mu)h \, d\lambda$$

belongs to $\widetilde{\mathcal{H}}$, where we have written $\delta_j = (\alpha_j, \beta_j)$. From this and Stone’s formula it follows

$$E(\delta_j)\gamma(\mu)h = E(\delta_j)E(\Delta)\gamma(\mu)h \in \widetilde{\mathcal{H}},$$

which proves (2.10) and, hence, yields the inclusion \subset in (2.5). Item (iii) is proved.

In order to verify (iv), assume that $Su = \lambda u$ for some $u \in \text{dom } S$ and $\lambda \in \Delta$. Then $A_0u = \lambda u$ and hence $u \in E(\Delta)\mathcal{H}$. On the other hand, for $g \in \text{ran } \Gamma_0$ and $\nu \in \mathbb{C} \setminus \mathbb{R}$ it follows together with Lemma 2.4 (i) that

$$(u, E(\Delta)\gamma(\nu)g) = (\gamma(\nu)^*u, g) = (\Gamma_1(A_0 - \bar{\nu})^{-1}u, g) = (\lambda - \bar{\nu})^{-1}(\Gamma_1u, g) = 0,$$

as $u \in \text{dom } S \subset \ker \Gamma_1$. Hence, $u \in E(\Delta)\mathcal{H}$ is orthogonal to the linear span of the elements $E(\Delta)\gamma(\nu)g$, $\nu \in \mathbb{C} \setminus \mathbb{R}$, $g \in \text{ran } \Gamma_0$, which is dense in $E(\Delta)\mathcal{H}$ by (2.5). This implies $u = 0$ and thus S does not possess eigenvalues in Δ .

It remains to show (v). The inclusion \supset in (2.8) is obvious. In order to prove the inclusion \subset it suffices to verify that the vectors $E(\Delta)\gamma(\nu)g$, $g \in \text{ran } \Gamma_0$, $\nu \in G$, span a dense set in $E(\Delta)\mathcal{H}$. Suppose that $E(\Delta)u$ is orthogonal to this set, that is,

$$0 = (E(\Delta)\gamma(\nu)g, E(\Delta)u) \tag{2.11}$$

holds for all $g \in \text{ran } \Gamma_0$ and all $\nu \in G$. Since $\rho(A_0) \ni \nu \mapsto \gamma(\nu)g$ is analytic for each $g \in \text{ran } \Gamma_0$ (see Lemma 2.4 (ii)) it follows that for each $g \in \text{ran } \Gamma_0$ the function $\nu \mapsto (E(\Delta)\gamma(\nu)g, E(\Delta)u)$ is analytic on $\rho(A_0)$, and hence (2.11) implies that this function is identically equal to zero. Now (2.5) yields $E(\Delta)u = 0$ and (v) follows. \square

3. Spectral properties of selfadjoint operators and corresponding Weyl functions

This section contains the main abstract results of this paper. We describe the spectral properties of a given selfadjoint operator by means of a corresponding Weyl function. For this we fix the following setting.

Assumption 3.1. Let S be a closed, densely defined, symmetric operator in the separable Hilbert space \mathcal{H} and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $\bar{T} = S^*$ with corresponding γ -field γ and Weyl function M . Moreover, let $A_0 = T \upharpoonright \ker \Gamma_0$ and denote by $E(\cdot)$ the spectral measure of A_0 .

3.1. Eigenvalues and corresponding eigenspaces

Let us start with a characterization of the isolated and embedded eigenvalues as well as the corresponding eigenspaces of a selfadjoint operator by means of an associated Weyl function. We write $s\text{-lim}$ for the strong limit of an operator function.

Theorem 3.2. *Let Assumption 3.1 be satisfied. Then $\lambda \in \mathbb{R}$ is an eigenvalue of A_0 such that $\mathcal{K} := \ker(A_0 - \lambda) \ominus \ker(S - \lambda) \neq \{0\}$ if and only if $R_\lambda M := s\text{-}\lim_{\eta \searrow 0} i\eta M(\lambda + i\eta) \neq 0$. If $\dim \mathcal{K} < \infty$ then the mapping*

$$\tau : \mathcal{K} \rightarrow \text{ran } R_\lambda M, \quad u \mapsto \Gamma_1 u, \tag{3.1}$$

is bijective; if $\dim \mathcal{K} = \infty$ then the mapping

$$\tau : \mathcal{K} \rightarrow \text{cl}_\tau(\text{ran } R_\lambda M), \quad u \mapsto \Gamma_1 u, \tag{3.2}$$

is bijective, where cl_τ denotes the closure in the normed space $\text{ran } \tau$, equipped with the norm in \mathcal{G} .

Remark 3.3. Recall that the limit $(R_\lambda M)g = \lim_{\eta \searrow 0} i\eta M(\lambda + i\eta)g$ exists for all $\lambda \in \mathbb{R}$ and all $g \in \text{ran } \Gamma_0$ by Lemma 2.4 (iv). Moreover, if λ is an isolated singularity of M , that is, there exists an open neighborhood \mathcal{O} of λ such that M is strongly holomorphic on $\mathcal{O} \setminus \{\lambda\}$ then $R_\lambda M \neq 0$ if and only if for some $g \in \text{ran } \Gamma_0$ the \mathcal{G} -valued function $\zeta \mapsto M(\zeta)g$ has a pole at λ . In this case $R_\lambda M$ coincides with the residue $\text{Res}_\lambda M$ of M at λ in the strong sense, i.e.,

$$(R_\lambda M)g = (\text{Res}_\lambda M)g = \frac{1}{2\pi i} \int_{\mathcal{C}} M(z)g \, dz, \quad g \in \text{ran } \Gamma_0,$$

where \mathcal{C} denotes the boundary of an open ball B such that M is strongly holomorphic in a neighborhood of \overline{B} except the point λ . We also remark that without additional assumptions the Weyl function is not able to distinguish between isolated and embedded eigenvalues of A_0 ; cf. Proposition 3.6 below.

Proof of Theorem 3.2. Let $\lambda \in \mathbb{R}$ be fixed. Note first that the mapping $\Gamma_1 \upharpoonright \mathcal{K}$ is injective. Indeed, for $u \in \mathcal{K} = \ker(A_0 - \lambda) \ominus \ker(S - \lambda)$ with $\Gamma_1 u = 0$ we have $u \in \ker \Gamma_0 \cap \ker \Gamma_1 = \text{dom } S$ and $Su = \lambda u$; hence $u = 0$. It is our aim to prove the inclusions

$$\text{ran } R_\lambda M \subset \text{ran}(\Gamma_1 \upharpoonright \mathcal{K}) \subset \overline{\text{ran } R_\lambda M}. \tag{3.3}$$

Note that the closure $\text{cl}_\tau(\text{ran } R_\lambda M)$ of $\text{ran } R_\lambda M$ in the normed space $\text{ran } \tau$, equipped with the norm of \mathcal{G} , coincides with $\overline{\text{ran } R_\lambda M} \cap \text{ran } \tau$. Hence (3.3) implies that the mapping τ in (3.1) and (3.2) is well-defined and bijective.

In order to verify (3.3) let $g \in \text{ran } \Gamma_0$ and denote by $E(\cdot)$ the spectral measure of A_0 . Then

$$\begin{aligned} & \left\| i\eta(A_0 - (\lambda + i\eta))^{-1} \gamma(\nu)g + E(\{\lambda\})\gamma(\nu)g \right\|^2 \\ &= \int_{\mathbb{R}} \left| \frac{i\eta}{t - (\lambda + i\eta)} + \mathbb{1}_{\{\lambda\}}(t) \right|^2 d(E(t)\gamma(\nu)g, \gamma(\nu)g) \rightarrow 0 \quad \text{as } \eta \searrow 0 \end{aligned} \tag{3.4}$$

holds for all $\nu \in \mathbb{C} \setminus \mathbb{R}$. Since by [Lemma 2.4](#) (i) the operator $\gamma(\nu)^*$ is bounded, it follows from [\(3.4\)](#) that

$$\lim_{\eta \searrow 0} i\eta \gamma(\nu)^*(A_0 - (\lambda + i\eta))^{-1} \gamma(\nu)g = -\gamma(\nu)^* E(\{\lambda\}) \gamma(\nu)g \tag{3.5}$$

holds for all $\nu \in \mathbb{C} \setminus \mathbb{R}$. Together with [Lemma 2.4](#) (v) we conclude that the limit on the left hand side of [\(3.5\)](#) coincides with

$$\lim_{\eta \searrow 0} i\eta \frac{M(\lambda + i\eta)g}{((\lambda + i\eta) - \nu)((\lambda + i\eta) - \bar{\nu})} = \frac{(R_\lambda M)g}{(\lambda - \nu)(\lambda - \bar{\nu})}. \tag{3.6}$$

With the help of [Lemma 2.4](#) (i), [\(3.5\)](#) and [\(3.6\)](#) we obtain

$$\begin{aligned} \Gamma_1 E(\{\lambda\}) \gamma(\nu)g &= \Gamma_1 (A_0 - \bar{\nu})^{-1} (A_0 - \bar{\nu}) E(\{\lambda\}) \gamma(\nu)g \\ &= (\lambda - \bar{\nu}) \gamma(\nu)^* E(\{\lambda\}) \gamma(\nu)g \\ &= -(\lambda - \bar{\nu}) \lim_{\eta \searrow 0} i\eta \gamma(\nu)^*(A_0 - (\lambda + i\eta))^{-1} \gamma(\nu)g \\ &= \frac{1}{\nu - \lambda} (R_\lambda M)g \end{aligned}$$

for all $\nu \in \mathbb{C} \setminus \mathbb{R}$. Denoting by P the orthogonal projection in \mathcal{H} onto $\mathcal{K} = \ker(A_0 - \lambda) \ominus \ker(S - \lambda)$ it follows

$$\Gamma_1 P \gamma(\nu)g = \frac{1}{\nu - \lambda} (R_\lambda M)g, \tag{3.7}$$

where we have used $\Gamma_1(\ker(S - \lambda)) = \{0\}$. From this the first inclusion in [\(3.3\)](#) follows immediately.

For the second inclusion in [\(3.3\)](#) note that the mapping $\Gamma_1 \upharpoonright \mathcal{K}$ is continuous as $\Gamma_1 u = \gamma(\mu)^*(A_0 - \bar{\mu})u = (\lambda - \bar{\mu})\gamma(\mu)^*u$ holds for all $u \in \mathcal{K}$ by [Lemma 2.4](#) (i). Moreover, for each $\nu \in \mathbb{C} \setminus \mathbb{R}$ the linear space $\{P\gamma(\nu)g : g \in \text{ran } \Gamma_0\}$ is dense in \mathcal{K} . In fact, fix $\nu \in \mathbb{C} \setminus \mathbb{R}$ and let $u \in \mathcal{K}$ be orthogonal to $P\gamma(\nu)g$ for all $g \in \text{ran } \Gamma_0$. Then

$$0 = (u, P\gamma(\nu)g) = (\gamma(\nu)^*u, g) = (\Gamma_1(A_0 - \bar{\nu})^{-1}u, g) = (\lambda - \bar{\nu})^{-1}(\Gamma_1 u, g)$$

by [Lemma 2.4](#) (i), which implies $\Gamma_1 u = 0$ as $\text{ran } \Gamma_0$ is dense. Hence we have $u \in \ker \Gamma_0 \cap \ker \Gamma_1 = \text{dom } S$ and this implies $u \in \mathcal{K} \cap \ker(S - \lambda)$, so that $u = 0$. Now the second inclusion in [\(3.3\)](#) follows together with [\(3.7\)](#) and the fact that $\Gamma_1 \upharpoonright \mathcal{K}$ is continuous. Hence the mapping τ in [\(3.2\)](#) is well-defined and bijective. If \mathcal{K} is finite-dimensional then clearly the closure in [\(3.2\)](#) can be omitted and we end up with the bijectivity of [\(3.1\)](#). \square

As an immediate consequence of [Theorem 3.2](#) all eigenvalues of A_0 which are not eigenvalues of S can be characterized as “generalized poles” of the Weyl function.

Corollary 3.4. *Let Assumption 3.1 be satisfied, and assume that $\lambda \in \mathbb{R}$ is not an eigenvalue of the symmetric operator S . Then λ is an eigenvalue of A_0 if and only if $R_\lambda M := \text{s-lim}_{\eta \searrow 0} i\eta M(\lambda + i\eta) \neq 0$. If the multiplicity of the eigenvalue λ is finite then the mapping*

$$\tau : \ker(A_0 - \lambda) \rightarrow \text{ran } R_\lambda M, \quad u \mapsto \Gamma_1 u,$$

is bijective; if the multiplicity of the eigenvalue λ is infinite then the mapping

$$\tau : \ker(A_0 - \lambda) \rightarrow \text{cl}_\tau(\text{ran } R_\lambda M), \quad u \mapsto \Gamma_1 u,$$

is bijective, where cl_τ denotes the closure in the normed space $\text{ran } \tau$, equipped with the norm in \mathcal{G} .

3.2. Continuous, absolutely continuous, and singular continuous spectra

In this subsection we describe the continuous, absolutely continuous, and singular continuous spectrum of a selfadjoint operator A_0 by means of the limits of an associated Weyl function M . Again we fix the setting in Assumption 3.1. It is clear that an additional minimality or simplicity condition must be imposed. Usually one assumes that the underlying symmetric operator S is simple; cf. [10]. However, for our purposes the weaker assumption of local simplicity in Section 2.3 is more appropriate: in order to characterize the spectrum of A_0 in an open interval $\Delta \subset \mathbb{R}$ we assume that

$$E(\Delta)\mathcal{H} = \text{clsp}\{E(\Delta)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}. \tag{3.8}$$

For instance, in Theorem 3.2 it turned out that an eigenvalue λ of A_0 with its full multiplicity can only be detected by the Weyl function if $\lambda \notin \sigma_p(S)$. This condition corresponds to the identity (3.8) with Δ replaced by $\{\lambda\}$; cf. Lemma 2.5 (iv).

In the next theorem we agree to say that the Weyl function M can be continued analytically to some point $\lambda \in \mathbb{R}$ if there exists an open neighborhood \mathcal{O} of λ in \mathbb{C} such that $\zeta \mapsto M(\zeta)g$ can be continued analytically to \mathcal{O} for all $g \in \text{ran } \Gamma_0$. We mention that the proof of (i) is similar to the proof of [23, Theorem 1.1].

Theorem 3.5. *Let Assumption 3.1 be satisfied, and let $\Delta \subset \mathbb{R}$ be an open interval such that the condition (3.8) is satisfied. Then the following assertions hold for each $\lambda \in \Delta$.*

- (i) $\lambda \in \rho(A_0)$ if and only if M can be continued analytically into λ .
- (ii) $\lambda \in \sigma_c(A_0)$ if and only if $\text{s-lim}_{\eta \searrow 0} i\eta M(\lambda + i\eta) = 0$ and M cannot be continued analytically into λ .

If S is simple then (i) and (ii) hold for all $\lambda \in \mathbb{R}$.

Proof. (i) Recall first that by [Lemma 2.4](#) (iv) the function $\lambda \mapsto M(\lambda)g$ is analytic on $\rho(A_0)$ for each $g \in \text{ran } \Gamma_0$, which proves the implication (\Rightarrow) . In order to verify the implication (\Leftarrow) in (i), let us assume that M can be continued analytically to some $\lambda \in \Delta$, that is, there exists an open neighborhood \mathcal{O} of λ in \mathbb{C} with $\mathcal{O} \cap \mathbb{R} \subset \Delta$ such that $\zeta \mapsto M(\zeta)g$ can be continued analytically to \mathcal{O} for each $g \in \text{ran } \Gamma_0$. Choose $a, b \in \mathbb{R}$ with $\lambda \in (a, b)$, $[a, b] \subset \mathcal{O}$, and $a, b \notin \sigma_p(A_0)$. The spectral projection $E((a, b))$ of A_0 corresponding to the interval (a, b) is given by Stone’s formula

$$E((a, b)) = \text{s-}\lim_{\delta \searrow 0} \frac{1}{2\pi i} \int_a^b ((A_0 - (t + i\delta))^{-1} - (A_0 - (t - i\delta))^{-1}) dt, \tag{3.9}$$

where the integral on the right-hand side is understood in the strong sense. Using the identity in [Lemma 2.4](#) (v) and [\(3.9\)](#) a straight forward calculation leads to

$$\begin{aligned} \|E((a, b))\gamma(\nu)g\|^2 &= (\gamma(\nu)^* E((a, b))\gamma(\nu)g, g) \\ &= \lim_{\delta \searrow 0} \frac{1}{2\pi i} \int_a^b \left((\gamma(\nu)^*(A_0 - (t + i\delta))^{-1}\gamma(\nu)g, g) \right. \\ &\quad \left. - (\gamma(\nu)^*(A_0 - (t - i\delta))^{-1}\gamma(\nu)g, g) \right) dt = 0 \end{aligned}$$

for all $g \in \text{ran } \Gamma_0$ and all $\nu \in \mathbb{C} \setminus \mathbb{R}$, since $\zeta \mapsto (M(\zeta)g, g)$ admits an analytic continuation into \mathcal{O} for all $g \in \text{ran } \Gamma_0$. Thus the assumption [\(3.8\)](#) and $[a, b] \subset \Delta$ together with [Lemma 2.5](#) (ii) imply $E((a, b)) = 0$. In particular, $\lambda \in \rho(A_0)$.

(ii) According to [Lemma 2.5](#) (iv) the condition [\(3.8\)](#) implies that S does not have eigenvalues in Δ . Hence item (ii) follows immediately from item (i) and [Corollary 3.4](#).

If S is simple then by [Lemma 2.5](#) (i) the assumption [\(3.8\)](#) is satisfied for $\Delta = \mathbb{R}$. Hence (i) and (ii) hold for all $\lambda \in \mathbb{R}$. \square

Now we return to the characterization of eigenvalues. We formulate a sufficient condition under which the Weyl function is able to distinguish between isolated and embedded eigenvalues.

Proposition 3.6. *Let [Assumption 3.1](#) be satisfied and let $\Delta \subset \mathbb{R}$ be an open interval. Assume that the condition [\(3.8\)](#) is satisfied and let $\lambda \in \Delta$. Then all assertions of [Corollary 3.4](#) hold for λ . Moreover, λ is an isolated eigenvalue of A_0 if and only if λ is a pole in the strong sense of M . In this case $R_\lambda M$ is the residue of M in the strong sense at λ ; cf. [Remark 3.3](#).*

Proof. Let $\lambda \in \mathbb{R}$ and let $\Delta \subset \mathbb{R}$ be an open interval with $\lambda \in \Delta$ such that [\(3.8\)](#) holds. Then $\lambda \notin \sigma_p(S)$ by [Lemma 2.5](#) (iv) and hence the assertions in [Corollary 3.4](#) hold for λ . Moreover, if λ is an isolated eigenvalue of A_0 then by [Lemma 2.4](#) (iv) there exists an open

neighborhood \mathcal{O} of λ such that $\zeta \mapsto M(\zeta)g$ is holomorphic on $\mathcal{O} \setminus \{\lambda\}$ for all $g \in \text{ran } \Gamma_0$. From [Corollary 3.4](#) we conclude that there exists $g \in \text{ran } \Gamma_0$ such that

$$\lim_{\eta \searrow 0} i\eta M(\lambda + i\eta)g \neq 0. \quad (3.10)$$

Hence [Lemma 2.4](#) (iv) implies that M has a pole of first order in the strong sense at λ . Conversely, if M has a pole (of first order) in the strong sense at λ then there exists $g \in \text{ran } \Gamma_0$ such that (3.10) holds. According to [Lemma 2.4](#) (iv) the order of the pole is one and, hence,

$$\lim_{\eta \searrow 0} i\eta M(\lambda + i\eta)g = (\text{Res}_\lambda M)g \neq 0.$$

It follows with the help of [Corollary 3.4](#) that λ is an eigenvalue of A_0 . Moreover, [Theorem 3.5](#) (i) implies that there exists an open neighborhood \mathcal{O} of λ in \mathbb{C} such that $\mathcal{O} \setminus \{\lambda\} \subset \rho(A_0)$. Hence λ is isolated in the spectrum of A_0 . This completes the proof. \square

Next we discuss the relation of the function M to the absolutely continuous and singular continuous spectrum of A_0 . In the special case of ordinary boundary triples and $\Delta = \mathbb{R}$ the following results reduce to those in [\[10\]](#). For our purposes a localized version and an extension to quasi boundary triples is necessary. The proofs presented here are somewhat more direct than those in [\[10\]](#); in particular, the integral representation of Nevanlinna functions and the corresponding measures are avoided.

In the following for a finite Borel measure μ on \mathbb{R} we denote the set of all growth points of μ by $\text{supp } \mu$, that is,

$$\text{supp } \mu = \{x \in \mathbb{R} : \mu((x - \varepsilon, x + \varepsilon)) > 0 \text{ for all } \varepsilon > 0\}.$$

Note that $\text{supp } \mu$ is closed with $\mu(\mathbb{R} \setminus \text{supp } \mu) = 0$ and that $\text{supp } \mu$ is minimal with this property, that is, each closed set $S \subset \mathbb{R}$ with $\mu(\mathbb{R} \setminus S) = 0$ satisfies $\text{supp } \mu \subset S$. Moreover, for a Borel set $\chi \subset \mathbb{R}$ we define the *absolutely continuous closure* (also called *essential closure*) by

$$\text{cl}_{\text{ac}}(\chi) := \{x \in \mathbb{R} : |(x - \varepsilon, x + \varepsilon) \cap \chi| > 0 \text{ for all } \varepsilon > 0\},$$

where $|\cdot|$ denotes the Lebesgue measure, and the *continuous closure* by

$$\text{cl}_c(\chi) := \{x \in \mathbb{R} : (x - \varepsilon, x + \varepsilon) \cap \chi \text{ is not countable for all } \varepsilon > 0\}. \quad (3.11)$$

Observe that $\text{cl}_{\text{ac}}(\chi)$ and $\text{cl}_c(\chi)$ both are closed and that $\text{cl}_{\text{ac}}(\chi) \subset \text{cl}_c(\chi) \subset \bar{\chi}$ holds, but in general the converse inclusions are not true. In fact, $\text{cl}_{\text{ac}}(\chi) = \emptyset$ if and only if $|\chi| = 0$, and $\text{cl}_c(\chi) = \emptyset$ if and only if χ is countable.

The following lemma can partly be found in, e.g., the monographs [\[24\]](#) or [\[59\]](#).

Lemma 3.7. *Let μ be a finite Borel measure on \mathbb{R} and denote by F its Stieltjes transform,*

$$F(\lambda) = \int_{\mathbb{R}} \frac{1}{t - \lambda} d\mu(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Then the limit $\operatorname{Im} F(x+i0) = \lim_{y \searrow 0} \operatorname{Im} F(x+iy)$ exists and is finite for Lebesgue almost all $x \in \mathbb{R}$. Let μ_{ac} and μ_s be the absolutely continuous and singular part, respectively, of μ in the Lebesgue decomposition $\mu = \mu_{ac} + \mu_s$, and decompose μ_s into the singular continuous part μ_{sc} and the pure point part. Then the following assertions hold.

- (i) $\operatorname{supp} \mu_{ac} = \operatorname{cl}_{ac}(\{x \in \mathbb{R} : 0 < \operatorname{Im} F(x+i0) < +\infty\})$.
- (ii) $\operatorname{supp} \mu_s \subset \overline{\{x \in \mathbb{R} : \operatorname{Im} F(x+i0) = +\infty\}}$.
- (iii) $\operatorname{supp} \mu_{sc} \subset \operatorname{cl}_c(\{x \in \mathbb{R} : \operatorname{Im} F(x+i0) = +\infty, \lim_{y \searrow 0} yF(x+iy) = 0\})$.

Proof. From [59, Lemma 3.14 and Theorem 3.23] it follows immediately that assertion (i) is true, that the limit $\operatorname{Im} F(x+i0)$ exists and is finite for Lebesgue almost all $x \in \mathbb{R}$, and that

$$\mu_s(\mathbb{R} \setminus \{x \in \mathbb{R} : \operatorname{Im} F(x+i0) = +\infty\}) = 0, \tag{3.12}$$

which implies (ii). In order to verify (iii) note first that $\lim_{y \searrow 0} yF(x+iy) = i\mu(\{x\})$ holds for all $x \in \mathbb{R}$ since

$$|yF(x+iy) - i\mu(\{x\})| \leq \int_{\mathbb{R}} \left| \frac{y}{t - (x+iy)} - i\mathbb{1}_{\{x\}}(t) \right| d\mu(t) \rightarrow 0, \quad y \searrow 0.$$

In particular, $\mu(\{x\}) \neq 0$ if and only if $\lim_{y \searrow 0} yF(x+iy) \neq 0$. Hence it follows from (3.12) and the definition of μ_{sc} that

$$\mu_{sc}(\mathbb{R} \setminus M_{sc}) = 0, \tag{3.13}$$

where

$$M_{sc} := \left\{ x \in \mathbb{R} : \operatorname{Im} F(x+i0) = +\infty, \lim_{y \searrow 0} yF(x+iy) = 0 \right\}.$$

For $x \in \mathbb{R} \setminus \operatorname{cl}_c(M_{sc})$ by definition there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap M_{sc}$ is countable; thus $\mu_{sc}((x - \varepsilon, x + \varepsilon) \cap M_{sc}) = 0$. With the help of (3.13) it follows

$$\mu_{sc}((x - \varepsilon, x + \varepsilon)) \leq \mu_{sc}((x - \varepsilon, x + \varepsilon) \cap M_{sc}) + \mu_{sc}(\mathbb{R} \setminus M_{sc}) = 0,$$

that is, $x \notin \operatorname{supp} \mu_{sc}$. \square

The absolutely continuous spectrum of a selfadjoint operator in some interval Δ can be characterized in the following way.

Theorem 3.8. *Let Assumption 3.1 be satisfied and let $\Delta \subset \mathbb{R}$ be an open interval such that the condition*

$$E(\delta)\mathcal{H} = \text{clsp}\{E(\delta)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\} \tag{3.14}$$

is satisfied for each open interval $\delta \subset \Delta$ with $\delta \cap \sigma_p(S) = \emptyset$. Then the absolutely continuous spectrum of A_0 in Δ is given by

$$\overline{\sigma_{\text{ac}}(A_0) \cap \Delta} = \overline{\bigcup_{g \in \text{ran } \Gamma_0} \text{cl}_{\text{ac}}(\{x \in \Delta : 0 < \text{Im}(M(x + i0)g, g) < +\infty\})}. \tag{3.15}$$

If S is simple then (3.15) holds for each open interval Δ , including the case $\Delta = \mathbb{R}$.

Proof. The proof of Theorem 3.8 consists of two separate steps in which the assertions (3.17) and (3.19) below will be shown. The identity (3.15) is then an immediate consequence of (3.17) and (3.19) (note that the right hand side in (3.19) does not depend on $\zeta \in \mathbb{C} \setminus \mathbb{R}$). We fix some notation first. Let us set

$$\mathcal{D}_\Delta := \{E(\Delta)\gamma(\zeta)g : \zeta \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\} \tag{3.16}$$

and define the measures $\mu_u := (E(\cdot)u, u)$ for $u \in \mathcal{H}$. Denote by P_{ac} the orthogonal projection in \mathcal{H} onto the absolutely continuous subspace \mathcal{H}_{ac} of A_0 . Observe that the spectral measure of the absolutely continuous part of A_0 is $E(\cdot)P_{\text{ac}}$ and that the absolutely continuous measures $\mu_{u,\text{ac}}$ are given by $\mu_{u,\text{ac}} = (E(\cdot)P_{\text{ac}}u, P_{\text{ac}}u) = \mu_{P_{\text{ac}}u}$.

Step 1. In this step the identity

$$\overline{\sigma_{\text{ac}}(A_0) \cap \Delta} = \overline{\bigcup_{u \in \mathcal{D}_\Delta} \text{supp } \mu_{u,\text{ac}}} \tag{3.17}$$

will be verified. First of all the open set $\Delta' := \Delta \setminus \overline{\sigma_p(S)}$ is the disjoint union of open intervals δ_j , $1 \leq j < N$, $N \in \mathbb{N} \cup \{\infty\}$, and for each δ_j we have

$$E(\delta_j)\mathcal{H} = \text{clsp}\{E(\delta_j)\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}$$

by assumption. With the help of Lemma 2.5 (iii) we conclude

$$E(\Delta')\mathcal{H} = \text{clsp}\{E(\Delta')\gamma(\nu)g : \nu \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}.$$

Since $\Delta' \subset \Delta$ it follows immediately that $E(\Delta')\mathcal{H} \subset \text{clsp } \mathcal{D}_\Delta$. Moreover, we have

$$P_{\text{ac}}E(\Delta)\mathcal{H} = P_{\text{ac}}E(\Delta')\mathcal{H} \subset P_{\text{ac}}(\text{clsp } \mathcal{D}_\Delta) \subset \text{clsp } P_{\text{ac}}\mathcal{D}_\Delta \subset P_{\text{ac}}E(\Delta)\mathcal{H}$$

and therefore

$$P_{ac}E(\Delta)\mathcal{H} = \text{clsp } P_{ac}\mathcal{D}_\Delta. \tag{3.18}$$

In order to verify (3.17), assume first that x does not belong to the left hand side of (3.17), that is, $x \notin \overline{\sigma_{ac}(A_0) \cap \Delta}$. Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap \Delta$ contains no absolutely continuous spectrum of A_0 . This yields

$$E((x - \epsilon, x + \epsilon) \cap \Delta)P_{ac} = 0$$

and for $u \in E(\Delta)\mathcal{H}$ one obtains

$$\begin{aligned} \mu_{u,ac}((x - \epsilon, x + \epsilon)) &= (E((x - \epsilon, x + \epsilon))P_{ac}u, P_{ac}u) \\ &= (E((x - \epsilon, x + \epsilon))P_{ac}E(\Delta)u, P_{ac}u) \\ &= (E((x - \epsilon, x + \epsilon) \cap \Delta)P_{ac}u, P_{ac}u) \\ &= 0. \end{aligned}$$

Therefore $(x - \epsilon, x + \epsilon) \cap \text{supp } \mu_{u,ac} = \emptyset$ for all $u \in E(\Delta)\mathcal{H}$, in particular, for all $u \in \mathcal{D}_\Delta$. Thus

$$x \notin \overline{\bigcup_{u \in \mathcal{D}_\Delta} \text{supp } \mu_{u,ac}}$$

and the inclusion \supset in (3.17) follows. For the converse inclusion assume that x does not belong to the right hand side of (3.17). Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus \text{supp } \mu_{u,ac}$ for all $u \in \mathcal{D}_\Delta$, that is,

$$\|E((x - \epsilon, x + \epsilon))P_{ac}u\|^2 = \mu_{u,ac}((x - \epsilon, x + \epsilon)) = 0$$

for all $u \in \mathcal{D}_\Delta$, and hence also for all $u \in \text{clsp } \mathcal{D}_\Delta$. With the help of (3.18) it follows

$$E((x - \epsilon, x + \epsilon) \cap \Delta)P_{ac}u = E((x - \epsilon, x + \epsilon))P_{ac}E(\Delta)u = 0$$

for all $u \in \mathcal{H}$. This shows that $(x - \epsilon, x + \epsilon) \cap \Delta$ does not contain absolutely continuous spectrum of A_0 , in particular, $x \notin \overline{\sigma_{ac}(A_0) \cap \Delta}$ and the inclusion \subset in (3.17) follows.

Step 2. In this step we show that the identity

$$\text{supp } \mu_{u,ac} = \text{cl}_{ac}(\{x \in \Delta : 0 < \text{Im}(M(x + i0)g, g) < +\infty\}) \tag{3.19}$$

holds for all $u = E(\Delta)\gamma(\zeta)g \in \mathcal{D}_\Delta$. Indeed, with the help of the formula (2.2) we compute

$$\begin{aligned} &\text{Im}(M(x + iy)g, g) \\ &= y\|\gamma(\zeta)g\|^2 + (|x - \zeta|^2 - y^2) \text{Im}((A_0 - (x + iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g) \\ &\quad + 2(x - \text{Re } \zeta)y \text{Re}((A_0 - (x + iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g), \end{aligned} \tag{3.20}$$

for all $x \in \mathbb{R}, y > 0, g \in \text{ran } \Gamma_0$, and $\zeta \in \mathbb{C} \setminus \mathbb{R}$. Moreover, dominated convergence implies that

$$y \operatorname{Re} \left((A_0 - (x + iy))^{-1} \gamma(\zeta)g, \gamma(\zeta)g \right) = \int_{\mathbb{R}} \frac{y(t-x)}{(t-x)^2 + y^2} d(E(t)\gamma(\zeta)g, \gamma(\zeta)g)$$

converges to zero as $y \searrow 0$. Therefore for $x \in \mathbb{R}$ (3.20) implies

$$\operatorname{Im}(M(x + i0)g, g) = |x - \zeta|^2 \operatorname{Im} \left((A_0 - (x + i0))^{-1} \gamma(\zeta)g, \gamma(\zeta)g \right), \tag{3.21}$$

in the sense that one of the limits exists if and only if the other limit exists, where $+\infty$ is allowed as (improper) limit.

For $u \in \mathcal{H}, x \in \mathbb{R}$, and $y > 0$ the imaginary part of the Stieltjes transform F_u of the measure $\mu_u = (E(\cdot)u, u)$ is given by

$$\begin{aligned} \operatorname{Im} F_u(x + iy) &= \operatorname{Im} \int_{\mathbb{R}} \frac{1}{t - (x + iy)} d(E(t)u, u) \\ &= \operatorname{Im} \left((A_0 - (x + iy))^{-1} u, u \right), \end{aligned} \tag{3.22}$$

and for $u \in E(\Delta)\mathcal{H}$ we obtain

$$\operatorname{Im} F_u(x + i0) = \begin{cases} \operatorname{Im} \left((A_0 - (x + i0))^{-1} u, u \right) & \text{if } x \in \Delta, \\ 0 & \text{if } x \notin \overline{\Delta}, \end{cases}$$

in particular, if $u = E(\Delta)\gamma(\zeta)g \in \mathcal{D}_\Delta$ then

$$\operatorname{Im} F_u(x + i0) = \begin{cases} \operatorname{Im} \left((A_0 - (x + i0))^{-1} \gamma(\zeta)g, \gamma(\zeta)g \right) & \text{if } x \in \Delta, \\ 0 & \text{if } x \notin \overline{\Delta}. \end{cases}$$

Taking into account (3.21) we then find

$$\operatorname{Im} F_u(x + i0) = \begin{cases} |x - \zeta|^{-2} \operatorname{Im}(M(x + i0)g, g) & \text{if } x \in \Delta, \\ 0 & \text{if } x \notin \overline{\Delta}, \end{cases} \tag{3.23}$$

for $u = E(\Delta)\gamma(\zeta)g \in \mathcal{D}_\Delta$. From Lemma 3.7 (i) we conclude together with (3.23) that

$$\begin{aligned} \operatorname{supp} \mu_{u,ac} &= \operatorname{cl}_{ac}(\{x \in \Delta : 0 < \operatorname{Im} F_u(x + i0) < +\infty\}) \\ &= \operatorname{cl}_{ac}(\{x \in \Delta : 0 < \operatorname{Im}(M(x + i0)g, g) < +\infty\}) \end{aligned}$$

holds for $u = E(\Delta)\gamma(\zeta)g \in \mathcal{D}_\Delta$, which shows (3.19). \square

Theorem 3.8 immediately implies the following two corollaries.

Corollary 3.9. *Let Assumption 3.1 be satisfied and assume that (3.14) holds for each open interval $\delta \subset \mathbb{R}$ such that $\delta \cap \sigma_p(S) = \emptyset$. Then*

$$\sigma_{ac}(A_0) = \overline{\bigcup_{g \in \text{ran } \Gamma_0} \text{cl}_{ac}(\{x \in \mathbb{R} : 0 < \text{Im}(M(x + i0)g, g) < +\infty\})}.$$

Corollary 3.10. *Let Assumption 3.1 be satisfied and let $\Delta \subset \mathbb{R}$ be an open interval such that the condition (3.8) holds. Then the absolutely continuous spectrum of A_0 in Δ is given by*

$$\overline{\sigma_{ac}(A_0) \cap \Delta} = \overline{\bigcup_{g \in \text{ran } \Gamma_0} \text{cl}_{ac}(\{x \in \Delta : 0 < \text{Im}(M(x + i0)g, g) < +\infty\})}.$$

In the next corollary a necessary and sufficient condition for the absence of absolutely continuous spectrum is given.

Corollary 3.11. *Let Assumption 3.1 be satisfied and let $\Delta \subset \mathbb{R}$ be an open interval. Assume that the condition (3.14) holds for each open interval $\delta \subset \Delta$ with $\delta \cap \sigma_p(S) = \emptyset$. Then $\sigma_{ac}(A_0) \cap \Delta = \emptyset$ if and only if for each $g \in \text{ran } \Gamma_0$ one has $\text{Im}(M(x + i0)g, g) = 0$ for almost all $x \in \Delta$.*

Proof. We make use of the fact that for $g \in \text{ran } \Gamma_0$

$$\text{cl}_{ac}(\{x \in \Delta : 0 < \text{Im}(M(x + i0)g, g) < +\infty\}) = \emptyset \tag{3.24}$$

if and only if

$$|\{x \in \Delta : 0 < \text{Im}(M(x + i0)g, g) < +\infty\}| = 0. \tag{3.25}$$

Assume first that $\sigma_{ac}(A_0) \cap \Delta = \emptyset$. Then (3.15) yields (3.24) for all $g \in \text{ran } \Gamma_0$, and hence (3.25) holds for all $g \in \text{ran } \Gamma_0$. Moreover, for $u = \gamma(\zeta)g$ and $x \in \mathbb{R}$ by (3.21) and (3.22) we have

$$\text{Im}(M(x + i0)g, g) = |x - \zeta|^2 \text{Im } F_u(x + i0),$$

and by Lemma 3.7 this limit exists and is finite for Lebesgue almost all $x \in \mathbb{R}$. Hence (3.25) implies that for all $g \in \text{ran } \Gamma_0$ one has $\text{Im}(M(x + i0)g, g) = 0$ for almost all $x \in \Delta$. For the converse implication assume that for every $g \in \text{ran } \Gamma_0$ one has $\text{Im}(M(x + i0)g, g) = 0$ for almost all $x \in \Delta$. Then (3.25) and hence also (3.24) holds for all $g \in \text{ran } \Gamma_0$. Thus (3.15) yields $\sigma_{ac}(A_0) \cap \Delta = \emptyset$. \square

Let us prove next inclusions for the singular and singular continuous spectra of A_0 . Recall the definition of the continuous closure $\text{cl}_c(\chi)$ of a Borel set χ in (3.11).

Theorem 3.12. *Let Assumption 3.1 be satisfied, and let $\Delta \subset \mathbb{R}$ be an open interval. Then the following assertions hold.*

(i) *If the condition (3.8) holds then the singular spectrum of A_0 in Δ satisfies*

$$(\sigma_s(A_0) \cap \Delta) \subset \overline{\bigcup_{g \in \text{ran } \Gamma_0} \{x \in \Delta : \text{Im}(M(x + i0)g, g) = +\infty\}}.$$

(ii) *If the condition (3.14) is satisfied for each open interval $\delta \subset \Delta$ with $\delta \cap \sigma_p(S) = \emptyset$ then the singular continuous spectrum of A_0 in Δ , $\sigma_{sc}(A_0) \cap \Delta$, is contained in the set*

$$\overline{\bigcup_{g \in \text{ran } \Gamma_0} \text{cl}_c(\{x \in \Delta : \text{Im}(M(x + i0)g, g) = +\infty, \lim_{y \searrow 0} y(M(x + iy)g, g) = 0\})}.$$

If S is simple then (i) and (ii) hold for each open interval Δ , including the case $\Delta = \mathbb{R}$.

Proof. We show the statements (i) and (ii) at once. Let us define

$$\mathcal{D}_\Delta := \{E(\Delta)\gamma(\zeta)g : \zeta \in \mathbb{C} \setminus \mathbb{R}, g \in \text{ran } \Gamma_0\}.$$

Note first that the same arguments as in Step 1 of the proof of Theorem 3.8 imply

$$\overline{\sigma_i(A_0) \cap \Delta} = \overline{\bigcup_{u \in \mathcal{D}_\Delta} \text{supp } \mu_{u,i}}, \quad i = s, sc. \tag{3.26}$$

In order to apply Lemma 3.7 (ii) and (iii), respectively, we calculate the limits that appear there. In fact, it follows from (2.2) that for each $g \in \text{ran } \Gamma_0$ and each $\zeta \in \mathbb{C} \setminus \mathbb{R}$

$$\lim_{y \searrow 0} \text{Im}(M(x + iy)g, g) = |x - \zeta|^2 \lim_{y \searrow 0} \text{Im}((A_0 - (x + iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g) \tag{3.27}$$

and

$$\lim_{y \searrow 0} y(M(x + iy)g, g) = |x - \zeta|^2 \lim_{y \searrow 0} y((A_0 - (x + iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g) \tag{3.28}$$

hold; cf. (3.21) for the first identity and the text below (3.21) for its interpretation as a possibly improper limit. Let $u = E(\Delta)\gamma(\zeta)g \in \mathcal{D}_\Delta$ and let

$$F_u(x + iy) = \int_{\mathbb{R}} \frac{1}{t - (x + iy)} d(E(t)u, u) = ((A_0 - (x + iy))^{-1}u, u)$$

be the Stieltjes transform of $\mu_u = (E(\cdot)u, u)$. Then

$$\text{Im } F_u(x + i0) = \text{Im}((A_0 - (x + i0))^{-1}E(\Delta)\gamma(\zeta)g, E(\Delta)\gamma(\zeta)g)$$

for all $x \in \mathbb{R}$. From this we conclude with the help of (3.27) that

$$\operatorname{Im} F_u(x + i0) = \begin{cases} |x - \zeta|^{-2} \operatorname{Im}(M(x + i0)g, g) & \text{if } x \in \Delta, \\ 0 & \text{if } x \notin \overline{\Delta}. \end{cases} \tag{3.29}$$

Similarly, from (3.28) we obtain

$$\lim_{y \searrow 0} y F_u(x + iy) = \begin{cases} |x - \zeta|^{-2} \lim_{y \searrow 0} y(M(x + iy)g, g) & \text{if } x \in \Delta, \\ 0 & \text{if } x \notin \overline{\Delta}. \end{cases} \tag{3.30}$$

It follows from (3.29), (3.30), and Lemma 3.7 that

$$\operatorname{supp} \mu_{u,s} \subset \overline{\{x \in \Delta : \operatorname{Im}(M(x + i0)g, g) = +\infty\}}$$

and

$$\operatorname{supp} \mu_{u,s} \subset \operatorname{cl}_c \left(\left\{ x \in \Delta : \operatorname{Im}(M(x + i0)g, g) = +\infty, \lim_{y \searrow 0} y(M(x + iy)g, g) = 0 \right\} \right)$$

for $u = E(\Delta)\gamma(\zeta)g \in \mathcal{D}_\Delta$. Thus the assertions of the theorem follow from (3.26). \square

We formulate two immediate corollaries which concern the singular continuous spectrum.

Corollary 3.13. *Let Assumption 3.1 be satisfied and assume that (3.14) holds for each open interval $\delta \subset \mathbb{R}$ such that $\delta \cap \sigma_p(S) = \emptyset$. Then the singular continuous spectrum $\sigma_{sc}(A_0)$ of A_0 is contained in the set*

$$\bigcup_{g \in \operatorname{ran} \Gamma_0} \overline{\operatorname{cl}_c(\{x \in \mathbb{R} : \operatorname{Im}(M(x + i0)g, g) = +\infty, \lim_{y \searrow 0} y(M(x + iy)g, g) = 0\})}.$$

Corollary 3.14. *Let Assumption 3.1 be satisfied, let $\Delta \subset \mathbb{R}$ be an open interval, and assume that the condition (3.8) holds. Then the singular continuous spectrum of A_0 in Δ , $\sigma_{sc}(A_0) \cap \Delta$, is contained in the set*

$$\bigcup_{g \in \operatorname{ran} \Gamma_0} \overline{\operatorname{cl}_c(\{x \in \Delta : \operatorname{Im}(M(x + i0)g, g) = +\infty, \lim_{y \searrow 0} y(M(x + iy)g, g) = 0\})}.$$

As a further immediate corollary of Theorem 3.12 we formulate a sufficient criterion for the absence of singular continuous spectrum in terms of the limiting behavior of the function M . The corresponding result for ordinary boundary triples (in the special case $\Delta = \mathbb{R}$) can be found in [10].

Corollary 3.15. *Let Assumption 3.1 be satisfied and let $\Delta \subset \mathbb{R}$ be an open interval such that the condition (3.14) is satisfied for each open interval $\delta \subset \Delta$ with $\delta \cap \sigma_p(S) = \emptyset$. If for each $g \in \operatorname{ran} \Gamma_0$ there exist at most countably many $x \in \Delta$ such that*

$$\text{Im}(M(x + iy)g, g) \rightarrow +\infty \quad \text{and} \quad y(M(x + iy)g, g) \rightarrow 0 \quad \text{as} \quad y \searrow 0$$

then $\sigma_{\text{sc}}(A_0) \cap \Delta = \emptyset$. If S is simple the assertion holds for each open interval Δ , including the case $\Delta = \mathbb{R}$.

As a further corollary of the theorems of this section we provide sufficient criteria for the spectrum of the operator A_0 to be purely absolutely continuous or purely singular continuous, respectively, in some set.

Corollary 3.16. *Let Assumption 3.1 be satisfied, let $\Delta \subset \mathbb{R}$ be an open interval such that the condition (3.8) is satisfied, and assume that*

$$\lim_{y \searrow 0} yM(x + iy)g = 0 \tag{3.31}$$

for all $g \in \text{ran } \Gamma_0$ and all $x \in \Delta$. Then the following assertions hold.

- (i) *If for each $g \in \text{ran } \Gamma_0$ there exist at most countably many $x \in \Delta$ such that $\text{Im}(M(x + i0)g, g) = +\infty$ then $\sigma(A_0) \cap \Delta = \sigma_{\text{ac}}(A_0) \cap \Delta$.*
- (ii) *If for each $g \in \text{ran } \Gamma_0$ one has $\text{Im}(M(x + i0)g, g) = 0$ for almost all $x \in \Delta$ then $\sigma(A_0) \cap \Delta = \sigma_{\text{sc}}(A_0) \cap \Delta$.*

In particular, if S is simple and Δ is an arbitrary open interval such that (3.31) holds for all $g \in \text{ran } \Gamma_0$ and all $x \in \Delta$ then (i) and (ii) are satisfied.

4. Second order elliptic differential operators on \mathbb{R}^n

In this section we show how the spectrum of a selfadjoint second order elliptic differential operator on \mathbb{R}^n , $n \geq 2$, can be described with the help of a Titchmarsh–Weyl function acting on an $n - 1$ -dimensional compact interface Σ which splits \mathbb{R}^n into a bounded domain Ω_i and an unbounded domain Ω_e with common boundary Σ .

We consider the differential expression

$$\mathcal{L} = - \sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^n \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \bar{a}_j \right) + a,$$

where $a_{jk}, a_j \in C^\infty(\mathbb{R}^n)$ together with their derivatives are bounded and satisfy $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \mathbb{R}^n$, $1 \leq j, k \leq n$, and $a \in L^\infty(\mathbb{R}^n)$ is real valued. Moreover, we assume that \mathcal{L} is uniformly elliptic on \mathbb{R}^n , that is, there exists $E > 0$ with

$$\sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k \geq E \sum_{k=1}^n \xi_k^2, \quad x \in \mathbb{R}^n, \quad \xi = (\xi_1, \dots, \xi_n)^\top \in \mathbb{R}^n. \tag{4.1}$$

The selfadjoint operator associated with \mathcal{L} in $L^2(\mathbb{R}^n)$ is given by

$$A_0 u = \mathcal{L} u, \quad \text{dom } A_0 = H^2(\mathbb{R}^n), \tag{4.2}$$

where $H^2(\mathbb{R}^n)$ is the usual L^2 -based Sobolev space of order 2 on \mathbb{R}^n . In Sections 4.1 and 4.2 two different choices of Titchmarsh–Weyl functions for the differential expression \mathcal{L} are studied.

4.1. A Weyl function corresponding to a transmission problem

We first consider a Weyl function for the operator A_0 which appears in transmission problems in connection with single layer potentials (see, e.g. [50, Chapter 6]) and which was also used in [2] to generalize the classical limit point/limit circle analysis from singular Sturm–Liouville theory to Schrödinger operators in \mathbb{R}^3 .

Let Σ be the boundary of a bounded C^∞ -domain $\Omega_i \subset \mathbb{R}^n$ and denote by Ω_e the corresponding exterior domain, that is, $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega_i}$. In the following we make use of operators induced by \mathcal{L} in $L^2(\Omega_i)$ and $L^2(\Omega_e)$, respectively. For $j = i, e$ we write \mathcal{L}_j for the restriction of the differential expression \mathcal{L} to functions on Ω_j . For functions in $L^2(\Omega_j)$ we use the index j and we write $u = u_i \oplus u_e$ for $u \in L^2(\mathbb{R}^n)$. As Σ is smooth, the selfadjoint Dirichlet operator associated with \mathcal{L}_j in $L^2(\Omega_j)$ is given by

$$A_{D,j} u_j = \mathcal{L}_j u_j, \quad \text{dom } A_{D,j} = \{u_j \in H^2(\Omega_j) : u_j|_\Sigma = 0\}, \quad j = i, e,$$

where $u_j|_\Sigma$ denotes the trace of u_j at $\Sigma = \partial\Omega_j$. Let $H^s(\Sigma)$ be the Sobolev spaces of orders $s \geq 0$ on Σ . We recall that for each $\lambda \in \rho(A_{D,j})$ and each $g \in H^{3/2}(\Sigma)$ there exists a unique solution $u_{\lambda,j} \in H^2(\Omega_j)$ of the boundary value problem $\mathcal{L}_j u_j = \lambda u_j$, $u_j|_\Sigma = g$. This implies that for each $\lambda \in \rho(A_{D,j})$ the Dirichlet-to-Neumann map

$$\Lambda_j(\lambda) : H^{3/2}(\Sigma) \rightarrow H^{1/2}(\Sigma), \quad u_{\lambda,j}|_\Sigma \mapsto \frac{\partial u_{\lambda,j}}{\partial \nu_{\mathcal{L}_j}} \Big|_\Sigma, \tag{4.3}$$

is well-defined; here the conormal derivative with respect to \mathcal{L}_j in the direction of the outer unit normal $\nu_j = (\nu_{j,1}, \dots, \nu_{j,n})^\top$ at $\Sigma = \partial\Omega_j$ is defined by

$$\frac{\partial u}{\partial \nu_{\mathcal{L}_j}} \Big|_\Sigma = \sum_{k,l=1}^n a_{kl} \nu_{j,k} \frac{\partial u}{\partial x_l} \Big|_\Sigma + \sum_{k=1}^n \overline{a_k} \nu_{j,k} u|_\Sigma.$$

Note that the outer unit normals at $\partial\Omega_i$ and $\partial\Omega_e$ coincide up to a minus sign. It will turn out below that the operator $\Lambda_i(\lambda) + \Lambda_e(\lambda)$ is invertible for all $\lambda \in \rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{D,e})$ and, hence, the operator function

$$\lambda \mapsto M(\lambda) = (\Lambda_i(\lambda) + \Lambda_e(\lambda))^{-1} \tag{4.4}$$

is well-defined on $\rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{D,\epsilon})$. We remark that the values $M(\lambda)$ are bounded operators in $L^2(\Sigma)$ with domain $H^{1/2}(\Sigma)$; cf. [Lemma 4.2](#) below for the details.

The following theorem is the main result of this section. It states that the absolutely continuous spectrum of A_0 can be recovered completely from the knowledge of the function M in [\(4.4\)](#), while the eigenvalues and corresponding eigenspaces may be only partially visible for the function M . This depends on the choice of the interface Σ and the fact that the symmetric operator

$$Su = \mathcal{L}u, \quad \text{dom } S = \{u \in H^2(\mathbb{R}^n) : u|_\Sigma = 0\}, \tag{4.5}$$

may have eigenvalues. In particular, in general S is not simple; cf. [Example 4.5](#) and [Example 4.6](#) below.

Theorem 4.1. *Let $A_0, \Sigma, S,$ and M be as above, let $\lambda, \mu \in \mathbb{R}$ such that $\lambda \notin \overline{\sigma_p(S)}, \mu \notin \sigma_p(S)$, and let $\Delta \subset \mathbb{R}$ be an open interval. Then the following assertions hold.*

- (i) $\mu \in \sigma_p(A_0)$ if and only if $R_\mu M := s\text{-}\lim_{\eta \searrow 0} i\eta M(\mu + i\eta) \neq 0$; if the multiplicity of the eigenvalue μ is finite then the mapping

$$\tau : \ker(A_0 - \mu) \rightarrow \text{ran } R_\mu M, \quad u \mapsto u|_\Sigma, \tag{4.6}$$

is bijective; if the multiplicity of the eigenvalue μ is infinite then the mapping

$$\tau : \ker(A_0 - \mu) \rightarrow \text{cl}_\tau(\text{ran } R_\mu M), \quad u \mapsto u|_\Sigma, \tag{4.7}$$

is bijective, where cl_τ denotes the closure in the normed space $\text{ran } \tau$, equipped with the norm in $L^2(\Sigma)$.

- (ii) λ is an isolated eigenvalue of A_0 if and only if λ is a pole in the strong sense of M . In this case [\(4.6\)](#) and [\(4.7\)](#) with $\mu = \lambda$ are bijective mappings and $R_\lambda M = \text{Res}_\lambda M$.
- (iii) $\lambda \in \rho(A_0)$ if and only if M can be continued analytically into λ .
- (iv) $\lambda \in \sigma_c(A_0)$ if and only if $s\text{-}\lim_{\eta \searrow 0} i\eta M(\lambda + i\eta) = 0$ and M cannot be continued analytically into λ .
- (v) The absolutely continuous spectrum $\sigma_{ac}(A_0)$ of A_0 in Δ is given by

$$\overline{\sigma_{ac}(A_0) \cap \Delta} = \overline{\bigcup_{g \in H^{1/2}(\Sigma)} \text{cl}_{ac}(\{x \in \Delta : 0 < \text{Im}(M(x + i0)g, g) < +\infty\})}$$

and, in particular, $\sigma_{ac}(A_0) \cap \Delta = \emptyset$ if and only if for each $g \in H^{1/2}(\Sigma)$ one has $\text{Im}(M(x + i0)g, g) = 0$ for almost all $x \in \Delta$.

- (vi) The singular continuous spectrum of A_0 in Δ , $\sigma_{sc}(A_0) \cap \Delta$, is contained in

$$\overline{\bigcup_{g \in H^{1/2}(\Sigma)} \text{cl}_c(\{x \in \Delta : \text{Im}(M(x + i0)g, g) = +\infty, \lim_{y \searrow 0} y(M(x + iy)g, g) = 0\})},$$

and, in particular, if for each $g \in H^{1/2}(\Sigma)$ there exist at most countably many $x \in \Delta$ such that $\text{Im}(M(x+iy)g, g) \rightarrow +\infty$ and $y(M(x+iy)g, g) \rightarrow 0$ as $y \searrow 0$ then $\sigma_{\text{sc}}(A_0) \cap \Delta = \emptyset$.

The proof of [Theorem 4.1](#) makes use of the following two lemmas and is given at the end of this subsection.

Lemma 4.2. *Let S be defined as in (4.5) and let*

$$\begin{aligned} Tu &= \mathcal{L}_i u_i \oplus \mathcal{L}_e u_e, \\ \text{dom } T &= \{u_i \oplus u_e \in H^2(\Omega_i) \oplus H^2(\Omega_e) : u_i|_\Sigma = u_e|_\Sigma\}. \end{aligned} \tag{4.8}$$

Then $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0, \Gamma_1 : \text{dom } T \rightarrow L^2(\Sigma), \quad \Gamma_0 u = \frac{\partial u_i}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma + \frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma, \quad \Gamma_1 u = u|_\Sigma,$$

is a quasi boundary triple for S^* such that $A_0 = T \upharpoonright \ker \Gamma_0$ and $\text{ran } \Gamma_0 = H^{1/2}(\Sigma)$. For all $\lambda \in \rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{D,e})$ the corresponding Weyl function coincides with the function M in (4.4), and $\text{dom } M(\lambda) = H^{1/2}(\Sigma)$.

Proof. The proof is similar to the proof of [5, Proposition 3.2]. For the convenience of the reader we provide the details. In order to show that $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for S^* we verify (i)–(iii) in the assumptions of [Proposition 2.2](#). Recall first that by the classical trace theorem the mapping

$$H^2(\Omega_j) \rightarrow H^{3/2}(\Sigma) \times H^{1/2}(\Sigma), \quad u_j \mapsto \left\{ u_j|_\Sigma, \frac{\partial u_j}{\partial \nu_{\mathcal{L}_j}} \Big|_\Sigma \right\}, \quad j = i, e,$$

is onto. Hence, for given $\varphi \in H^{1/2}(\Sigma)$ and $\psi \in H^{3/2}(\Sigma)$ there exist $u_j \in H^2(\Omega_j)$ such that

$$\frac{\partial u_i}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma = \varphi, \quad \frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma = 0, \quad \text{and} \quad u_i|_\Sigma = \psi = u_e|_\Sigma,$$

and it follows $u_i \oplus u_e \in \text{dom } T$, $\Gamma_0(u_i \oplus u_e) = \varphi$, and $\Gamma_1(u_i \oplus u_e) = \psi$. This implies that $\text{ran}(\Gamma_0, \Gamma_1)^\top = H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)$. In particular, $\text{ran}(\Gamma_0, \Gamma_1)^\top$ is dense in $L^2(\Sigma) \times L^2(\Sigma)$. Furthermore, $C^\infty(\mathbb{R}^n \setminus \Sigma)$ is a dense subspace of $L^2(\mathbb{R}^n)$ which is contained in $\ker \Gamma_0 \cap \ker \Gamma_1$. Thus (i) in [Proposition 2.2](#) holds. Next we verify the identity (2.1) for $u = u_i \oplus u_e, v = v_i \oplus v_e \in \text{dom } T$. With the help of Green’s identity and $u|_\Sigma = u_j|_\Sigma, v|_\Sigma = v_j|_\Sigma, j = i, e$, we compute

$$\begin{aligned}
 (Tu, v) - (u, Tv) &= (\mathcal{L}_e u_e, v_e) - (u_e, \mathcal{L}_e v_e) + (\mathcal{L}_i u_i, v_i) - (u_i, \mathcal{L}_i v_i) \\
 &= \left(u_e|_\Sigma, \frac{\partial v_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma \right) - \left(\frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma, v_e|_\Sigma \right) \\
 &\quad + \left(u_i|_\Sigma, \frac{\partial v_i}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma \right) - \left(\frac{\partial u_i}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma, v_i|_\Sigma \right) \\
 &= \left(u|_\Sigma, \frac{\partial v_i}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma + \frac{\partial v_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma \right) - \left(\frac{\partial u_i}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma + \frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma, v|_\Sigma \right) \\
 &= (\Gamma_1 u, \Gamma_0 v) - (\Gamma_0 u, \Gamma_1 v).
 \end{aligned}$$

We have shown that (ii) in Proposition 2.2 holds. Finally it is not difficult to see that $\text{dom } A_0 = H^2(\mathbb{R}^n)$ is contained in $\ker \Gamma_0$, that is, assumption (iii) in Proposition 2.2 is satisfied. Therefore we obtain from Proposition 2.2 that $T \upharpoonright (\ker \Gamma_0 \cap \ker \Gamma_1)$ is a densely defined, closed, symmetric operator in $L^2(\mathbb{R}^n)$, that $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for its adjoint and that $A_0 = T \upharpoonright \ker \Gamma_0$. In particular, $T \upharpoonright \ker \Gamma_0$ is defined on $H^2(\mathbb{R}^n)$. Hence $T \upharpoonright (\ker \Gamma_0 \cap \ker \Gamma_1)$ coincides with the symmetric operator S in (4.5) and $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $\bar{T} = S^*$. It remains to check that the corresponding Weyl function has the form (4.4). For this let $\lambda \in \rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{D,e})$ and let $u_\lambda = u_{\lambda,i} \oplus u_{\lambda,e} \in \ker(T - \lambda)$, that is, $u_{\lambda,j} \in H^2(\Omega_j)$, $j = i, e$, $u_{\lambda,i}|_\Sigma = u_{\lambda,e}|_\Sigma$, and $\mathcal{L}_j u_{\lambda,j} = \lambda u_{\lambda,j}$, $j = i, e$. Then we have

$$(\Lambda_i(\lambda) + \Lambda_e(\lambda))\Gamma_1 u_\lambda = \frac{\partial u_{\lambda,i}}{\partial \nu_{\mathcal{L}_i}} \Big|_\Sigma + \frac{\partial u_{\lambda,e}}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma = \Gamma_0 u_\lambda. \tag{4.9}$$

Note further that $\Lambda_i(\lambda) + \Lambda_e(\lambda)$ is injective for all $\lambda \in \rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{D,e})$. In fact, assume $\Gamma_1 u_\lambda \in \ker(\Lambda_i(\lambda) + \Lambda_e(\lambda))$. Then (4.9) implies $u_\lambda \in \ker \Gamma_0 = \text{dom } A_0$, and it follows $u_\lambda \in \ker(A_0 - \lambda)$. Since $\lambda \in \rho(A_0)$ we obtain $u_\lambda = 0$ and, hence, $\Gamma_1 u_\lambda = 0$. Therefore it follows from (4.9) that the Weyl function corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ coincides with the function M in (4.4). \square

In the next lemma it is shown that S satisfies the local simplicity in the assumptions of the results in Section 3.

Lemma 4.3. *Let A_0 be the selfadjoint elliptic operator in (4.2) with spectral measure $E(\cdot)$ and let S be the symmetric operator in (4.5). Let $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple in Lemma 4.2 and let γ be the corresponding γ -field. Then*

$$E(\delta)L^2(\mathbb{R}^n) = \text{clsp}\{E(\delta)\gamma(\nu)g : g \in H^{1/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R}\}$$

holds for every open interval $\delta \subset \mathbb{R}$ such that $\delta \cap \sigma_p(S) = \emptyset$.

Proof. For $j = i, e$ we consider the densely defined, closed, symmetric operators

$$S_j u_j = \mathcal{L}_j u_j, \quad \text{dom } S_j = \left\{ u_j \in H^2(\Omega_j) : u_j|_\Sigma = \frac{\partial u_j}{\partial \nu_{\mathcal{L}_j}} \Big|_\Sigma = 0 \right\},$$

in $L^2(\Omega_j)$ and the operators

$$T_j u_j = \mathcal{L}_j u_j, \quad \text{dom } T_j = H^2(\Omega_j),$$

in $L^2(\Omega_j)$. It is not difficult to verify that $\{L^2(\Sigma), \Gamma_0^j, \Gamma_1^j\}$, where

$$\Gamma_0^j, \Gamma_1^j : \text{dom } T_j \rightarrow L^2(\Sigma), \quad \Gamma_0^j u_j = u_j|_\Sigma, \quad \Gamma_1^j u_j = -\frac{\partial u_j}{\partial \nu_{\mathcal{L}_j}} \Big|_\Sigma,$$

is a quasi boundary triple for S_j^* , $j = i, e$; cf. [3, Proposition 4.1]. For $\lambda \in \rho(A_{D,j})$, $j = i, e$, the corresponding γ -fields are given by

$$\gamma_j(\lambda) : L^2(\Sigma) \supset H^{3/2}(\Sigma) \rightarrow L^2(\Omega_j), \quad \varphi \mapsto \gamma_j(\lambda)\varphi = u_{\lambda,j},$$

where $u_{\lambda,j}$ is the unique solution in $H^2(\Omega_j)$ of $\mathcal{L}_j u_j = \lambda u_j$, $u_j|_\Sigma = \varphi$. It follows in the same way as in [8, Proposition 2.2] that S_e is simple; the simplicity of S_i follows from a unique continuation argument, see, e.g. [7, Proposition 2.5]. Therefore we have

$$L^2(\Omega_j) = \text{clsp}\{\gamma_j(\nu)g : g \in H^{3/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R}\}, \quad j = i, e,$$

and hence

$$\begin{aligned} L^2(\mathbb{R}^n) &= L^2(\Omega_i) \oplus L^2(\Omega_e) \\ &= \text{clsp}\{\gamma_i(\mu)g \oplus \gamma_e(\nu)h : g, h \in H^{3/2}(\Sigma), \mu, \nu \in \mathbb{C} \setminus \mathbb{R}\}. \end{aligned} \tag{4.10}$$

Here and in the following \oplus denotes the orthogonality of $L^2(\Omega_i)$ and $L^2(\Omega_e)$ in $L^2(\mathbb{R}^n)$.

Let now $\delta \subset \mathbb{R}$ be an open interval such that $\delta \cap \sigma_p(S) = \emptyset$ and let T be as in (4.8). Since

$$\{\gamma_i(\nu)g \oplus \gamma_e(\nu)g : g \in H^{3/2}(\Sigma)\} = \ker(T - \nu) = \text{ran } \gamma(\nu), \quad \nu \in \mathbb{C} \setminus \mathbb{R}, \tag{4.11}$$

we have to verify that

$$\mathcal{H}_\delta := \text{clsp}\{E(\delta)(\gamma_i(\nu)g \oplus \gamma_e(\nu)g) : g \in H^{3/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R}\} = E(\delta)L^2(\mathbb{R}^n).$$

We note first that the inclusion $\mathcal{H}_\delta \subset E(\delta)L^2(\mathbb{R}^n)$ is obviously true. For the opposite inclusion we conclude from (4.10) that it suffices to verify

$$E(\delta)(\gamma_i(\mu)g \oplus 0) \in \mathcal{H}_\delta, \quad g \in H^{3/2}(\Sigma), \mu \in \mathbb{C} \setminus \mathbb{R}, \tag{4.12}$$

and

$$E(\delta)(0 \oplus \gamma_e(\nu)h) \in \mathcal{H}_\delta, \quad h \in H^{3/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R}. \tag{4.13}$$

Let us show the statements in (4.12)–(4.13). We start with (4.13). Let us fix $\mu \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 2.4 (ii) we have

$$\gamma_j(\nu)h = (I + (\nu - \mu)(A_{D,j} - \nu)^{-1})\gamma_j(\mu)h, \quad h \in H^{3/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R},$$

$j = i, e$. From this it follows

$$\begin{aligned} \mathcal{H}_\delta &= \text{clsp} \{ E(\delta)(\gamma_i(\nu)h \oplus \gamma_e(\nu)h) : h \in H^{3/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R} \} \\ &= \text{clsp} \left\{ E(\delta)(\gamma_i(\mu)h \oplus \gamma_e(\mu)h), \right. \\ &\quad \left. E(\delta)((A_{D,i} - \nu)^{-1}\gamma_i(\mu)h \oplus (A_{D,e} - \nu)^{-1}\gamma_e(\mu)h) : h \in H^{3/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R} \right\}. \end{aligned}$$

Since $A_{D,i}$ and $A_{D,e}$ are both semibounded from below we may choose $\lambda_0 \in \mathbb{R}$ such that $\sigma(A_{D,j}) \subset (\lambda_0, \infty)$, $j = i, e$. Recall that the spectrum of $A_{D,i}$ is purely discrete and let $\lambda_1 < \lambda_2 < \dots$ be the distinct eigenvalues of $A_{D,i}$. Then for all $\eta, \varepsilon > 0$ and $k = 0, 1, 2, \dots$ the function

$$E(\delta) \left[\int_{\lambda_k + \eta}^{\lambda_{k+1} - \eta} ((A_{D,i} - (\lambda + i\varepsilon))^{-1} - (A_{D,i} - (\lambda - i\varepsilon))^{-1})\gamma_i(\mu)h \, d\lambda \oplus \int_{\lambda_k + \eta}^{\lambda_{k+1} - \eta} ((A_{D,e} - (\lambda + i\varepsilon))^{-1} - (A_{D,e} - (\lambda - i\varepsilon))^{-1})\gamma_e(\mu)h \, d\lambda \right]$$

belongs to \mathcal{H}_δ , and as $(\lambda_k, \lambda_{k+1}) \subset \rho(A_{D,i})$, Stone’s formula implies

$$E(\delta)(0 \oplus E_e((\lambda_k, \lambda_{k+1}))\gamma_e(\mu)h) \in \mathcal{H}_\delta, \tag{4.14}$$

where $E_e(\cdot)$ is the spectral measure of $A_{D,e}$. Next we show that for the eigenvalues λ_k , $k = 1, 2, \dots$, of $A_{D,i}$ the property

$$E(\delta)(0 \oplus E_e(\{\lambda_k\})\gamma_e(\mu)h) \in \mathcal{H}_\delta \tag{4.15}$$

holds. For this consider the element

$$u = 0 \oplus E_e(\{\lambda_k\})\gamma_e(\mu)h$$

for some fixed $h \in H^{3/2}(\Sigma)$. Clearly, as $u \in \ker((A_{D,i} \oplus A_{D,e}) - \lambda_k)$ and as $A_{D,i} \oplus A_{D,e}$ is a selfadjoint extension of the symmetric operator S in (4.5) we may write u in the form $u = u_D \tilde{\oplus} u_S$ with $u_S \in \ker(S - \lambda_k)$ and

$$u_D \in \ker((A_{D,i} \oplus A_{D,e}) - \lambda_k) \tilde{\ominus} \ker(S - \lambda_k), \tag{4.16}$$

where $\tilde{\oplus}$ and $\tilde{\ominus}$ indicate the orthogonality of subspaces in $\ker((A_{D,i} \oplus A_{D,e}) - \lambda_k)$. Then for each $v \in \bigcap_{\nu \in \mathbb{C} \setminus \mathbb{R}} \text{ran}(S - \nu)$ and each $\nu \in \mathbb{C} \setminus \mathbb{R}$ one has

$$\begin{aligned} (v, u_D) &= ((S - \nu)(S - \nu)^{-1}v, u_D) \\ &= ((S - \nu)^{-1}v, ((A_{D,i} \oplus A_{D,e}) - \bar{\nu})u_D) \\ &= (\lambda_k - \nu)((S - \nu)^{-1}v, u_D). \end{aligned} \tag{4.17}$$

Since the limit

$$y := \lim_{\eta \searrow 0} \eta(S - (\lambda_k + i\eta))^{-1}v = \lim_{\eta \searrow 0} \eta((A_{D,i} \oplus A_{D,e}) - (\lambda_k + i\eta))^{-1}v$$

exists and

$$\begin{aligned} (y, (S^* - \lambda_k)w) &= \lim_{\eta \searrow 0} \eta((S - (\lambda_k + i\eta))^{-1}v, (S^* - \lambda_k)w) \\ &= \lim_{\eta \searrow 0} \eta((S - \lambda_k)(S - (\lambda_k + i\eta))^{-1}v, w) \\ &= \lim_{\eta \searrow 0} \eta[(v, w) + (i\eta(S - (\lambda_k + i\eta))^{-1}v, w)] \\ &= 0 \end{aligned}$$

holds for all $w \in \text{dom } S^*$ we conclude that

$$y = \lim_{\eta \searrow 0} \eta(S - (\lambda_k + i\eta))^{-1}v \in (\text{ran}(S^* - \lambda_k))^\perp = \ker(S - \lambda_k).$$

In particular, (4.16) implies $(y, u_D) = 0$. Therefore we obtain from the identity (4.17) with $\nu = \lambda_k + i\eta$ in the limit

$$(v, u_D) = -i \lim_{\eta \searrow 0} \eta((S - (\lambda_k + i\eta))^{-1}v, u_D) = -i(y, u_D) = 0.$$

This shows that u_D is orthogonal to $\bigcap_{\nu \in \mathbb{C} \setminus \mathbb{R}} \text{ran}(S - \nu)$ and hence

$$u_D \in \text{clsp}\{\ker(S^* - \nu) : \nu \in \mathbb{C} \setminus \mathbb{R}\} = \text{clsp}\{\ker(T - \nu) : \nu \in \mathbb{C} \setminus \mathbb{R}\}.$$

Therefore (4.11) implies

$$u_D \in \text{clsp}\{\gamma_i(\nu)h \oplus \gamma_e(\nu)h : h \in H^{3/2}(\Sigma), \nu \in \mathbb{C} \setminus \mathbb{R}\}. \tag{4.18}$$

Note that if the eigenvalue λ_k of $A_{D,i}$ is contained in the interval δ then by assumption $\lambda_k \notin \sigma_p(S)$ and hence $u = u_D$ in this case. If $\lambda_k \notin \delta$ then $u_S \in \ker(S - \lambda_k) \subset \ker(A_0 - \lambda_k)$

implies that u_S is orthogonal to $\text{ran } E(\delta)$, so that $E(\delta)u_S = 0$. Summing up we have for any eigenvalue $\lambda_k, k = 1, 2, \dots$, of $A_{D,i}$ that

$$E(\delta)(0 \oplus E_e(\{\lambda_k\})\gamma_e(\mu)h) = E(\delta)u = E(\delta)(u_S \tilde{\oplus} u_D) = E(\delta)u_D \in \mathcal{H}_\delta$$

by (4.18). We have shown (4.15).

Let $m \in \mathbb{N}$. Then we have

$$E_e((-\infty, \lambda_m))\gamma_e(\mu)h = \sum_{k=1}^{m-1} E_e(\{\lambda_k\})\gamma_e(\mu)h + \sum_{k=0}^{m-1} E_e((\lambda_k, \lambda_{k+1}))\gamma_e(\mu)h$$

and from (4.14) and (4.15) we conclude

$$E(\delta)(0 \oplus E_e(-\infty, \lambda_m)\gamma_e(\mu)h) \in \mathcal{H}_\delta.$$

Taking the limit $m \nearrow +\infty$ we obtain $E(\delta)(0 \oplus \gamma_e(\mu)h) \in \mathcal{H}_\delta$. We have proved (4.13).

For (4.12) observe that for $\mu \in \mathbb{C} \setminus \mathbb{R}$ fixed, $g \in H^{3/2}(\Sigma)$ and $k = 1, 2, \dots$

$$E(\delta)(E_i(\{\lambda_k\})\gamma_i(\mu)g \oplus 0) \in \mathcal{H}_\delta$$

can be verified in the same way as (4.15), where $E_i(\cdot)$ is the spectral measure of $A_{D,i}$. Hence for $m \in \mathbb{N}$ we conclude

$$E(\delta)(E_i((-\infty, \lambda_m))\gamma_i(\mu)g \oplus 0) \in \mathcal{H}_\delta$$

and in the limit $m \nearrow +\infty$ we obtain (4.12).

Now (4.12)–(4.13) together with (4.10) imply the inclusion $E(\delta)L^2(\mathbb{R}^n) \subset \mathcal{H}_\delta$. This completes the proof of Lemma 4.3. \square

As a consequence of Lemma 4.3 we obtain the following corollary.

Corollary 4.4. *The operator S in (4.5) is simple if and only if $\sigma_p(S) = \emptyset$.*

Proof of Theorem 4.1. Let $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple for $\bar{T} = S^*$ in Lemma 4.2. Then $T \upharpoonright \ker \Gamma_0$ corresponds to the selfadjoint elliptic differential operator A_0 in (4.2) and the associated Weyl function coincides with the operator function M in (4.4). Taking Lemma 4.3 into account, item (i) follows from Corollary 3.4 and items (ii)–(iv) are consequences of Theorem 3.5 and Proposition 3.6 when choosing an open interval $\delta \ni \lambda$ with $\delta \cap \sigma_p(S) = \emptyset$. Moreover, item (v) follows from Theorem 3.8 and Corollary 3.11, and item (vi) is due to Theorem 3.12 and Corollary 3.15. \square

We point out that in the case that the symmetric operator S is simple the assertions in Theorem 4.1 hold for all $\lambda, \mu \in \mathbb{R}$. On the other hand, without further assumptions, it

may happen that S possesses eigenvalues. In this case at least the parts of the eigenspaces of A_0 which do not belong to S can be characterized in terms of the function M ; cf. [Theorem 3.2](#). The next examples illustrate that a proper choice of the interface Σ may avoid eigenvalues of S .

Example 4.5. Assume that \mathcal{L} acts as the Laplacian outside some compact set $K \subset \mathbb{R}^n$ and choose Σ to be the boundary of any smooth, bounded domain $\Omega_i \supset K$. Then S does not have any eigenvalues. Indeed, if $u \in H^2(\mathbb{R}^n)$ satisfies $\mathcal{L}u = \lambda u$ on \mathbb{R}^n and $u|_\Sigma = 0$ then $u|_{\Omega_e}$ belongs to $\ker(A_{D,e} - \lambda)$ and must vanish. Then a unique continuation argument implies $u = 0$. Hence S is simple by [Corollary 4.4](#) and the assertions in [Theorem 4.1](#) hold for all $\lambda, \mu \in \mathbb{R}$.

Example 4.6. Let the coefficients of \mathcal{L} be chosen in such a way that for some bounded, smooth domain $\Omega_i \subset \mathbb{R}^n$ the operator $A_{D,i}$ in $L^2(\Omega_i)$ is strictly positive; for instance this happens if $-\frac{2}{E} \sum_{j=1}^n \|a_j\|_\infty^2 + \text{ess inf } a \geq 0$ on Ω_i , where E is an ellipticity constant for \mathcal{L} , see [\(4.1\)](#). If we choose $\Sigma = \partial\Omega_i$ then S has no non-positive eigenvalues, otherwise $Su = \lambda u$ for some $\lambda \leq 0$ and $u \in \text{dom } S$ with $u \neq 0$, and a unique continuation argument yields that u_i is nontrivial, thus u_i is an eigenfunction of $A_{D,i}$ corresponding to the eigenvalue $\lambda \leq 0$, a contradiction. Hence in this situation all non-positive eigenvalues of A_0 and the corresponding eigenspaces can be described completely in terms of the function M .

4.2. A block operator matrix Weyl function associated with a decoupled system

In this section we consider a different Weyl function for the operator A_0 , which corresponds to a symmetric operator which is always simple, independently of the choice of the interface Σ . This symmetric operator is the orthogonal sum of the minimal symmetric realizations S_i and S_e of \mathcal{L} in $L^2(\Omega_i)$ and $L^2(\Omega_e)$, respectively, in the proof of [Lemma 4.3](#), and hence an infinite dimensional restriction of the symmetric operator in [\(4.5\)](#); it can be viewed as a decoupled symmetric operator. Let Λ_i and Λ_e be the Dirichlet-to-Neumann maps for the interior and exterior elliptic boundary value problem, respectively, defined in [\(4.3\)](#), and let

$$A_{N,e}u_e = \mathcal{L}_e u_e, \quad \text{dom } A_{N,e} = \left\{ u_e \in H^2(\Omega_e) : \frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_\Sigma = 0 \right\},$$

be the selfadjoint realization of \mathcal{L}_e in $L^2(\Omega_e)$ with Neumann boundary conditions. In [Lemma 4.8](#) below it will turn out that the function

$$\lambda \mapsto \widetilde{M}(\lambda) = \begin{pmatrix} \Lambda_i(\lambda) & 1 \\ 1 & -\Lambda_e(\lambda)^{-1} \end{pmatrix}^{-1} \quad \text{in } L^2(\Sigma) \times L^2(\Sigma) \tag{4.19}$$

is well defined on $\rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{N,e})$ and can be viewed as the Weyl function of a quasi boundary triple for $(S_i \oplus S_e)^*$, where A_0 in [\(4.2\)](#) corresponds to the kernel of the

first boundary mapping. We mention that a scalar analog of the function \widetilde{M} in (4.19) appears in connection with λ -dependent Sturm–Liouville boundary value problems in [22] (see also [60]), in [36,38,42] for Hamiltonian systems, and in more general abstract form in [6,17,19].

In the present setting Lemma 4.8 and Lemma 4.9 below combined with the results in Section 3 lead to an improvement of items (i)–(iv) in Theorem 4.1. The assertions (v) and (vi) in Theorem 4.1 remain valid with M and $H^{1/2}(\Sigma)$ replaced by \widetilde{M} and $H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)$, respectively.

Theorem 4.7. *Let $A_0, \Sigma,$ and \widetilde{M} be as above and let $\lambda \in \mathbb{R}$. Then the following assertions hold.*

- (i) $\lambda \in \sigma_p(A_0)$ if and only if $R_\lambda \widetilde{M} := s\text{-}\lim_{\eta \searrow 0} i\eta \widetilde{M}(\lambda + i\eta) \neq 0$; if the multiplicity of the eigenvalue λ is finite then the mapping

$$\tau : \ker(A_0 - \lambda) \rightarrow \text{ran } R_\lambda \widetilde{M}, \quad u \mapsto \left(\begin{array}{c} u_i|_\Sigma \\ \frac{\partial u_e}{\partial \nu \zeta_e}|_\Sigma \end{array} \right), \tag{4.20}$$

is bijective; if the multiplicity of the eigenvalue λ is infinite then the mapping

$$\tau : \ker(A_0 - \lambda) \rightarrow \text{cl}_\tau(\text{ran } R_\lambda \widetilde{M}), \quad u \mapsto \left(\begin{array}{c} u_i|_\Sigma \\ \frac{\partial u_e}{\partial \nu \zeta_e}|_\Sigma \end{array} \right), \tag{4.21}$$

is bijective, where cl_τ denotes the closure in the normed space $\text{ran } \tau$, equipped with the norm in $L^2(\Sigma) \times L^2(\Sigma)$.

- (ii) λ is an isolated eigenvalue of A_0 if and only if λ is a pole in the strong sense of \widetilde{M} . In this case (4.20) and (4.21) are bijective mappings and $R_\lambda \widetilde{M} = \text{Res}_\lambda \widetilde{M}$.
- (iii) $\lambda \in \rho(A_0)$ if and only if \widetilde{M} can be continued analytically into λ .
- (iv) $\lambda \in \sigma_c(A_0)$ if and only if $s\text{-}\lim_{\eta \searrow 0} i\eta \widetilde{M}(\lambda + i\eta) = 0$ and \widetilde{M} cannot be continued analytically into λ .
- (v) The absolutely continuous spectrum $\sigma_{ac}(A_0)$ of A_0 in Δ is given by

$$\overline{\sigma_{ac}(A_0) \cap \Delta} = \overline{\bigcup_{g \in H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)} \text{cl}_{ac}(\{x \in \Delta : 0 < \text{Im}(\widetilde{M}(x + i0)g, g) < +\infty\})}$$

and, in particular, $\sigma_{ac}(A_0) \cap \Delta = \emptyset$ if and only if for each $g \in H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)$ one has $\text{Im}(\widetilde{M}(x + i0)g, g) = 0$ for almost all $x \in \Delta$.

- (vi) The singular continuous spectrum of A_0 in Δ , $\sigma_{sc}(A_0) \cap \Delta$, is contained in

$$\overline{\bigcup_{g \in H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)} \text{cl}_c(\{x \in \Delta : \text{Im}(\widetilde{M}(x + i0)g, g) = +\infty, \lim_{y \searrow 0} y(\widetilde{M}(x + iy)g, g) = 0\})},$$

and, in particular, if for each $g \in H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)$ there exist at most countably many $x \in \Delta$ such that $\text{Im}(\widetilde{M}(x + iy)g, g) \rightarrow +\infty$ and $y(\widetilde{M}(x + iy)g, g) \rightarrow 0$ as $y \searrow 0$ then $\sigma_{\text{sc}}(A_0) \cap \Delta = \emptyset$.

We provide a quasi boundary triple such that \widetilde{M} in (4.19) is the corresponding Weyl function. As indicated above we make use of the densely defined, closed, symmetric operators

$$S_j u_j = \mathcal{L}_j u_j, \quad \text{dom } S_j = \left\{ u_j \in H^2(\Omega_j) : u_j|_{\Sigma} = \frac{\partial u_j}{\partial \nu_{\mathcal{L}_j}} \Big|_{\Sigma} = 0 \right\},$$

in $L^2(\Omega_j)$ for $j = i, e$, which appeared already in the proof of Lemma 4.3 and which are both simple. Besides the operators S_j also the operators

$$T_j u_j = \mathcal{L}_j u_j, \quad \text{dom } T_j = H^2(\Omega_j),$$

appear in the formulation of the next lemma.

Lemma 4.8. *The triple $\{L^2(\Sigma) \times L^2(\Sigma), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where $\widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \text{dom}(T_i \oplus T_e) \rightarrow L^2(\Sigma) \times L^2(\Sigma)$ and*

$$\widetilde{\Gamma}_0 u = \begin{pmatrix} \frac{\partial u_i}{\partial \nu_{\mathcal{L}_i}} \Big|_{\Sigma} + \frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_{\Sigma} \\ u_i|_{\Sigma} - u_e|_{\Sigma} \end{pmatrix}, \quad \widetilde{\Gamma}_1 u = \begin{pmatrix} u_i|_{\Sigma} \\ \frac{\partial u_e}{\partial \nu_{\mathcal{L}_e}} \Big|_{\Sigma} \end{pmatrix},$$

is a quasi boundary triple for $S_i^* \oplus S_e^*$ such that $(T_i \oplus T_e) \upharpoonright \ker \widetilde{\Gamma}_0$ coincides with the operator A_0 in (4.2) and $\text{ran } \widetilde{\Gamma}_0 = H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)$. For all $\lambda \in \rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{N,e})$ the corresponding Weyl function coincides with the function \widetilde{M} in (4.19).

Proof. The proof of Lemma 4.8 follows the same strategy as the proof of Lemma 4.2 and some details are left to the reader. Well known properties of traces of H^2 -functions yield

$$\text{ran}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top = (H^{1/2}(\Sigma) \times H^{3/2}(\Sigma)) \times (H^{3/2}(\Sigma) \times H^{1/2}(\Sigma)),$$

which is dense in $(L^2(\Sigma) \times L^2(\Sigma)) \times (L^2(\Sigma) \times L^2(\Sigma))$. Moreover, $C_0^\infty(\mathbb{R}^n \setminus \Sigma)$ is a dense subspace of $L^2(\mathbb{R}^n)$ which is contained in $\ker \widetilde{\Gamma}_0 \cap \ker \widetilde{\Gamma}_1$. Green’s identity implies that (2.1) holds, and as $H^2(\mathbb{R}^n)$ is contained in $\ker \widetilde{\Gamma}_0$ the selfadjoint operator A_0 is contained in $(T_i \oplus T_e) \upharpoonright \ker \widetilde{\Gamma}_0$. Hence the assumptions (i)–(iii) in Proposition 2.2 are satisfied and it follows that $\{L^2(\Sigma) \times L^2(\Sigma), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a quasi boundary triple for $S_i^* \oplus S_e^*$ such that $A_0 = (T_i \oplus T_e) \upharpoonright \ker \widetilde{\Gamma}_0$.

Let us verify that the corresponding Weyl function is given by \widetilde{M} in (4.19). For this let $\lambda \in \rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{N,e})$ and let $u_\lambda = u_{\lambda,i} \oplus u_{\lambda,e} \in \text{dom}(T_i \oplus T_e)$ be such that $\mathcal{L}_j u_{\lambda,j} = \lambda u_{\lambda,j}$, $j = i, e$. Then we have

$$\begin{aligned}
\begin{pmatrix} \Lambda_i(\lambda) & 1 \\ 1 & -\Lambda_e(\lambda)^{-1} \end{pmatrix} \widetilde{\Gamma}_1 u_\lambda &= \begin{pmatrix} \Lambda_i(\lambda) & 1 \\ 1 & -\Lambda_e(\lambda)^{-1} \end{pmatrix} \begin{pmatrix} u_{\lambda,i}|_\Sigma \\ \frac{\partial u_{\lambda,e}}{\partial \nu_{\mathcal{L}_e}}|_\Sigma \end{pmatrix} \\
&= \begin{pmatrix} \Lambda_i(\lambda) u_{\lambda,i}|_\Sigma + \frac{\partial u_{\lambda,e}}{\partial \nu_{\mathcal{L}_e}}|_\Sigma \\ u_{\lambda,i}|_\Sigma - \Lambda_e(\lambda)^{-1} \frac{\partial u_{\lambda,e}}{\partial \nu_{\mathcal{L}_e}}|_\Sigma \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u_{\lambda,i}}{\partial \nu_{\mathcal{L}_i}}|_\Sigma + \frac{\partial u_{\lambda,e}}{\partial \nu_{\mathcal{L}_e}}|_\Sigma \\ u_{\lambda,i}|_\Sigma - u_{\lambda,e}|_\Sigma \end{pmatrix} \\
&= \widetilde{\Gamma}_0 u_\lambda.
\end{aligned}$$

By the definition of the Weyl function we obtain that the function \widetilde{M} in (4.19) coincides with the Weyl function associated to the quasi boundary triple $\{L^2(\Sigma) \times L^2(\Sigma), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for all $\lambda \in \rho(A_0) \cap \rho(A_{D,i}) \cap \rho(A_{N,e})$. \square

The next lemma is a direct consequence of the fact that the symmetric operators S_i and S_e are simple; cf. [7, Proposition 2.5] and [8, Proposition 2.2].

Lemma 4.9. *The symmetric operator $S_i \oplus S_e$ is simple.*

Proof of Theorem 4.7. Let $\{L^2(\Sigma) \times L^2(\Sigma), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be the quasi boundary triple in Lemma 4.8. Then $(T_i \oplus T_e) \upharpoonright \ker \widetilde{\Gamma}_0$ corresponds to the selfadjoint elliptic differential operator A_0 in (4.2) and the associated Weyl function coincides with the operator function \widetilde{M} in (4.19). Taking Lemma 4.9 into account, item (i) follows from Corollary 3.4 and items (ii)–(iv) are consequences of Theorem 3.5 and Proposition 3.6. Item (v) follows from Theorem 3.8 and Corollary 3.11, and item (vi) is a consequence of Theorem 3.12 and Corollary 3.15. \square

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