PAPER • OPEN ACCESS

Inverse problems with partial data for elliptic operators on unbounded Lipschitz domains

View the article online for updates and enhancements.

Inverse problems with partial data for elliptic operators on unbounded Lipschitz domains

Jussi Behrndt[1] and **Jonathan Rohleder**[2,3]

[1] Institut für Angewandte Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
[2] Stockholms universitet, Matematiska institutionen, 10691 Stockholm, Sweden

E-mail: behrndt@tugraz.at and jonathan.rohleder@math.su.se

CrossMark

Abstract

For a second order formally symmetric elliptic differential expression we show that the knowledge of the Dirichlet-to-Neumann map or Robin-to-Dirichlet map for suitably many energies on an arbitrarily small open subset of the boundary determines the self-adjoint operator with a Dirichlet boundary condition or with a (possibly non-self-adjoint) Robin boundary condition uniquely up to unitary equivalence. These results hold for general Lipschitz domains, which can be unbounded and may have a non-compact boundary, and under weak regularity assumptions on the coefficients of the differential expression.

Keywords: Dirichlet-to-Neumann map, elliptic differential operator, inverse problem, Calderón problem, Gelfand problem

1. Introduction

Let $\mathcal{L}$ be a uniformly elliptic, formally symmetric differential expression of the form

$$\mathcal{L} = -\sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k + \sum_{j=1}^{n} \left(a_j \partial_j - \partial_j \overline{a_j} \right) + a \tag{1.1}$$

on a possibly unbounded Lipschitz domain Ω. For appropriate $\lambda \in \mathbb{C}$, the corresponding Dirichlet-to-Neumann map is given by

[3] Author to whom any correspondence should be addressed.

$$M(\lambda) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad u_\lambda|_{\partial\Omega} \mapsto \partial_{\mathcal{L}} u_\lambda|_{\partial\Omega},$$

where $u_\lambda \in H^1(\Omega)$ solves the differential equation $\mathcal{L}u = \lambda u$, $u_\lambda|_{\partial\Omega}$ denotes the trace of u_λ on the boundary $\partial\Omega$ and $\partial_{\mathcal{L}} u_\lambda|_{\partial\Omega}$ is the conormal derivative of u_λ on $\partial\Omega$ with respect to $\mathcal{L}$. In the present paper it will be shown that the partial knowledge of $M(\lambda)$ on an arbitrarily small nonempty, relatively open subset ω of $\partial\Omega$ for a set of points λ with an accumulation point determines the self-adjoint Dirichlet operator

$$A_{\mathrm{D}}u = \mathcal{L}u, \quad \mathrm{dom}\, A_{\mathrm{D}} = \left\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), u|_{\partial\Omega} = 0 \right\},$$

and other realizations of $\mathcal{L}$ with (possibly non-self-adjoint) Robin boundary conditions uniquely up to unitary equivalence in $L^2(\Omega)$. We impose weak regularity assumptions on the coefficients, that is, $a_{jk}, a_j : \overline{\Omega} \to \mathbb{C}$ are bounded Lipschitz functions, $1 \leqslant j, k \leqslant n$, and $a : \Omega \to \mathbb{R}$ is measurable and bounded. We emphasize that Ω is an unbounded Lipschitz domain without any additional geometric restrictions, and that ω may be a bounded subset of $\partial\Omega$ even in the case that $\partial\Omega$ is unbounded.

The interplay between elliptic differential operators and their corresponding Dirichlet-to-Neumann maps is of particular interest for spectral theory and inverse problems, among them the famous Calderón problem, the multidimensional Gelfand inverse boundary spectral problem, and inverse scattering problems on Riemannian manifolds. In his famous paper [20] Calderón asked whether the uniformly positive coefficient γ in the differential expression $-\nabla \cdot \gamma \nabla$ on a bounded domain Ω is uniquely determined by the Dirichlet-to-Neumann map on the boundary $\partial\Omega$ or on parts of the boundary; this corresponds to the case $a_{jk} = \gamma \delta_{jk}$, $a_j = a = 0$ in (1.1), and γ describes the isotropic conductivity of an inhomogeneous body. There is an extensive literature on this topic and uniqueness of the coefficient γ from the knowledge of $M(0)$ has been shown under rather general regularity assumptions, see, e.g. [7, 60, 61, 63, 73] and [19, 35, 45, 62] for results with partial data, as well as [3, 23, 55, 71, 72, 74] for the more general case of an anisotropic conductivity ($a_j = a = 0$ in (1.1)) and the surveys [75–77]. If Ω is an unbounded domain the situation is much more difficult since, very roughly speaking, the spectrum contains continuous parts. For conductivities that are constant outside compact sets, special unbounded domains (infinite slabs or transversally anisotropic geometries), and magnetic Schrödinger operators, uniqueness results were shown in [21, 22, 24, 34, 44, 46–48, 53, 54, 56, 65, 69].

In Gelfand's inverse boundary spectral problem—which is a variant of the inverse problems discussed in the present paper for bounded domains—one reconstructs from the given boundary spectral data on a compact manifold (consisting of eigenvalues and boundary data of eigenfunctions of a self-adjoint elliptic operator) the manifold and its metric (up to gauge equivalence) with the help of the boundary control method; see [2, 12–15, 41, 42, 50] and [49, 51] for the non-self-adjoint case. There is also a strong recent interest in closely related problems in inverse scattering theory on compact and non-compact Riemannian manifolds; here the main theme is the reconstruction of the manifold and its Riemannian metric from the knowledge of the scattering matrix for the Laplace–Beltrami operator, see e.g. [15, 36–40, 42, 52].

The inverse problems discussed in this paper are of a somewhat more abstract, but also more general nature. In sections 3 and 4 it will be shown that the knowledge of the Dirichlet-to-Neumann map for a suitable set of points λ with an accumulation point on an arbitrarily small open subset of the boundary determines the self-adjoint Dirichlet operator and other non-self-adjoint realizations with mixed Dirichlet–Robin boundary conditions up to unitary equivalence. We treat here the general case of an unbounded Lipschitz domain without any additional geometric restrictions and assume weak regularity assumptions on the coefficients

of the elliptic differential expression. We emphasize that unitary equivalence determines the spectral properties, so that, in particular, the isolated and embedded eigenvalues, continuous, essential, absolutely continuous and singular continuous spectra are uniquely determined by the partial knowledge of the Dirichlet-to-Neumann map. Finally, in section 5 another variant of our uniqueness result is provided for self-adjoint Robin realizations, where instead of the Dirichlet-to-Neumann map a Robin-to-Dirichlet map on an open subset of the boundary is considered. The main results in this paper complement earlier results for bounded domains from [9], see also [64], where the uniqueness problem is substantially easier since all spectral singularities are discrete eigenvalues, and hence poles of the Dirichlet-to-Neumann map. Our proofs in the present paper are based on more elaborate methods from the extension theory of symmetric operators and the spectral theory of elliptic operators; related techniques were also developed and used in [10, 11] for the spectral analysis of Schrödinger and more general elliptic operators. In this context we also refer the reader to [1, 27, 28, 30–33, 57, 58, 66–68] for some recent related papers on spectral theory of elliptic differential operators, to the classical contributions [29, 78], and to [4–6, 16–18, 26] for operator-theoretic approaches to Dirichlet-to-Neumann and Robin-to-Dirichlet maps.

2. Preliminaries

In this section we provide some preliminaries on elliptic differential operators on possibly unbounded Lipschitz domains. Throughout this paper we assume that $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a connected Lipschitz domain in the sense of, e.g. [70, VI.3], that is, Ω is an open, connected set with a nonempty boundary $\partial\Omega$ and there exist $\varepsilon > 0$, $N \in \mathbb{N}$, $M > 0$ and (finitely or infinitely many) open sets $U_1, U_2, \ldots$ with the following properties.

 (i) For each $x \in \partial\Omega$ there exists j such that the open ball $B(x, \varepsilon)$ of radius ε centered at x is contained in U_j.
 (ii) No point of $\mathbb{R}^n$ is contained in more than N of the U_j.
(iii) For each j there exists a function $\zeta_j : \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$|\zeta_j(x) - \zeta_j(y)| \leqslant M|x - y|, \quad x, y \in \mathbb{R}^{n-1},$$

such that (up to a possible rotation of coordinates) the Lipschitz hypographs

$$\Omega_j := \left\{ (x_1, \ldots, x_n)^\top \in \mathbb{R}^n : x_n < \zeta_j(x_1, \ldots, x_{n-1}) \right\}$$

satisfy $U_j \cap \Omega = U_j \cap \Omega_j$.

We are particularly interested in the case that Ω is unbounded. Note that the boundary $\partial\Omega$ may be noncompact. It can be described by the graphs of countably many Lipschitz functions with a joint Lipschitz constant.

In the following we denote by $H^s(\Omega)$ and $H^t(\partial\Omega)$ the Sobolev spaces of order $s \in \mathbb{R}$ on Ω and of order $t \in [-1, 1]$ on its boundary $\partial\Omega$, respectively. We point out that under the above assumptions on Ω many typical properties of Sobolev spaces on bounded Lipschitz domains and their boundaries remain true. For instance, by the same proofs as provided in [59, theorem 3.37 and theorem 3.40] for bounded domains, one verifies that there exists a continuous, surjective trace operator from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$ and that its kernel coincides with $H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. In the following we denote the trace of a function $u \in H^1(\Omega)$ by $u|_{\partial\Omega}$.

On Ω let us consider the differential expression $\mathcal{L}$ in (1.1) satisfying the uniform ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geqslant E \sum_{k=1}^{n} \xi_k^2, \quad \xi = (\xi_1,\ldots\xi_n)^{\top} \in \mathbb{R}^n, \; x \in \overline{\Omega}, \tag{2.1}$$

for some $E > 0$. We assume that

$$a_{jk}, a_j : \overline{\Omega} \to \mathbb{C} \text{ are bounded Lipschitz functions}, \quad 1 \leqslant j, k \leqslant n, \tag{2.2}$$

$$a_{jk}(x) = \overline{a_{kj}(x)}, \quad x \in \overline{\Omega}, \tag{2.3}$$

and that

$$a : \Omega \to \mathbb{R} \text{ is measurable and bounded.} \tag{2.4}$$

In the following we make use of the conormal derivative (with respect to $\mathcal{L}$). For a function $u \in H^1(\Omega)$ such that $\mathcal{L}u \in L^2(\Omega)$ in the sense of distributions, the conormal derivative of u at $\partial\Omega$ with respect to $\mathcal{L}$ is defined as the unique $\psi \in H^{-1/2}(\partial\Omega)$ which satisfies the identity

$$\mathfrak{a}[u,v] = (\mathcal{L}u,v)_{L^2(\Omega)} + (\psi, v|_{\partial\Omega})_{\partial\Omega}$$

for all $v \in H^1(\Omega)$, where $(\cdot,\cdot)_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$, $(\cdot,\cdot)_{\partial\Omega}$ denotes the (sesquilinear) duality of $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, and

$$\mathfrak{a}[u,v] = \int_{\Omega} \left(\sum_{j,k=1}^{n} a_{jk}\partial_k u \cdot \overline{\partial_j v} + \sum_{j=1}^{n} \left(a_j\partial_j u \cdot \overline{v} + \overline{a_j}u \cdot \overline{\partial_j v}\right) + au\overline{v} \right) \mathrm{d}x; \tag{2.5}$$

see [59, lemma 4.3]. We shall use the notation $\psi = \partial_{\mathcal{L}}u|_{\partial\Omega}$.

3. An inverse problem for the Dirichlet operator with partial Dirichlet-to-Neumann data

In this section we prove that the partial knowledge of the Dirichlet-to-Neumann map determines the Dirichlet realization of $\mathcal{L}$ in $L^2(\Omega)$ uniquely up to unitary equivalence. Recall first that (2.2)–(2.4) ensure that the Dirichlet operator

$$A_{\mathrm{D}}u = \mathcal{L}u, \quad \operatorname{dom} A_{\mathrm{D}} = \left\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), u|_{\partial\Omega} = 0 \right\}, \tag{3.1}$$

is a semibounded self-adjoint operator in $L^2(\Omega)$ since it corresponds to the closed semibounded sesquilinear form

$$\mathfrak{a}_{\mathrm{D}}[u,v] := \mathfrak{a}[u,v], \qquad u,v \in \operatorname{dom}\mathfrak{a}_{\mathrm{D}} = H_0^1(\Omega),$$

via the first representation theorem; see [43, theorem VI.2.1] and [25, chapter VI].

In order to define the Dirichlet-to-Neumann map associated with $\mathcal{L}$ on the boundary of the unbounded Lipschitz domain Ω we need the following lemma, which is well known for bounded domains and remains valid in the unbounded case. For the convenience of the reader we provide a short proof. By $\rho(A_{\mathrm{D}})$ we denote the resolvent set of A_{D}, i.e. the complement of the spectrum.

Lemma 3.1. *For each $\lambda \in \rho(A_{\mathrm{D}})$ and each $\varphi \in H^{1/2}(\partial\Omega)$ the boundary value problem*

$$\mathcal{L}u = \lambda u, \qquad u|_{\partial\Omega} = \varphi, \tag{3.2}$$

has a unique solution $u_\lambda \in H^1(\Omega)$.

Proof. Let $\lambda \in \rho(A_{\mathrm{D}})$ and $\varphi \in H^{1/2}(\partial\Omega)$. Since the trace map is surjective from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ there exists (a non-unique) $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = \varphi$. Let $\mathfrak{a}$ be the symmetric sesquilinear form on $H^1(\Omega)$ defined in (2.5). It follows from (2.2) and (2.4) that there exists $C > 0$ such that

$$|\mathfrak{a}[u,v]| \leqslant C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad u,v \in H^1(\Omega), \tag{3.3}$$

where $\|\cdot\|_{H^1(\Omega)}$ denotes the norm in $H^1(\Omega)$. In particular, the antilinear mapping

$$F_{w,\zeta} : H_0^1(\Omega) \to \mathbb{C}, \quad v \mapsto \mathfrak{a}[w,v] + \zeta(w,v)_{L^2(\Omega)},$$

is bounded on $H_0^1(\Omega)$ for each $\zeta \in \mathbb{R}$; hence $F_{w,\zeta}$ belongs to the antidual of $H_0^1(\Omega)$. Moreover, it follows from (3.3) and the ellipticity condition (2.1) that we can fix $\zeta_0 \in \mathbb{R}$ such that

$$\mathfrak{a}[u,v] + \zeta_0(u,v)_{L^2(\Omega)}, \quad u,v \in H_0^1(\Omega), \tag{3.4}$$

defines an inner product on $H_0^1(\Omega)$ with an induced norm that is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$. In particular, $H_0^1(\Omega)$ equipped with the inner product in (3.4) is a Hilbert space. By the Fréchet–Riesz theorem there exists a unique $u_0 \in H_0^1(\Omega)$ such that

$$\mathfrak{a}[u_0,v] + \zeta_0(u_0,v)_{L^2(\Omega)} = F_{w,\zeta_0}(v) = \mathfrak{a}[w,v] + \zeta_0(w,v)_{L^2(\Omega)}, \quad v \in H_0^1(\Omega).$$

Consequently, $\mathfrak{a}[u_0 - w, v] + \zeta_0(u_0 - w, v)_{L^2(\Omega)} = 0$ for all $v \in H_0^1(\Omega)$, which implies $\mathcal{L}(u_0 - w) + \zeta_0(u_0 - w) = 0$ in the distributional sense. For $\lambda \in \rho(A_{\mathrm{D}})$ it follows, in particular, that $(\mathcal{L} - \lambda)(u_0 - w) \in L^2(\Omega)$. Let us set

$$u_\lambda = u_0 - w - (A_{\mathrm{D}} - \lambda)^{-1}(\mathcal{L} - \lambda)(u_0 - w) \in H^1(\Omega).$$

Then $u_\lambda|_{\partial\Omega} = w|_{\partial\Omega} = \varphi$ and $(\mathcal{L} - \lambda)u_\lambda = 0$. Thus u_λ is a solution of (3.2).

In order to prove uniqueness let $v_\lambda \in H^1(\Omega)$ be a further solution of (3.2). Then we have

$$\mathcal{L}(u_\lambda - v_\lambda) = \lambda(u_\lambda - v_\lambda) \quad \text{and} \quad (u_\lambda - v_\lambda)|_{\partial\Omega} = 0,$$

that is, $(u_\lambda - v_\lambda) \in \ker(A_{\mathrm{D}} - \lambda)$. Since $\lambda \in \rho(A_{\mathrm{D}})$, it follows $u_\lambda = v_\lambda$. $\qquad\square$

Lemma 3.1 ensures that the Dirichlet-to-Neumann map in the following definition is well-defined.

Definition 3.2. For $\lambda \in \rho(A_{\mathrm{D}})$ the *Dirichlet-to-Neumann map* $M(\lambda)$ is defined by

$$M(\lambda) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \qquad M(\lambda)u_\lambda|_{\partial\Omega} := \partial_{\mathcal{L}}u_\lambda|_{\partial\Omega},$$

for each $u_\lambda \in H^1(\Omega)$ satisfying $\mathcal{L}u_\lambda = \lambda u_\lambda$.

For $\lambda \in \rho(A_{\mathrm{D}})$ we will also make use of the *Poisson operator* $\gamma(\lambda)$ defined by

$$\gamma(\lambda) : H^{1/2}(\partial\Omega) \to L^2(\Omega), \qquad \gamma(\lambda)u_\lambda|_{\partial\Omega} := u_\lambda, \tag{3.5}$$

for any $u_\lambda \in H^1(\Omega)$ such that $\mathcal{L}u_\lambda = \lambda u_\lambda$; see lemma 3.1.

We collect some properties of the Dirichlet-to-Neumann map and the Poisson operator in the following lemma. Its proof is analogous to the case of a bounded Lipschitz domain carried out in [9, lemma 2.4].

Lemma 3.3. *For $\lambda, \mu \in \rho(A_D)$ let $\gamma(\lambda), \gamma(\mu)$ be the Poisson operators and let $M(\lambda), M(\mu)$ be the Dirichlet-to-Neumann maps. Then the following assertions hold.*

 (i) $\gamma(\lambda)$ is bounded and its adjoint $\gamma(\lambda)^* : L^2(\Omega) \to H^{-1/2}(\partial\Omega)$ is given by

$$\gamma(\lambda)^* u = -\partial_\mathcal{L}\big((A_D - \overline{\lambda})^{-1}u\big)|_{\partial\Omega}, \quad u \in L^2(\Omega).$$

 (ii) The identity

$$\gamma(\lambda) = \big(I + (\lambda - \mu)(A_D - \lambda)^{-1}\big)\gamma(\mu)$$

 holds.

(iii) $M(\lambda)$ is a bounded operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, the operator function $\lambda \mapsto M(\lambda)$ is holomorphic on $\rho(A_D)$, and

$$(\text{Im}\,\mu)\|\gamma(\mu)\varphi\|_{L^2(\Omega)}^2 = -\text{Im}(M(\mu)\varphi, \varphi)_{\partial\Omega}$$

holds for all $\varphi \in H^{1/2}(\partial\Omega)$.

The next theorem is the main result in this section; one can view it as a generalized variant of the multidimensional Gelfand inverse boundary spectral problem with partial data on arbitrary unbounded Lipschitz domains. Instead of determining coefficients up to gauge equivalence here an operator uniqueness result is obtained. Roughly speaking theorem 3.4 states that the knowledge of the Dirichlet-to-Neumann map $M(\lambda)$ on a nonempty open subset ω of the boundary $\partial\Omega$ for sufficiently many λ determines the Dirichlet operator uniquely up to unitary equivalence. For bounded Lipschitz domains such a result was shown in [9], see also [64].

Theorem 3.4. *Let $\mathcal{L}_1, \mathcal{L}_2$ be two uniformly elliptic differential expressions on Ω of the form (1.1) with coefficients $a_{jk,1}, a_{j,1}, a_1$ and $a_{jk,2}, a_{j,2}, a_2$, respectively, satisfying (2.2)–(2.4). Denote by $A_{D,1}, A_{D,2}$ and $M_1(\lambda), M_2(\lambda)$ the corresponding self-adjoint Dirichlet operators and Dirichlet-to-Neumann maps, respectively. Assume that $\omega \subset \partial\Omega$ is an open, nonempty set such that*

$$(M_1(\lambda)\varphi, \varphi)_{\partial\Omega} = (M_2(\lambda)\varphi, \varphi)_{\partial\Omega}, \quad \varphi \in H^{1/2}(\partial\Omega), \text{ supp } \varphi \subset \omega, \qquad (3.6)$$

holds for all $\lambda \in \mathcal{D}$, where $\mathcal{D} \subset \rho(A_{D,1}) \cap \rho(A_{D,2})$ is a set with an accumulation point in $\rho(A_{D,1}) \cap \rho(A_{D,2})$. Then there exists a unitary operator U in $L^2(\Omega)$ such that

$$A_{D,2} = U A_{D,1} U^* \qquad\qquad\qquad (3.7)$$

holds.

Before we provide a proof of the theorem, let us point out that unitary equivalence of self-adjoint operators implies that their spectra coincide.

Corollary 3.5. *Let the assumptions be as in theorem 3.4. Then $\mu \in \mathbb{R}$ belongs to the point (discrete, essential, continuous, absolutely continuous, singular continuous) spectrum of $A_{D,1}$ if and only if μ belongs to the point (discrete, essential, continuous, absolutely continuous, singular continuous) spectrum of $A_{D,2}$, respectively.*

Proof of theorem 3.4.   The proof will be carried out in two steps. In the first step an iso-metric operator defined on a subspace of $L^2(\Omega)$ is constructed; this step follows the strategy of the proof of [9, theorem 1.3] but is given here for completeness. In the second step we show that this operator extends to a unitary operator such that (3.7) holds.

Step 1.   Let $\mathcal{L}_1$, $\mathcal{L}_2$ be differential expressions as in the theorem and let $A_{D,1}, A_{D,2}$ and $M_1(\lambda)$, $M_2(\lambda)$ be the corresponding Dirichlet operators and Dirichlet-to-Neumann maps, respectively. Moreover, denote by $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ the corresponding Poisson operators as in (3.5). Assume that (3.6) holds for all $\lambda \in \mathcal{D}$. Since $(M_i(\cdot)\varphi, \varphi)_{\partial\Omega}$ is holomorphic on $\rho(A_{D,i})$ for all $\varphi \in H^{1/2}(\partial\Omega)$ with supp $\varphi \subset \omega$, $i = 1, 2$, and $\mathcal{D}$ has an accumulation point in $\rho(A_{D,1}) \cap \rho(A_{D,2})$, it follows that

$$(M_1(\lambda)\varphi, \varphi)_{\partial\Omega} = (M_2(\lambda)\varphi, \varphi)_{\partial\Omega}, \quad \varphi \in H^{1/2}(\partial\Omega), \text{ supp } \varphi \subset \omega,$$

holds for all $\lambda \in \rho(A_{D,1}) \cap \rho(A_{D,2})$. With lemma 3.3 (iii) for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and all $\varphi \in H^{1/2}(\partial\Omega)$ with supp $\varphi \subset \omega$ we obtain

$$\begin{aligned}
\|\gamma_1(\mu)\varphi\|_{L^2(\Omega)}^2 &= -\frac{\mathrm{Im}(M_1(\mu)\varphi, \varphi)_{\partial\Omega}}{\mathrm{Im}\,\mu} \\
&= -\frac{\mathrm{Im}(M_2(\mu)\varphi, \varphi)_{\partial\Omega}}{\mathrm{Im}\,\mu} = \|\gamma_2(\mu)\varphi\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.8}$$

Let us define a linear mapping V in $L^2(\Omega)$ on the domain

$$\mathrm{dom}\, V = \mathrm{span}\{\gamma_1(\mu)\varphi : \varphi \in H^{1/2}(\partial\Omega), \text{ supp }\varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R}\} \tag{3.9}$$

by setting

$$V\gamma_1(\mu)\varphi = \gamma_2(\mu)\varphi, \quad \varphi \in H^{1/2}(\partial\Omega), \text{ supp }\varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R}, \tag{3.10}$$

and extending it by linearity to all of dom V. It follows from (3.8) that V is a well-defined, isometric operator in $L^2(\Omega)$ with

$$\mathrm{ran}\, V = \mathrm{span}\{\gamma_2(\mu)\varphi : \varphi \in H^{1/2}(\partial\Omega), \text{ supp }\varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R}\}.$$

Moreover, if we fix $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then by lemma 3.3 (ii) we have $\mathrm{ran}(A_{D,1} - \lambda)^{-1}\gamma_1(\mu) \subset \mathrm{dom}\, V$ and

$$\begin{aligned}
V(A_{D,1} - \lambda)^{-1}\gamma_1(\mu)\varphi &= V\frac{\gamma_1(\lambda)\varphi - \gamma_1(\mu)\varphi}{\lambda - \mu} = \frac{\gamma_2(\lambda)\varphi - \gamma_2(\mu)\varphi}{\lambda - \mu} \\
&= (A_{D,2} - \lambda)^{-1}\gamma_2(\mu)\varphi = (A_{D,2} - \lambda)^{-1}V\gamma_1(\mu)\varphi
\end{aligned}$$

for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ with $\mu \neq \lambda$ and all $\varphi \in H^{1/2}(\partial\Omega)$ with supp $\varphi \subset \omega$. By linearity this implies

$$V(A_{D,1} - \lambda)^{-1} \upharpoonright H_\lambda = (A_{D,2} - \lambda)^{-1}V \upharpoonright H_\lambda, \tag{3.11}$$

where H_λ is the subspace of dom V given by

$$H_\lambda = \mathrm{span}\{\gamma_1(\mu)\varphi : \varphi \in H^{1/2}(\partial\Omega), \text{ supp }\varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R}, \mu \neq \lambda\}. \tag{3.12}$$

Step 2.   Let us show that the linear space dom V in (3.9) is dense in $L^2(\Omega)$. For this choose a Lipschitz domain $\widetilde{\Omega}$ such that $\Omega \subset \widetilde{\Omega}$, $\partial\Omega \setminus \omega \subset \partial\widetilde{\Omega}$, and $\widetilde{\Omega} \setminus \Omega$ contains an open ball

$\mathcal{O}$, and such that $\mathcal{L}_1$ admits a uniformly elliptic, formally symmetric extension $\widetilde{\mathcal{L}}_1$ to $\widetilde{\Omega}$ with coefficients satisfying (2.2)–(2.4) on $\widetilde{\Omega}$. Let $\widetilde{A}_{\mathrm{D},1}$ denote the self-adjoint Dirichlet operator associated with $\widetilde{\mathcal{L}}_1$ in $L^2(\widetilde{\Omega})$,

$$\widetilde{A}_{\mathrm{D},1}\widetilde{u} = \widetilde{\mathcal{L}}_1\widetilde{u}, \quad \mathrm{dom}\,\widetilde{A}_{\mathrm{D},1} = \left\{\widetilde{u} \in H^1(\widetilde{\Omega}) : \widetilde{\mathcal{L}}_1\widetilde{u} \in L^2(\widetilde{\Omega}), \widetilde{u}|_{\partial\widetilde{\Omega}} = 0\right\}.$$

Since $\widetilde{A}_{\mathrm{D},1}$ is semibounded from below, we can assume without loss of generality that this operator has a positive lower bound η. In fact, when a constant is added to the zero order term of $\mathcal{L}_1$ (and $\widetilde{\mathcal{L}}_1$) the linear space $\mathrm{dom}\,V$ in (3.9) remains the same.

For each $\widetilde{v} \in L^2(\widetilde{\Omega})$ such that $\widetilde{v}$ vanishes on Ω we define

$$\widetilde{u}_{\mu,\widetilde{v}} = (\widetilde{A}_{\mathrm{D},1} - \mu)^{-1}\widetilde{v}, \quad \mu \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, denote by $u_{\mu,\widetilde{v}}$ the restriction of $\widetilde{u}_{\mu,\widetilde{v}}$ to Ω. Then $u_{\mu,\widetilde{v}} \in H^1(\Omega)$, $\mathcal{L}_1 u_{\mu,\widetilde{v}} = \mu u_{\mu,\widetilde{v}}$, and $\mathrm{supp}\,(u_{\mu,\widetilde{v}}|_{\partial\Omega}) \subset \omega$, that is, with $\varphi := u_{\mu,\widetilde{v}}|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ we have $u_{\mu,\widetilde{v}} = \gamma(\mu)\varphi$ and $\mathrm{supp}\,\varphi \subset \omega$; in particular, $u_{\mu,\widetilde{v}} \in \mathrm{dom}\,V$ holds for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and all $\widetilde{v} \in L^2(\widetilde{\Omega})$ with $\widetilde{v}|_{\Omega} = 0$.

Let $u \in L^2(\Omega)$ such that u is orthogonal to $\mathrm{dom}\,V$. Then the extension $\widetilde{u}$ of u by zero to $\widetilde{\Omega}$ satisfies

$$0 = (u, u_{\overline{\mu},\widetilde{v}})_{L^2(\Omega)} = \left(\widetilde{u}, (\widetilde{A}_{\mathrm{D},1} - \overline{\mu})^{-1}\widetilde{v}\right)_{L^2(\widetilde{\Omega})} = \left((\widetilde{A}_{\mathrm{D},1} - \mu)^{-1}\widetilde{u}, \widetilde{v}\right)_{L^2(\widetilde{\Omega})}$$

for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and all $\widetilde{v} \in L^2(\widetilde{\Omega})$ with $\widetilde{v}|_{\Omega} = 0$. Hence

$$\left((\widetilde{A}_{\mathrm{D},1} - \mu)^{-1}\widetilde{u}\right)\big|_{\widetilde{\Omega}\setminus\Omega} = 0, \quad \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{3.13}$$

Following an idea from [8, section 3] we define the operator semigroup

$$T(t) = \mathrm{e}^{-t\sqrt{\widetilde{A}_{\mathrm{D},1}}}, \quad t \geqslant 0,$$

generated by the square root of $\widetilde{A}_{\mathrm{D},1}$. Then $t \mapsto T(t)\widetilde{u}$ is twice differentiable with

$$\partial_t^2 T(t)\widetilde{u} = \widetilde{A}_{\mathrm{D},1}T(t)\widetilde{u}, \quad t > 0,$$

from which we conclude

$$\left(-\partial_t^2 + \widetilde{\mathcal{L}}_1\right)T(t)\widetilde{u} = 0, \quad x \in \widetilde{\Omega}, t > 0, \tag{3.14}$$

in the distributional sense. Note that

$$(x,t) \mapsto \left(\mathrm{e}^{-t\sqrt{\widetilde{A}_{\mathrm{D},1}}}\widetilde{u}\right)(x) \in L^2(\widetilde{\Omega} \times (0,\infty)).$$

Since the differential expression $\mathcal{L}_1$ is uniformly elliptic on $\widetilde{\Omega}$, regularity theory implies $\mathrm{e}^{-t\sqrt{\widetilde{A}_{\mathrm{D},1}}}\widetilde{u} \in H^2_{\mathrm{loc}}(\widetilde{\Omega} \times (0,\infty))$. For any real numbers a, b, $a < b$, which are no eigenvalues of $\widetilde{A}_{\mathrm{D},1}$ the Stone formula

$$E_1((a,b))\widetilde{u} = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i}\left(\int_a^b \left(\widetilde{A}_{\mathrm{D},1} - (z + i\varepsilon)\right)^{-1} - \left(\widetilde{A}_{\mathrm{D},1} - (z - i\varepsilon)\right)^{-1}\mathrm{d}z\right)\widetilde{u}$$

for the spectral measure $E_1(\cdot)$ of $\widetilde{A}_{\mathrm{D},1}$ and (3.13) imply $(E_1((a,b))\widetilde{u})|_{\widetilde{\Omega}\setminus\Omega} = 0$. Thus, in particular, for each $t \geqslant 0$

$$\left(\mathrm{e}^{-t\sqrt{\widetilde{A}_{\mathrm{D},1}}}\widetilde{u}\right)\bigg|_{\widetilde{\Omega}\setminus\Omega} = \left(\int_\eta^\infty \mathrm{e}^{-t\sqrt{z}}\mathrm{d}E_1(z)\widetilde{u}\right)\bigg|_{\widetilde{\Omega}\setminus\Omega} = 0. \tag{3.15}$$

By (3.15), $\mathrm{e}^{-t\sqrt{\widetilde{A}_{\mathrm{D},1}}}\widetilde{u}$ vanishes on the nonempty, open set $\mathcal{O}\times(0,\infty)$, and (3.14) and unique continuation yield $T(t)\widetilde{u} = 0$ identically on $\widetilde{\Omega}$ for all $t > 0$, see, e.g. [79]. Thus, taking the limit $t\searrow 0$ we obtain $\widetilde{u} = 0$ and, hence, $u = 0$. Thus dom V is dense in $L^2(\Omega)$. Analogously one shows that ran V is dense in $L^2(\Omega)$.

To summarize, the operator V in (3.10) is densely defined and isometric in $L^2(\Omega)$ with a dense range. Hence it extends by continuity to a unitary operator $U : L^2(\Omega) \to L^2(\Omega)$. Moreover, note that the space $H_\lambda \subset \mathrm{dom}\,V$ in (3.12) is dense in $L^2(\Omega)$ as well since $(\gamma(\mu)\varphi, u)_{L^2(\Omega)} = 0$ for all $\mu \in \mathbb{C}\setminus\mathbb{R}$ with $\mu \neq \lambda$ and all $\varphi \in H^{1/2}(\partial\Omega)$ with $\mathrm{supp}\,\varphi \subset \omega$ implies, by continuity, $(\gamma(\mu)\varphi, u)_{L^2(\Omega)} = 0$ for all $\mu \in \mathbb{C}\setminus\mathbb{R}$ and all $\varphi \in H^{1/2}(\partial\Omega)$ with $\mathrm{supp}\,\varphi \subset \omega$ and hence $u = 0$. Therefore the identity (3.11) extends to

$$U(A_{\mathrm{D},1} - \lambda)^{-1} = (A_{\mathrm{D},2} - \lambda)^{-1}U,$$

which implies $U\mathrm{dom}\,A_{\mathrm{D},1} = \mathrm{dom}\,A_{\mathrm{D},2}$ and $A_{\mathrm{D},2} = UA_{\mathrm{D},1}U^*$. This completes the proof of theorem 3.4. $\qquad\square$

4. An inverse problem for a mixed non-self-adjoint Dirichlet–Robin operator with partial Dirichlet-to-Neumann data

In this section we consider non-self-adjoint operators with mixed Dirichlet–Robin boundary conditions. We shall provide a variant of theorem 3.4 for m-sectorial elliptic operators satisfying a Robin boundary condition on an open subset $\omega \subset \partial\Omega$ and Dirichlet boundary conditions on $\partial\Omega \setminus \omega$. Here the knowledge of the Dirichlet-to-Neumann map is assumed locally at the same subset ω of $\partial\Omega$ on which the Robin condition is given.

In order to define the operators under consideration, let us set

$$H_\omega^{1/2} = \overline{\left\{\varphi \in H^{1/2}(\partial\Omega) : \mathrm{supp}\,\varphi \subset \omega\right\}},$$

where the closure is taken in $H^{1/2}(\partial\Omega)$. Let $\theta \in L^\infty(\partial\Omega)$ be a complex-valued function such that $\theta|_{\partial\Omega\setminus\omega} = 0$, and consider the quadratic form

$$\mathfrak{a}_{\theta,\omega}[u,v] = \mathfrak{a}[u,v] + (\theta u|_{\partial\Omega}, v|_{\partial\Omega})_{\partial\Omega}, \quad \mathrm{dom}\,\mathfrak{a}_{\theta,\omega} = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} \in H_\omega^{1/2}\right\},$$

where $\mathfrak{a}$ is given in (2.5). One verifies that $\mathfrak{a}_{\theta,\omega}$ is a densely defined, sectorial, closed form in $L^2(\Omega)$ and gives rise to the m-sectorial operator

$$A_{\theta,\omega}u = \mathcal{L}u,$$
$$\mathrm{dom}\,A_{\theta,\omega} = \left\{u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), \partial_\mathcal{L}u|_\omega + \theta u|_\omega = 0, u|_{\partial\Omega} \in H_\omega^{1/2}\right\}; \tag{4.1}$$

this operator realization of $\mathcal{L}$ in $L^2(\Omega)$ is subject to a Dirichlet boundary condition on $\partial\Omega \setminus \omega$ and the Robin boundary condition $\partial_\mathcal{L}u|_\omega + \theta u|_\omega = 0$ on ω, which is understood as

$$\left(\partial_\mathcal{L}u|_{\partial\Omega} + \theta u|_{\partial\Omega}, \varphi\right)_{\partial\Omega} = 0, \qquad \varphi \in H^{1/2}(\partial\Omega), \mathrm{supp}\,\varphi \subset \omega. \tag{4.2}$$

Note also that for a real-valued $\theta \in L^\infty(\partial\Omega)$ such that $\theta|_{\partial\Omega\setminus\omega} = 0$ the operator $A_{\theta,\omega}$ in (4.1) is self-adjoint in $L^2(\Omega)$ and semibounded from below.

Theorem 4.1. *Let $\mathcal{L}_1, \mathcal{L}_2$ be two uniformly elliptic differential expressions on Ω of the form (1.1) with coefficients $a_{jk,1}, a_{j,1}, a_1$ and $a_{jk,2}, a_{j,2}, a_2$, respectively, satisfying (2.2)–(2.4), and let $M_1(\lambda), M_2(\lambda)$ be the corresponding Dirichlet-to-Neumann maps. Assume that $\omega \subset \partial\Omega$ is an open, nonempty set such that*

$$(M_1(\lambda)\varphi, \varphi)_{\partial\Omega} = (M_2(\lambda)\varphi, \varphi)_{\partial\Omega}, \qquad \varphi \in H^{1/2}(\partial\Omega), \operatorname{supp}\varphi \subset \omega, \qquad (4.3)$$

holds for all $\lambda \in \mathcal{D}$, where $\mathcal{D} \subset \rho(A_{\mathrm{D},1}) \cap \rho(A_{\mathrm{D},2})$ is a set with an accumulation point in $\rho(A_{\mathrm{D},1}) \cap \rho(A_{\mathrm{D},2})$. Let $\theta \in L^{\infty}(\partial\Omega)$ be a complex-valued function such that $\theta|_{\partial\Omega\setminus\omega} = 0$ and denote by $A_{\theta,\omega,1}$ and $A_{\theta,\omega,2}$ the m-sectorial operators associated with $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, as in (4.1). Then there exists a unitary operator U in $L^2(\Omega)$ (the same as in theorem 3.4) such that

$$A_{\theta,\omega,2} = UA_{\theta,\omega,1}U^*$$

holds.

Theorem 4.1 is essentially a consequence of theorem 3.4 and the following proposition, which relates the resolvent of the Dirichlet operator A_{D} in (3.1) to the resolvent of the operator $A_{\theta,\omega}$ via a perturbation term containing the Dirichlet-to-Neumann map and the function θ. We shall restrict elements in $H^{-1/2}(\partial\Omega)$ to ω and use the operator

$$P_{\omega} : H^{-1/2}(\partial\Omega) \to \left\{\psi|_{\omega} : \psi \in H^{-1/2}(\partial\Omega)\right\}, \quad P_{\omega}\psi = \psi|_{\omega}; \qquad (4.4)$$

here the restriction $\psi|_{\omega}$ is defined by $(\psi|_{\omega}, \varphi) := (\psi, \varphi)_{\partial\Omega}$ for all $\varphi \in H^{1/2}(\partial\Omega)$ with $\operatorname{supp}\varphi \subset \omega$. One can view P_{ω} as the dual of the embedding operator from $H_{\omega}^{1/2}$ into $H^{1/2}(\partial\Omega)$.

Proposition 4.2. *Let $\omega \subset \partial\Omega$ be an open, nonempty set, let $\theta \in L^{\infty}(\partial\Omega)$ be a complex-valued function such that $\theta|_{\partial\Omega\setminus\omega} = 0$, and let $A_{\theta,\omega}$ be the m-sectorial operator defined in (4.1). Then the operator $P_{\omega}(\theta + M(\lambda)) \upharpoonright H_{\omega}^{1/2}$ is injective for all $\lambda \in \rho(A_{\theta,\omega}) \cap \rho(A_{\mathrm{D}})$ and the identity*

$$(A_{\theta,\omega} - \lambda)^{-1} = (A_{\mathrm{D}} - \lambda)^{-1} + \gamma(\lambda)\left(P_{\omega}(\theta + M(\lambda)) \upharpoonright H_{\omega}^{1/2}\right)^{-1} P_{\omega}\gamma(\overline{\lambda})^* \quad (4.5)$$

holds for all $\lambda \in \rho(A_{\theta,\omega}) \cap \rho(A_{\mathrm{D}})$.

Proof. We verify first that $P_{\omega}(\theta + M(\lambda)) \upharpoonright H_{\omega}^{1/2}$ is injective for $\lambda \in \rho(A_{\theta,\omega}) \cap \rho(A_{\mathrm{D}})$. Indeed, assume that $\psi \in H_{\omega}^{1/2}$ is such that $P_{\omega}(\theta + M(\lambda))\psi = 0$, that is,

$$\left((\theta + M(\lambda))\psi, \varphi\right)_{\partial\Omega} = 0, \quad \varphi \in H^{1/2}(\partial\Omega), \ \operatorname{supp}\varphi \subset \omega.$$

Then $u_{\lambda} := \gamma(\lambda)\psi$ satisfies $\mathcal{L}u_{\lambda} = \lambda u_{\lambda}, u_{\lambda}|_{\partial\Omega} \in H_{\omega}^{1/2}$, and

$$\left(\theta u_{\lambda}|_{\partial\Omega} + \partial_{\mathcal{L}}u_{\lambda}|_{\partial\Omega}, \varphi\right)_{\partial\Omega} = 0, \quad \varphi \in H^{1/2}(\partial\Omega), \ \operatorname{supp}\varphi \subset \omega,$$

which implies $u_{\lambda} \in \ker(A_{\theta,\omega} - \lambda)$ by (4.1) and (4.2). Together with $\lambda \in \rho(A_{\theta,\omega})$ it follows $u_{\lambda} = 0$ and, thus, $\psi = u_{\lambda}|_{\partial\Omega} = 0$.

Let us now come to the proof of (4.5). For this let $v \in L^2(\Omega)$ be arbitrary. Since $\lambda \in \rho(A_{\theta,\omega}) \cap \rho(A_{\mathrm{D}})$, we can define

$$u = (A_{\theta,\omega} - \lambda)^{-1}v - (A_{\mathrm{D}} - \lambda)^{-1}v \quad \text{and} \quad z = (A_{\theta,\omega} - \lambda)^{-1}v. \qquad (4.6)$$

Then $u \in H^1(\Omega)$ with $\mathcal{L}u = \lambda u$, $z \in \operatorname{dom} A_{\theta,\omega}$, and $u|_{\partial\Omega} = z|_{\partial\Omega} \in H_\omega^{1/2}$. Moreover,

$$\partial_{\mathcal{L}} u|_{\partial\Omega} = \partial_{\mathcal{L}} z|_{\partial\Omega} - \partial_{\mathcal{L}}\big((A_{\mathrm{D}} - \lambda)^{-1}v\big)|_{\partial\Omega} = \partial_{\mathcal{L}} z|_{\partial\Omega} + \gamma(\overline{\lambda})^* v$$

by lemma 3.3 (i). For all $\psi \in H^{1/2}(\partial\Omega)$ with $\operatorname{supp}\psi \subset \omega$ we then obtain

$$
\begin{aligned}
\big(\gamma(\overline{\lambda})^* v, \psi\big)_{\partial\Omega} &= \big(\partial_{\mathcal{L}} u|_{\partial\Omega} - \partial_{\mathcal{L}} z|_{\partial\Omega}, \psi\big)_{\partial\Omega} \\
&= \big(M(\lambda)u|_{\partial\Omega} - \partial_{\mathcal{L}} z|_{\partial\Omega}, \psi\big)_{\partial\Omega} = \big((M(\lambda) + \theta)z|_{\partial\Omega}, \psi\big)_{\partial\Omega}.
\end{aligned}
$$

Hence $P_\omega \gamma(\overline{\lambda})^* v = P_\omega(\theta + M(\lambda))z|_{\partial\Omega}$, that is, $P_\omega \gamma(\overline{\lambda})^* v \in \operatorname{ran}(P_\omega(\theta + M(\lambda)) \restriction H_\omega^{1/2})$ and

$$\big(P_\omega(\theta + M(\lambda)) \restriction H_\omega^{1/2}\big)^{-1} P_\omega \gamma(\overline{\lambda})^* v = z|_{\partial\Omega} = u|_{\partial\Omega}.$$

It follows

$$\gamma(\lambda)\big(P_\omega(\theta + M(\lambda)) \restriction H_\omega^{1/2}\big)^{-1} P_\omega \gamma(\overline{\lambda})^* v = \gamma(\lambda)u|_{\partial\Omega} = u,$$

which, together with the definition of u in (4.6), completes the proof of (4.5). $\qquad\square$

Proof of theorem 4.1. Let U be the unitary operator in $L^2(\Omega)$ constructed in the proof of theorem 3.4, which satisfies

$$U\gamma_1(\mu)\varphi = \gamma_2(\mu)\varphi \tag{4.7}$$

for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and all $\varphi \in H^{1/2}(\partial\Omega)$ with $\operatorname{supp}\varphi \subset \omega$ as well as

$$U(A_{\mathrm{D},1} - \lambda)^{-1} = (A_{\mathrm{D},2} - \lambda)^{-1} U \tag{4.8}$$

for $\lambda \in \rho(A_{\mathrm{D},1}) \cap \rho(A_{\mathrm{D},2})$. Let us fix $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cap \rho(A_{\theta,\omega,1}) \cap \rho(A_{\theta,\omega,2})$. Then with P_ω in (4.4) the identity

$$P_\omega \gamma_1(\overline{\lambda})^* = P_\omega \gamma_2(\overline{\lambda})^* U \tag{4.9}$$

holds. In fact, for $u \in L^2(\Omega)$ and $\psi \in H^{1/2}(\partial\Omega)$ with $\operatorname{supp}\psi \subset \omega$ we have

$$(\gamma_1(\overline{\lambda})^* u, \psi)_{\partial\Omega} = (u, \gamma_1(\overline{\lambda})\psi)_{L^2(\Omega)} = (u, U^*\gamma_2(\overline{\lambda})\psi)_{L^2(\Omega)} = (\gamma_2(\overline{\lambda})^* U u, \psi)_{\partial\Omega}$$

taking into account (4.7); this yields (4.9). Using proposition 4.2, (4.8), the assumption (4.3), and (4.9), we obtain

$$
\begin{aligned}
U(A_{\theta,\omega,1} - \lambda)^{-1} &= U(A_{\mathrm{D},1} - \lambda)^{-1} + U\gamma_1(\lambda)\big(P_\omega(\theta + M_1(\lambda)) \restriction H_\omega^{1/2}\big)^{-1} P_\omega \gamma_1(\overline{\lambda})^* \\
&= (A_{\mathrm{D},2} - \lambda)^{-1} U + \gamma_2(\lambda)\big(P_\omega(\theta + M_2(\lambda)) \restriction H_\omega^{1/2}\big)^{-1} P_\omega \gamma_2(\overline{\lambda})^* U \\
&= (A_{\theta,\omega,2} - \lambda)^{-1} U.
\end{aligned}
$$

This yields $A_{\theta,\omega,2} = U A_{\theta,\omega,1} U^*$ and completes the proof. $\qquad\square$

5. An inverse problem for a self-adjoint Robin operator with partial Robin-to-Dirichlet data

In this section we turn to an inverse problem for elliptic differential operators with Robin boundary conditions on the whole boundary of the unbounded Lipschitz domain Ω. In contrast to the previous section we restrict ourselves to self-adjoint boundary conditions. More specifically, for a real-valued function $\theta \in L^\infty(\partial\Omega)$ we consider the densely defined, semibounded, closed form

$$\mathfrak{a}_\theta[u,v] = \mathfrak{a}[u,v] + (\theta u|_{\partial\Omega}, v|_{\partial\Omega})_{\partial\Omega}, \quad \operatorname{dom}\mathfrak{a}_\theta = H^1(\Omega),$$

in $L^2(\Omega)$ and the corresponding semibounded, self-adjoint Robin operator

$$A_\theta u = \mathcal{L}u, \quad \operatorname{dom}A_\theta = \left\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), \partial_\mathcal{L}u|_{\partial\Omega} + \theta u|_{\partial\Omega} = 0 \right\}.$$

Our aim is to prove that this operator is determined uniquely up to unitary equivalence by the knowledge of a corresponding Robin-to-Dirichlet map on any nonempty, open subset of the boundary.

The following lemma prepares the definition of the Robin-to-Dirichlet map. It can be proved analogously to lemma 3.1.

Lemma 5.1. *For each $\lambda \in \rho(A_\theta)$ and each $\psi \in H^{-1/2}(\partial\Omega)$ the boundary value problem*

$$\mathcal{L}u = \lambda u, \qquad \partial_\mathcal{L}u|_{\partial\Omega} + \theta u|_{\partial\Omega} = \psi,$$

has a unique solution $u_\lambda \in H^1(\Omega)$.

Due to lemma 5.1 the following definition makes sense.

Definition 5.2. For $\lambda \in \rho(A_\theta)$ the *Robin-to-Dirichlet map* $M_\theta(\lambda)$ is defined by

$$M_\theta(\lambda) : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \quad M_\theta(\lambda)\big(\partial_\mathcal{L}u_\lambda|_{\partial\Omega} + \theta u_\lambda|_{\partial\Omega}\big) := u_\lambda|_{\partial\Omega},$$

for each $u_\lambda \in H^1(\Omega)$ satisfying $\mathcal{L}u_\lambda = \lambda u_\lambda$.

For $\lambda \in \rho(A_\theta)$ we also define the *Poisson operator for the Robin problem* $\gamma_\theta(\lambda)$ by

$$\gamma_\theta(\lambda) : H^{-1/2}(\partial\Omega) \to L^2(\Omega), \quad \gamma_\theta(\lambda)\big(\partial_\mathcal{L}u_\lambda|_{\partial\Omega} + \theta u_\lambda|_{\partial\Omega}\big) := u_\lambda, \qquad (5.1)$$

for any $u_\lambda \in H^1(\Omega)$ such that $\mathcal{L}u_\lambda = \lambda u_\lambda$.

In order to prove the main result of this section we collect some properties of $\gamma_\theta(\lambda)$ and $M_\theta(\lambda)$, which are analogs of the statements in lemma 3.3. Their proofs are similar to those in [9, lemma 2.4] and are not repeated here.

Lemma 5.3. *For $\lambda, \mu \in \rho(A_\theta)$ let $\gamma_\theta(\lambda), \gamma_\theta(\mu)$ be the Poisson operators for the Robin problem and let $M_\theta(\lambda), M_\theta(\mu)$ be the Robin-to-Dirichlet maps. Then the following assertions hold.*

(i) $\gamma_\theta(\lambda)$ is bounded and the identity

$$\gamma_\theta(\lambda) = \big(I + (\lambda - \mu)(A_\theta - \lambda)^{-1}\big)\gamma_\theta(\mu)$$

holds.

(ii) $M_\theta(\lambda)$ is a bounded operator from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$, the operator function $\lambda \mapsto M_\theta(\lambda)$ is holomorphic on $\rho(A_\theta)$, and

$$(\operatorname{Im}\mu)\|\gamma_\theta(\mu)\varphi\|^2_{L^2(\Omega)} = \operatorname{Im}(M_\theta(\mu)\varphi, \varphi)_{\partial\Omega}$$

holds for all $\varphi \in H^{-1/2}(\partial\Omega)$.

For $\varphi \in H^{-1/2}(\partial\Omega)$ and an open set $\nu \subset \partial\Omega$ we shall say that φ vanishes on ν if $(\varphi, \eta)_{\partial\Omega} = 0$ for all $\eta \in H^{1/2}(\partial\Omega)$ with supp $\eta \subset \nu$. As usual, we define the support supp $\varphi \subset \partial\Omega$ of φ to be the complement of the union of all open sets on which φ vanishes.

The main result of this section is the following.

Theorem 5.4. *Let $\mathcal{L}_1, \mathcal{L}_2$ be two uniformly elliptic differential expressions on Ω of the form (1.1) with coefficients $a_{jk,1}, a_{j,1}, a_1$ and $a_{jk,2}, a_{j,2}, a_2$, respectively, satisfying (2.2)–(2.4). Let $\theta_1, \theta_2 \in L^\infty(\partial\Omega)$ be real-valued and let A_{θ_1}, A_{θ_2} and $M_{\theta_1}(\lambda), M_{\theta_2}(\lambda)$ denote the corresponding self-adjoint Robin operators and Robin-to-Dirichlet maps, respectively. Assume that $\omega \subset \partial\Omega$ is an open, nonempty set such that*

$$(M_{\theta_1}(\lambda)\varphi, \varphi)_{\partial\Omega} = (M_{\theta_2}(\lambda)\varphi, \varphi)_{\partial\Omega}, \quad \varphi \in H^{-1/2}(\partial\Omega), \text{ supp } \varphi \subset \omega, \quad (5.2)$$

holds for all $\lambda \in \mathcal{D}$, where $\mathcal{D} \subset \rho(A_{\theta_1}) \cap \rho(A_{\theta_2})$ is a set with an accumulation point in $\rho(A_{\theta_1}) \cap \rho(A_{\theta_2})$. Then there exists a unitary operator U in $L^2(\Omega)$ such that

$$A_{\theta_2} = U A_{\theta_1} U^*$$

holds.

Proof. The proof of theorem 5.4 is a modification of the proof of theorem 3.4 and we will leave some details to the reader. For any $\mu \in \mathbb{C} \setminus \mathbb{R}$ and let $\gamma_{\theta_i}(\mu)$ be the Poisson operator for the Robin problem as defined in (5.1), $i = 1, 2$. We define a linear mapping V in $L^2(\Omega)$ on the domain

$$\operatorname{dom} V = \operatorname{span}\{\gamma_{\theta_1}(\mu)\varphi : \varphi \in H^{-1/2}(\partial\Omega), \text{ supp } \varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R}\}$$

setting

$$V\gamma_{\theta_1}(\mu)\varphi = \gamma_{\theta_2}(\mu)\varphi, \quad \varphi \in H^{-1/2}(\partial\Omega), \text{ supp } \varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R},$$

and extending this operator by linearity to all of dom V. Clearly, we have

$$\operatorname{ran} V = \operatorname{span}\{\gamma_{\theta_2}(\mu)\varphi : \varphi \in H^{-1/2}(\partial\Omega), \text{ supp } \varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R}\}.$$

As in Step 1 of the proof of theorem 3.4 we conclude from (5.2) with the help of lemma 5.3 (i) and (ii) (instead of lemma 3.3 (ii) and (iii)) that V is well-defined, isometric, and satisfies

$$V(A_{\theta_1} - \lambda)^{-1} \upharpoonright H_\lambda = (A_{\theta_2} - \lambda)^{-1} V \upharpoonright H_\lambda \qquad (5.3)$$

for each fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$, where H_λ is the subspace of dom V given by

$$H_\lambda = \operatorname{span}\{\gamma_{\theta_1}(\mu)\varphi : \varphi \in H^{1/2}(\partial\Omega), \text{ supp } \varphi \subset \omega, \mu \in \mathbb{C} \setminus \mathbb{R}, \mu \neq \lambda\}.$$

Let us now check that dom V is dense in $L^2(\Omega)$. Let $\widetilde{\Omega}$ and $\widetilde{\mathcal{L}}_1$ be defined as in Step 2 of the proof of theorem 3.4 above with the additional condition that there exist $\omega_0 \subset \partial\Omega$ such that $\overline{\omega}_0 \subset \omega$ and still $\partial\Omega \setminus \omega_0 \subset \partial\widetilde{\Omega}$. Define the real-valued function $\widetilde{\theta}_1 \in L^\infty(\partial\widetilde{\Omega})$ by

$$\widetilde{\theta}_1 = \begin{cases} \theta_1 & \text{on } \partial\Omega \setminus \omega_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the operator

$$\widetilde{A}_{\widetilde{\theta}_1}\widetilde{u} = \widetilde{\mathcal{L}}_1\widetilde{u}, \quad \mathrm{dom}\,\widetilde{A}_{\widetilde{\theta}_1} = \{\widetilde{u} \in H^1(\widetilde{\Omega}) : \widetilde{\mathcal{L}}_1\widetilde{u} \in L^2(\widetilde{\Omega}), \ \partial_{\widetilde{\mathcal{L}}_1}\widetilde{u}|_{\partial\widetilde{\Omega}} + \widetilde{\theta}_1\widetilde{u}|_{\partial\widetilde{\Omega}} = 0\},$$

in $L^2(\widetilde{\Omega})$ is self-adjoint and semibounded from below; as in the proof of theorem 3.4 one argues that $\widetilde{A}_{\widetilde{\theta}_1}$ can be assumed to be uniformly positive. For each $\widetilde{v} \in L^2(\widetilde{\Omega})$ such that $\widetilde{v}$ vanishes on Ω, we define

$$\widetilde{u}_{\mu,\widetilde{v}} = (\widetilde{A}_{\widetilde{\theta}_1} - \mu)^{-1}\widetilde{v}, \quad \mu \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, we denote by $u_{\mu,\widetilde{v}}$ the restriction of $\widetilde{u}_{\mu,\widetilde{v}}$ to Ω. Then $\mathcal{L}_1 u_{\mu,\widetilde{v}} = \mu u_{\mu,\widetilde{v}}$ and by construction

$$\mathrm{supp}\,\big(\partial_{\mathcal{L}_1}u_{\mu,\widetilde{v}}|_{\partial\Omega} + \theta_1 u_{\mu,\widetilde{v}}|_{\partial\Omega}\big) \subset \overline{\omega_0} \subset \omega. \tag{5.4}$$

In fact, to justify (5.4) consider $x \in \partial\Omega \setminus \overline{\omega_0}$, choose an open set $\nu \subset \partial\Omega \setminus \overline{\omega_0}$ with $x \in \nu$, and let $\varphi \in H^{1/2}(\partial\Omega)$ with $\mathrm{supp}\,\varphi \subset \nu$. The first inclusion in (5.4) follows if we show

$$(\partial_{\mathcal{L}_1}u_{\mu,\widetilde{v}}|_{\partial\Omega} + \theta_1 u_{\mu,\widetilde{v}}|_{\partial\Omega}, \varphi)_{\partial\Omega} = 0. \tag{5.5}$$

Choose $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = \varphi$ so that, in particular, $w|_{\omega_0} = 0$. Hence the extension $\widetilde{w}$ by zero of w onto $\widetilde{\Omega}$ satisfies $\widetilde{w} \in H^1(\widetilde{\Omega})$ and $\mathrm{supp}\,(\widetilde{w}|_{\partial\widetilde{\Omega}}) \subset \partial\Omega \setminus \overline{\omega_0}$. Now it follows from the definition of the conormal derivative that

$$\begin{aligned}
(\partial_{\mathcal{L}_1}u_{\mu,\widetilde{v}}|_{\partial\Omega} &+ \theta_1 u_{\mu,\widetilde{v}}|_{\partial\Omega}, \varphi)_{\partial\Omega} \\
&= -(\mathcal{L}_1 u_{\mu,\widetilde{v}}, w)_{L^2(\Omega)} + \mathfrak{a}[u_{\mu,\widetilde{v}}, w] + (\theta_1 u_{\mu,\widetilde{v}}|_{\partial\Omega}, w|_{\partial\Omega})_{\partial\Omega} \\
&= -(\widetilde{\mathcal{L}}_1\widetilde{u}_{\mu,\widetilde{v}}, \widetilde{w})_{L^2(\widetilde{\Omega})} + \widetilde{\mathfrak{a}}[\widetilde{u}_{\mu,\widetilde{v}}, \widetilde{w}] + (\widetilde{\theta}_1\widetilde{u}_{\mu,\widetilde{v}}|_{\partial\widetilde{\Omega}}, \widetilde{w}|_{\partial\widetilde{\Omega}})_{\partial\widetilde{\Omega}} \\
&= (\partial_{\widetilde{\mathcal{L}}_1}\widetilde{u}_{\mu,\widetilde{v}}|_{\partial\widetilde{\Omega}} + \widetilde{\theta}_1\widetilde{u}_{\mu,\widetilde{v}}|_{\partial\widetilde{\Omega}}, \widetilde{w}|_{\partial\widetilde{\Omega}})_{\partial\widetilde{\Omega}} = 0,
\end{aligned}$$

which proves (5.5) and therefore (5.4) holds. Now it follows in the same way as in the proof of theorem 3.4 that $u_{\mu,\widetilde{v}} \in \mathrm{dom}\,V$ for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and all $\widetilde{v} \in L^2(\widetilde{\Omega})$ with $\widetilde{v}|_\Omega = 0$.

If we choose $u \in L^2(\Omega)$ being orthogonal to $\mathrm{dom}\,V$ and denote by $\widetilde{u}$ the extension of u by zero to $\widetilde{\Omega}$ then we obtain

$$0 = (u, u_{\overline{\mu},\widetilde{v}})_{L^2(\Omega)} = \big(\widetilde{u}, (\widetilde{A}_{\widetilde{\theta}_1} - \overline{\mu})^{-1}\widetilde{v}\big)_{L^2(\widetilde{\Omega})} = \big((\widetilde{A}_{\widetilde{\theta}_1} - \mu)^{-1}\widetilde{u}, \widetilde{v}\big)_{L^2(\widetilde{\Omega})}$$

for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and all $\widetilde{v} \in L^2(\widetilde{\Omega})$ which vanish on Ω, that is,

$$\big((\widetilde{A}_{\widetilde{\theta}_1} - \mu)^{-1}\widetilde{u}\big)|_{\widetilde{\Omega}\setminus\Omega} = 0$$

for all $\mu \in \mathbb{C} \setminus \mathbb{R}$. Proceeding further as in Step 2 of the proof of theorem 3.4 it can be concluded that $\mathrm{e}^{-t\sqrt{\widetilde{A}_{\widetilde{\theta}_1}}}\widetilde{u}$ vanishes on an open, nonempty subset of $\widetilde{\Omega} \times (0, \infty)$ and by unique continuation it follows $\mathrm{e}^{-t\sqrt{\widetilde{A}_{\widetilde{\theta}_1}}}\widetilde{u} = 0$ on $\widetilde{\Omega}$ for each $t > 0$. Hence, $u = 0$, which implies that $\mathrm{dom}\,V$ is dense in $L^2(\Omega)$. Analogously one shows that $\mathrm{ran}\,V$ is dense in $L^2(\Omega)$.

Now it follows in the same way as in the end of Step 2 of the proof of theorem 3.4 that the isometric operator V extends by continuity to a unitary operator $U : L^2(\Omega) \to L^2(\Omega)$ and that (5.3) extends to

$$U(A_{\theta_1} - \lambda)^{-1} = (A_{\theta_2} - \lambda)^{-1}U.$$

This yields $A_{\theta_2} = UA_{\theta_1}U^*$ and hence completes the proof of the theorem. □

Acknowledgments

ORCID iDs

Jussi Behrndt ● https://orcid.org/0000-0002-3442-6777
Jonathan Rohleder ● https://orcid.org/0000-0003-1354-5387

References

[1] Abels H, Grubb G and Wood I 2014 Extension theory and Kreĭn-type resolvent formulas for nonsmooth boundary value problems *J. Funct. Anal.* **266** 4037–100
[2] Anderson M, Katsuda A, Kurylev Y, Lassas M and Taylor M 2004 Boundary regularity for the Ricci equation, geometric convergence, and Gelfand's inverse boundary problem *Inventiones Math.* **158** 261–321
[3] Astala K, Lassas M and Päivärinta L 2005 Calderón's inverse problem for anisotropic conductivity in the plane *Commun. PDE* **30** 207–24
[4] Arendt W and ter Elst A F M 2011 The Dirichlet-to-Neumann operator on rough domains *J. Differ. Equ.* **251** 2100–24
[5] Arendt W and ter Elst A F M 2015 The Dirichlet-to-Neumann operator on exterior domains *Potential Anal.* **43** 313–40
[6] Arendt W, ter Elst A F M, Kennedy J B and Sauter M 2014 The Dirichlet-to-Neumann operator via hidden compactness *J. Funct. Anal.* **266** 1757–86
[7] Astala K and Päivärinta L 2006 Calderón's inverse conductivity problem in the plane *Ann. Math.* **163** 265–99
[8] Bär C and Strohmaier A 2001 Semi-bounded restrictions of Dirac type operators and the unique continuation property *Differ. Geom. Appl.* **15** 175–82
[9] Behrndt J and Rohleder J 2012 An inverse problem of Calderón type with partial data *Commun. PDE* **37** 1141–59
[10] Behrndt J and Rohleder J 2015 Spectral analysis of self-adjoint elliptic differential operators, Dirichlet-to-Neumann maps, and abstract Weyl functions *Adv. Math.* **285** 1301–38
[11] Behrndt J and Rohleder J 2016 Titchmarsh–Weyl theory for Schrödinger operators on unbounded domains *J. Spectr. Theory* **6** 67–87
[12] Belishev M I 1987 An approach to multidimensional inverse problems for the wave equation *Dokl. Akad. Nauk SSSR* **297** 524–7
     Belishev M I 1988 *Sov. Math. Dokl.* **36** 481–4 (transl.)
[13] Belishev M I 1997 Boundary control in reconstruction of manifolds and metrics (the BC method) *Inverse Problems* **13** R1–45
[14] Belishev M I 2007 Recent progress in the boundary control method *Inverse Problems* **23** R1–67
[15] Belishev M I and Kurylev Y 1992 To the reconstruction of a Riemannian manifold via its spectral data (BC-method) *Commun. PDE* **17** 767–804

[16] Brown B M, Grubb G and Wood I G 2009 *M*-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems *Math. Nachr.* **282** 314–47

[17] Brown B M, Marletta M, Naboko S and Wood I G 2008 Boundary triplets and *M*-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices *J. Lond. Math. Soc.* **77** 700–18

[18] Brown B M, Marletta M, Naboko S and Wood I G 2017 Inverse problems for boundary triples with applications *Stud. Math.* **237** 241–75

[19] Bukhgeim A L and Uhlmann G 2002 Recovering a potential from partial Cauchy data *Commun. PDE* **27** 653–68

[20] Calderón A P 1980 On an inverse boundary value problem *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)* (Rio de Janeiro: Soc. Brasil. Mat.) pp 65–73

[21] Caro P and Marinov K 2016 Stability of inverse problems in an infinite slab with partial data *Commun. PDE* **41** 683–704

[22] Choulli M and Soccorsi E 2015 An inverse anisotropic conductivity problem induced by twisting a homogeneous cylindrical domain *J. Spectr. Theory* **5** 295–329

[23] Dos Santos Ferreira D, Kenig C E, Salo M and Uhlmann G 2009 Limiting Carleman weights and anisotropic inverse problems *Inventory Math.* **178** 119–71

[24] Dos Santos Ferreira D, Kurylev Y, Lassas M and Salo M 2016 The Calderón problem in transversally anisotropic geometries *J. Eur. Math. Soc.* **18** 2579–626

[25] Edmunds D E and Evans W D 1987 *Spectral Theory, Differential Operators* (Oxford: Clarendon)

[26] Gesztesy F and Mitrea M 2008 Generalized Robin boundary conditions, Robin-to-Dirichlet maps, Kreĭn-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains *Perspectives in Partial Differential Equations, Harmonic Analysis and Applications* (*Proc. Symp. in Pure Mathematics* vol 79) (Providence, RI: American Mathematical Society) pp 105–73

[27] Gesztesy F and Mitrea M 2009 Nonlocal Robin Laplacians and some remarks on a paper by Filonov on eigenvalue inequalities *J. Differ. Equ.* **247** 2871–96

[28] Gesztesy F and Mitrea M 2011 A description of all self-adjoint extensions of the Laplacian and Kreĭn-type resolvent formulas on non-smooth domains *J. Anal. Math.* **113** 53–172

[29] Grubb G 1968 A characterization of the non-local boundary value problems associated with an elliptic operator *Ann. Scuola Norm. Suppl. Pisa* **22** 425–513

[30] Grubb G 2008 Krein resolvent formulas for elliptic boundary problems in nonsmooth domains *Rend. Semin. Mat. Univ. Politec. Torino* **66** 271–97

[31] Grubb G 2011 Spectral asymptotics for Robin problems with a discontinuous coefficient *J. Spectral Theory* **1** 155–77

[32] Grubb G 2011 Perturbation of essential spectra of exterior elliptic problems *Appl. Anal.* **90** 103–23

[33] Grubb G 2012 Extension theory for elliptic partial differential operators with pseudodifferential methods *Operator Methods for Boundary Value Problems* (*London Mathematical Society Lecture Note Series* vol 404) (Cambridge: Cambridge University Press) pp 221–58

[34] Ikehata M 2001 Inverse conductivity problem in the infinite slab *Inverse Problems* **17** 437–54

[35] Imanuvilov O, Uhlmann G and Yamamoto M 2010 The Calderón problem with partial data in two dimensions *J. Am. Math. Soc.* **23** 655–91

[36] Isozaki H 2004 Inverse spectral problems on hyperbolic manifolds and their applications to inverse boundary value problems *Am. J. Math.* **126** 1261–313

[37] Isozaki H and Kurylev Y 2014 Introduction to spectral theory, inverse problem on asymptotically hyperbolic manifolds *MSJ Memoirs 32* (Tokyo: Mathematical Society of Japan)

[38] Isozaki H, Kurylev Y and Lassas M 2010 Forward and inverse scattering on manifolds with asymptotically cylindrical ends *J. Funct. Anal.* **258** 2060–118

[39] Isozaki H, Kurylev Y and Lassas M 2014 Recent progress of inverse scattering theory on non-compact manifolds *Inverse problems and Applications* (*Contemporary Mathematics* vol 615) (Providence, RI: American Mathematical Society) pp 143–63

[40] Isozaki H, Kurylev Y and Lassas M 2017 Conic singularities, generalized scattering matrix, and inverse scattering on asymptotically hyperbolic surfaces *J. Reine Angew. Math.* **724** 53–103

[41] Katchalov A and Kurylev Y 1998 Multidimensional inverse problem with incomplete boundary spectral data *Commun. PDE* **23** 55–95

[42] Katchalov A, Kurylev Y and Lassas M 2001 *Inverse Boundary Spectral Problems* (*Chapman, Hall/ CRC Monographs and Surveys in Pure and Applied Mathematics* vol 123) (London: Chapman and Hall)

[43] Kato T 1995 *Perturbation Theory for Linear Operators* (Berlin: Springer)
[44] Kavian O, Kian Y and Soccorsi E 2015 Uniqueness and stability results for an inverse spectral problem in a periodic waveguide *J. Math. Pures Appl.* **104** 1160–89
[45] Kenig C E, Sjöstrand J and Uhlmann G 2007 The Calderón problem with partial data *Ann. Math.* **165** 567–91
[46] Kian Y Recovery of non compactly supported coefficients of elliptic equations on an infinite waveguide *J. Inst. Math. Jussieu* 1–28
[47] Kian Y Determination of non-compactly supported electromagnetic potentials in unbounded closed waveguide to appear in *Rev. Mat. Ib.* (https://doi.org/10.4171/RMI/1143)
[48] Krupchyk K, Lassas M and Uhlmann G 2012 Inverse problems with partial data for a magnetic Schrödinger operator in an infinite slab and on a bounded domain *Commun. Math. Phys.* **312** 87–126
[49] Kurylev Y and Lassas M 1997 The multidimensional Gelfand inverse problem for non-self-adjoint operators *Inverse Problems* **13** 1495–501
[50] Kurylev Y and Lassas M 2006 Multidimensional Gelfand inverse boundary spectral problem: uniqueness and stability *Cubo* **8** 41–59
[51] Lassas M 1998 Inverse boundary spectral problem for non-self-adjoint Maxwell's equations with incomplete data *Commun. PDE* **23** 629–48
[52] Lassas M, Salo M and Uhlmann G 2015 *Wave Phenomena* (*Handbook of Mathematical Methods in Imaging* vol 1–3) (New York: Springer) pp 1205–52
[53] Lassas M, Taylor M and Uhlmann G 2003 The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary *Commun. Anal. Geom.* **11** 207–21
[54] Lassas M and Uhlmann G 2001 Determining Riemannian manifold from boundary measurements *Ann. Sci. Ec. Norm. Super.* **34** 771–87
[55] Lee J and Uhlmann G 1989 Determining anisotropic real-analytic conductivities by boundary measurements *Commun. Pure Appl. Math.* **42** 1097–112
[56] Li X and Uhlmann G 2010 Inverse problems with partial data in a slab *Inverse Problems Imaging* **4** 449–62
[57] Malamud M M 2010 Spectral theory of elliptic operators in exterior domains *Russ. J. Math. Phys.* **17** 96–125
[58] Mantile A, Posilicano A and Sini M 2016 Self-adjoint elliptic operators with boundary conditions on not closed hypersurfaces *J. Differ. Equ.* **261** 1–55
[59] McLean W 2000 *Strongly Elliptic Systems, Boundary Integral Equations* (Cambridge: Cambridge University Press)
[60] Nachman A 1988 Reconstructions from boundary measurements *Ann. Math.* **128** 531–76
[61] Nachman A 1996 Global uniqueness for a two-dimensional inverse boundary value problem *Ann. Math.* **143** 71–96
[62] Nachman A and Street B 2010 Reconstruction in the Calderón problem with partial data *Commun. PDE* **35** 375–90
[63] Nachman A, Sylvester J and Uhlmann G 1988 An n-dimensional Borg–Levinson theorem *Commun. Math. Phys.* **115** 595–605
[64] Ouhabaz E M 2018 A milder version of Calderón's inverse problem for anisotropic conductivities and partial data *J. Spectr. Theory* **8** 435–57
[65] Pohjola V 2014 An inverse problem for the magnetic Schrödinger operator on a half space with partial data *Inverse Problems Imaging* **8** 1169–89
[66] Posilicano A 2008 Self-adjoint extensions of restrictions *Oper. Matrices* **2** 1–24
[67] Posilicano A and Raimondi L 2009 Krein's resolvent formula for self-adjoint extensions of symmetric second-order elliptic differential operators *J. Phys. A: Math. Theor.* **42** 015204
[68] Post O 2016 Boundary pairs associated with quadratic forms *Math. Nachr.* **289** 1052–99
[69] Salo M and Wang J-N 2006 Complex spherical waves and inverse problems in unbounded domains *Inverse Problems* **22** 2299–309
[70] Stein E 1970 *Singular Integrals, Differentiability Properties of Functions* (Princeton, NJ: Princeton University Press)
[71] Sun Z and Uhlmann G 2003 Anisotropic inverse problems in two dimensions *Inverse Problems* **19** 1001–10
[72] Sylvester J 1990 An anisotropic inverse boundary value problem *Commun. Pure Appl. Math.* **43** 201–32

[73] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problem *Ann. Math.* **125** 153–69

[74] Sylvester J and Uhlmann G 1991 Inverse problems in anisotropic media *Contemp. Math.* **122** 105–17

[75] Uhlmann G 2009 Electrical impedance tomography and Calderón's problem *Inverse Problems* **25** 123011

[76] Uhlmann G 2014 Inverse problems: seeing the unseen *Bull. Math. Sci.* **4** 209–79

[77] Uhlmann G 2014 30 years of Calderón's problem *Séminaire Laurent Schwartz-Équations aux dérivés Partielles et Applications 2012–2013, Exp. No. XIII, 25 pp., Sémin. Équ. Dériv. Partielles* (Palaiseau: École Polytech.)

[78] Vishik M I 1952 On general boundary problems for elliptic differential equations [Russian] *Trudy Moskov. Mat. Obšč.* **1** 187–246

[79] Wolff T H 1993 Recent work on sharp estimates in second-order elliptic unique continuation problems *J. Geom. Anal.* **3** 621–50