On generalized resolvents of symmetric operators of defect one with finitely many negative squares

Jussi Behrndt and Carsten Trunk

Abstract

Jussi Behrndt and Carsten Trunk (2005). On generalized resolvents of symmetric operators of defect one with finitely many negative squares. *Proceedings of the University of Vaasa, Reports* xxx, pp–pp.

For a closed symmetric operator A of defect one with finitely many negative squares in a Krein space we establish a bijective correspondence between the compressed resolvents of minimal selfadjoint exit space extensions of A with finitely many negative squares and a special subclass of meromorphic functions in $\mathbb{C}\backslash\mathbb{R}$.

Jussi Behrndt, Department of Mathematics MA 6–4, Straße des 17. Juni 136, TU Berlin, 10623 Berlin, Germany, E-mail: behrndt@math.tu-berlin.de
Carsten Trunk, Department of Mathematics MA 6–3, Straße des 17. Juni 136, TU Berlin, 10623 Berlin, Germany, E-mail: trunk@math.tu-berlin.de

Keywords: generalized resolvents, boundary value spaces, definitizable operators, Krein spaces.
Mathematics Subject Classification (2000): 47B50; 47B25.

1. Introduction

For a closed densely defined symmetric operator A with equal defect numbers in the Hilbert space $\mathfrak{K}$ let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary value space for the adjoint operator A^* and let A_0 be the restriction of A^* to $\ker \Gamma_0$, $A_0 := A^*|\ker \Gamma_0$. If γ and M are the corresponding γ-field and Weyl function, respectively, then it is well known that the Krein-Naimark formula

$$P_{\mathfrak{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^* \qquad (1)$$

establishes a bijective correspondence between the compressed resolvents of minimal selfadjoint exit space extensions $\widetilde{A}$ of A in $\mathfrak{K} \times \mathfrak{H}$, where $\mathfrak{H}$ is a Hilbert space, and the so-called Nevanlinna families τ.

The aim of this note is to give a similar correspondence for a class of symmetric operators in Krein spaces. More precisely, if A is a closed symmetric operator of defect one with finitely many negative squares acting in a Krein space $\mathcal{K}$ and if A has a selfadjoint extension A_0 in $\mathcal{K}$ with nonempty resolvent set we prove in Theorem 3 that the formula (1) establishes a bijective correspondence between the compressed resolvents of minimal selfadjoint exit space extensions $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$ having also finitely many negative squares and scalar functions τ belonging to some classes $D_{\widehat{\kappa}}$, $\widehat{\kappa} \in \{0, 1, \dots\}$. Moreover we show how the number $\widehat{\kappa}$ is related to the number of negative squares of $\widetilde{A}$. Here the exit space $\mathcal{H}$ is in general a Krein space and the classes $D_{\widehat{\kappa}}$ are subclasses of the so-called definitizable functions (cf. [Jonas (2000)]). The classes $D_{\widehat{\kappa}}$ where introduced and studied in connection with eigenvalue dependent boundary value problems by the authors in [Behrndt and Trunk (2005)]. Roughly speaking a function τ belongs to some class $D_{\widehat{\kappa}}$ if $\lambda \mapsto \lambda\tau(\lambda)$ is a generalized Nevanlinna function.

Our approach is based on [Derkach, Hassi, Malamud and de Snoo (2000)]; see also [Hassi, Kaltenbäck and de Snoo (1997) and (1998)]. For the special case that the exit space $\mathcal{H}$ is a Pontryagin space Theorem 3 follows from [Derkach (1998)]. In this situation the functions $\tau \in D_{\widehat{\kappa}}$ belong to certain subclasses of the generalized Nevanlinna functions.

2. Preliminaries

Let throughout this paper $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space. The linear space of all bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we write $\mathcal{L}(\mathcal{K})$. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K}^2$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse, the multivalued part etc. we refer to [Dijksma and de Snoo (1987)].

Let S be a linear relation in $\mathcal{K}$. The *adjoint* $S^+ \in \widetilde{\mathcal{C}}(\mathcal{K})$ of S is defined as

$$S^+ := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \,\middle|\, [f', h] = [f, h'] \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in S \right\}.$$

The linear relation S is said to be *symmetric* (*selfadjoint*) if $S \subset S^+$ (resp. $S = S^+$). For a closed linear relation S in $\mathcal{K}$ the resolvent set $\rho(S)$ of $S \in \widetilde{\mathcal{C}}(\mathcal{K})$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $(S - \lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. For the definition of the point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ we refer to [Dijksma et al. (1987)].

For the description of the selfadjoint extensions of closed symmetric relations we use the so-called boundary value spaces.

Definition 1. *Let A be a closed symmetric relation in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a* boundary value space for A^+ *if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exist mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G} \times \mathcal{G}$ is surjective, and the relation*

$$[f', g] - [f, g'] = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})$$

holds for all $\hat{f} = \binom{f}{f'}, \hat{g} = \binom{g}{g'} \in A^+$.

For basic facts on boundary value spaces and further references see e.g. [Derkach (1999)]. We recall only a few important consequences. Let A be a closed symmetric relation and assume that there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+. Then $A_0 := \ker \Gamma_0$ and $A_1 := \ker \Gamma_1$ are selfadjoint extensions of A. The mapping $\Gamma = \binom{\Gamma_0}{\Gamma_1}$ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \{\hat{f} \in A^+ \,|\, \Gamma \hat{f} \in \Theta\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{2}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between $\widetilde{\mathcal{C}}(\mathcal{G})$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (2) gives a one-to-one correspondence between the closed symmetric (selfadjoint) extensions of A and the closed symmetric (resp. selfadjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker(\Gamma_1 - \Theta\Gamma_0). \tag{3}$$

Let $\mathcal{N}_{\lambda,A^+} := \ker(A^+ - \lambda)$ be the defect subspace of A and $\hat{\mathcal{N}}_{\lambda,A^+} := \{\binom{f}{\lambda f} \,|\, f \in \mathcal{N}_{\lambda,A^+}\}$. Now we assume, in addition, that the selfadjoint relation A_0 has a nonempty resolvent set. For each $\lambda \in \rho(A_0)$ the relation A^+ can be written as a direct sum of (the subspaces) A_0 and $\hat{\mathcal{N}}_{\lambda,A^+}$. Denote by π_1 the orthogonal projection onto the first component of $\mathcal{K}^2$. The functions

$$\gamma(\lambda) := \pi_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}, \mathcal{K}) \quad \text{and} \quad M(\lambda) := \Gamma_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}) \tag{4}$$

are defined and holomorphic on $\rho(A_0)$ and are called the *γ-field* and the *Weyl function* corresponding to A and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$.

Let $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and let A_Θ be the corresponding extension of A via (2). For $\lambda \in \rho(A_0)$ we have

$$\lambda \in \rho(A_\Theta) \quad \text{if and only if} \quad 0 \in \rho(\Theta - M(\lambda)). \tag{5}$$

Moreover the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{6}$$

holds for $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$ (cf. [Derkach (1999)]).

Recall, that a piecewise meromorphic function G in $\mathbb{C}\backslash\mathbb{R}$ belongs to the *generalized Nevanlinna class* $N_{\kappa'}$, $\kappa' \in \mathbb{N}_0$, if G is symmetric with respect to the real axis, that is $G(\overline{\lambda}) = \overline{G(\lambda)}$ for all points λ of holomorphy of G, and the so-called Nevanlinna kernel

$$N_G(\lambda, \mu) := \frac{G(\lambda) - G(\overline{\mu})}{\lambda - \overline{\mu}}$$

has κ negative squares (see e.g. [Krein and Langer (1977)]). The subclasses $D_{\widehat{\kappa}}$, $\widehat{\kappa} \in \mathbb{N}_0$, (see Definition 2) of the so-called definitizable functions (cf. [Jonas (2000)]) were introduced and studied in [Behrndt et al. (2005)].

Definition 2. *Let τ be a piecewise meromorphic function in $\mathbb{C}\backslash\mathbb{R}$ which is symmetric with respect to the real axis and let $\lambda_0 \in \mathbb{C}$ be a point of holomorphy of τ. We say that τ belongs to the class $D_{\widehat{\kappa}}$, $\widehat{\kappa} \in \mathbb{N}_0$, if there exists a generalized Nevanlinna function $G \in N_{\widehat{\kappa}}$ holomorphic at λ_0 and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$ such that*

$$\frac{\lambda}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds for all points λ where τ, G and g are holomorphic.

Let A be a closed symmetric relation in $\mathcal{K}$. We say that A has *defect* $m \in \mathbb{N} \cup \{\infty\}$ if there exists a selfadjoint extension $\widehat{A}$ in $\mathcal{K}$ such that $\dim(\widehat{A}/A) = m$. If J is a fundamental symmetry in $\mathcal{K}$ then A has defect m if and only if the deficiency indices $n_\pm(JA) = \dim\ker((JA)^* \mp i)$ of the symmetric relation JA in the Hilbert space $(\mathcal{K}, [J\cdot, \cdot])$ are equal to m. A closed symmetric relation A in the Krein space $(\mathcal{K}, [\cdot, \cdot])$

is said to have κ *negative squares*, $\kappa \in \mathbb{N}_0$, if the hermitian form $\langle \cdot, \cdot \rangle$ on A, defined by

$$\left\langle \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \right\rangle := [f, g'], \qquad \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \in A,$$

has κ negative squares, that is, there exists a κ-dimensional subspace $\mathcal{M}$ in A such that $\langle \hat{v}, \hat{v} \rangle < 0$ if $\hat{v} \in \mathcal{M}$, $\hat{v} \neq 0$, but no $\kappa + 1$ dimensional subspace with this property. If, in addition, the defect of A is one and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary value space for A^+ such that the resolvent set of $A_0 = \ker \Gamma_0$ is nonempty, then the corresponding Weyl function M belongs to some subclass $D_{\widehat{\kappa}}$, $\widehat{\kappa} \leq \kappa + 1$.

Conversely, by [Behrndt et al. (2005)] each function $\tau \in D_{\widehat{\kappa}}$ which is not equal to a constant is a Weyl function corresponding to a symmetric operator T in some Krein space $\mathcal{H}$ and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ such that the selfadjoint relation $\ker \Gamma_0'$ has $\widehat{\kappa}$ negative squares.

3. A class of generalized resolvents of symmetric operators with finitely many negative squares

Let A be a not necessarily densely defined symmetric operator in the Krein space $\mathcal{K}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ and let $\mathcal{H}$ be a further Krein space. A selfadjoint extension $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$ is said to be an *exit space extension of* A and $\mathcal{H}$ is called the *exit space*. The exit space extension $\widetilde{A}$ of A is said to be *minimal* if $\rho(\widetilde{A})$ is nonempty and

$$\mathcal{K} \times \mathcal{H} = \operatorname{clsp} \left\{ \mathcal{K}, (\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} \mid \lambda \in \rho(\widetilde{A}) \right\}$$

holds. The elements of $\mathcal{K} \times \mathcal{H}$ will be written in the form $\{k, h\}$, $k \in \mathcal{K}$, $h \in \mathcal{H}$. Let $P_{\mathcal{K}} : \mathcal{K} \times \mathcal{H} \to \mathcal{H}$, $\{k, h\} \mapsto k$, be the projection onto the first component of $\mathcal{K} \times \mathcal{H}$. Then the compression

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}, \qquad \lambda \in \rho(\widetilde{A}),$$

of the resolvent of $\widetilde{A}$ to $\mathcal{K}$ is said to be a *generalized resolvent* of A.

In the proof of Theorem 3 below we will deal with direct products of linear relations. The following notation will be used. If U is a relation in $\mathcal{K}$ and V is a relation in $\mathcal{H}$

we shall write $U \times V$ for the direct product of U and V which is a relation in $\mathcal{K} \times \mathcal{H}$,

$$U \times V = \left\{ \begin{pmatrix} \{f_1, f_2\} \\ \{f_1', f_2'\} \end{pmatrix} \;\middle|\; \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in U, \begin{pmatrix} f_2 \\ f_2' \end{pmatrix} \in V \right\}.$$

For the pair $\binom{\{f_1,f_2\}}{\{f_1',f_2'\}}$ we shall also write $\{\hat{f}_1, \hat{f}_2\}$, where $\hat{f}_1 = \binom{f_1}{f_1'}$ and $\hat{f}_2 = \binom{f_2}{f_2'}$.

Theorem 3. *Let A be a symmetric operator of defect one with finitely many negative squares and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ with corresponding γ-field γ and Weyl function M. Assume that $A_0 = \ker \Gamma_0$ has a nonempty resolvent set. Then the following holds.*

(i) *The formula*

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{7}$$

establishes a bijective correspondence between the compressed resolvents of minimal selfadjoint exit space extensions $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$ which have finitely many negative squares and the functions τ from the class $\bigcup_{\widehat{\kappa}=0}^{\infty} D_{\widehat{\kappa}} \cup \big\{ \binom{0}{c} \mid c \in \mathbb{C} \big\}$.

(ii) *Assume that A has κ negative squares. If $\widetilde{A}$ is a minimal selfadjoint exit space extension with $\widetilde{\kappa}$ negative squares in $\mathcal{K} \times \mathcal{H}$, $\mathcal{H} \neq \{0\}$, then τ belongs to $D_{\widehat{\kappa}}$, where*

$$0 \leq \widehat{\kappa} \in \{\widetilde{\kappa} - \kappa - 2, \ldots, \widetilde{\kappa} - \kappa + 1\}.$$

Conversely, if $\tau \in D_{\widehat{\kappa}}$, $\widehat{\kappa} \in \mathbb{N}_0$, then the corresponding selfadjoint exit space extension $\widetilde{A}$ in $\mathcal{K} \times \mathcal{H}$ has

$$0 \leq \widetilde{\kappa} \in \{\kappa + \widehat{\kappa} - 1, \ldots, \kappa + \widehat{\kappa} + 2\}$$

negative squares.

Proof. Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let $\widetilde{A}$ be a minimal selfadjoint exit space extension of A in $\mathcal{K} \times \mathcal{H}$ which has $\widetilde{\kappa}$ negative square. The linear relations

$$S := \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} \;\middle|\; \begin{pmatrix} \{k, 0\} \\ \{k', 0\} \end{pmatrix} \in \widetilde{A} \right\} \quad \text{and} \quad T := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \;\middle|\; \begin{pmatrix} \{0, h\} \\ \{0, h'\} \end{pmatrix} \in \widetilde{A} \right\}$$

are closed and symmetric in $\mathcal{K}$ and $\mathcal{H}$, respectively. As S is an extension of A either S is of defect one and coincides with A or S is selfadjoint in $\mathcal{K}$. It follows from [Strauss (1962)], [Remark 5.3, Derkach et al. (2000)] that in the first case T is also of defect one and in the second case T is selfadjoint in $\mathcal{H}$.

If S and T are both selfadjoint, then $S\times T$ coincides with $\widetilde{A}$. As $\widetilde{A}$ is a minimal exit space extension we have

$$\mathcal{H} = \operatorname{clsp}\left\{P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}|_{\mathcal{K}} \mid \lambda\in\rho(\widetilde{A})\right\} = \{0\}.$$

Hence $\widetilde{A}$ is a selfadjoint extension of A in $\mathcal{K}$ and there exists a constant $\tau\in\mathbb{R}\cup\left\{\binom{0}{c}\mid c\in\mathbb{C}\right\}$ such that $\widetilde{A} = \binom{\Gamma_0}{\Gamma_1}^{-1}\{-\tau\}$ and by (6) we have

$$(\widetilde{A}-\lambda)^{-1} = (A_0-\lambda)^{-1} - \gamma(\lambda)\big(M(\lambda)+\tau\big)^{-1}\gamma(\overline{\lambda})^{+}.$$

If S and T are both of defect one we have $A=S$ and it follows from [§5, Derkach et al. (2000)] that A^+ and T^+ can be written as

$$A^+ = \left\{\begin{pmatrix}k\\k'\end{pmatrix} \;\Big|\; \begin{pmatrix}\{k,h\}\\\{k',h'\}\end{pmatrix}\in\widetilde{A}\right\} \quad\text{and}\quad T^+ = \left\{\begin{pmatrix}h\\h'\end{pmatrix} \;\Big|\; \begin{pmatrix}\{k,h\}\\\{k',h'\}\end{pmatrix}\in\widetilde{A}\right\}.$$

Let

$$\widehat{P}_{\mathcal{K}}:\widetilde{A}\to A^+,\ \begin{pmatrix}\{k,h\}\\\{k',h'\}\end{pmatrix}\mapsto\begin{pmatrix}k\\k'\end{pmatrix} \quad\text{and}\quad \widehat{P}_{\mathcal{H}}:\widetilde{A}\to T^+,\ \begin{pmatrix}\{k,h\}\\\{k',h'\}\end{pmatrix}\mapsto\begin{pmatrix}h\\h'\end{pmatrix}.$$

In the sequel we denote the elements in A^+ and T^+ by $\hat{f}_1$ and $\hat{f}_2$, respectively. It follows as in [Theorem 5.4, Derkach et al. (2000)] that $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$, where

$$\Gamma_0' := -\Gamma_0\widehat{P}_{\mathcal{K}}\widehat{P}_{\mathcal{H}}^{-1} \quad\text{and}\quad \Gamma_1' := \Gamma_1\widehat{P}_{\mathcal{K}}\widehat{P}_{\mathcal{H}}^{-1},$$

is a boundary value space for T^+. $\widetilde{A}$ is the canonical selfadjoint extension of the symmetric relation $A\times T$ in $\mathcal{K}\times\mathcal{H}$ given by

(8) $$\widetilde{A} = \left\{\{\hat{f}_1,\hat{f}_2\}\in A^+\times T^+ \mid \Gamma_0\hat{f}_1+\Gamma_0'\hat{f}_2 = \Gamma_1\hat{f}_1-\Gamma_1'\hat{f}_2 = 0\right\}.$$

Since $A\times T$ is of defect two, A has κ negative squares and $\widetilde{A}$ has $\widetilde{\kappa}$ negative squares we conclude that T has

$$0\le\kappa'\in\left\{\widetilde{\kappa}-\kappa-2,\widetilde{\kappa}-\kappa-1,\widetilde{\kappa}-\kappa\right\}$$

negative squares.

For $\lambda\in\rho(\widetilde{A})$ the relation

$$\operatorname{ran}\big(P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}|_{\mathcal{K}}\big) = \mathcal{N}_{\lambda,T^+} = \ker(T^+-\lambda),$$

holds (cf. [Lemma 2.14, Derkach, Hassi, Malamud and de Snoo (2005)]. Since $\widetilde{A}$ is a minimal exit space extension we have

(9) $$\mathcal{H} = \operatorname{clsp}\left\{P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}|_{\mathcal{K}} \mid \lambda\in\rho(\widetilde{A})\right\} = \operatorname{clsp}\left\{\mathcal{N}_{\lambda,T^+} \mid \lambda\in\rho(\widetilde{A})\right\}$$

and this implies that T is an operator.

Let

$$\hat{\mathcal{N}}_{\infty,T^+} := \left\{ \begin{pmatrix} 0 \\ f \end{pmatrix} \in T^+ \right\} \quad \text{and} \quad \mathcal{F}_{\Pi'} := \begin{pmatrix} \Gamma'_0 \\ \Gamma'_1 \end{pmatrix} \hat{\mathcal{N}}_{\infty,T^+},$$

where $\mathcal{F}_{\Pi'} \subset \mathbb{C}^2$ is the so-called forbidden relation (cf. [Derkach (1999)]). As T is an operator of defect one the dimension of $\mathcal{F}_{\Pi'}$ is less or equal to one. We choose $\alpha \in \mathbb{R}$ such that

$$\left\{ \begin{pmatrix} x \\ \alpha x \end{pmatrix} \;\middle|\; x \in \mathbb{C} \right\} \cap \mathcal{F}_{\Pi'} = \{0\}$$

and define $T_\alpha := \ker(\Gamma'_1 - \alpha\Gamma'_0)$. Then T_α is selfadjoint and by [Proposition 2.1, Derkach (1999)] T_α is an operator. From $\{0\} = \operatorname{mul} T_\alpha = (\operatorname{dom} T_\alpha)^{[\perp]}$ we conclude that T_α is densely defined.

We claim that $\rho(T_\alpha)$ is nonempty. In fact, for $\lambda \in \rho(\widetilde{A})$ we have $\operatorname{ran}(\widetilde{A} - \lambda) = \mathcal{K} \times \mathcal{H}$ and since $A \times T$ is of defect two also the range of $(A \times T) - \lambda$ is closed. Therefore $\operatorname{ran}(T - \lambda)$, $\lambda \in \rho(\widetilde{A})$, is closed in $\mathcal{H}$ and the same holds true for $\operatorname{ran}(T_\alpha - \lambda)$. Assume now $\rho(T_\alpha) = \emptyset$. Then

$$\rho(\widetilde{A}) \subset \big(\sigma_p(T_\alpha) \cup \sigma_r(T_\alpha)\big)$$

and as $\lambda \in \sigma_r(T_\alpha)$ implies $\overline{\lambda} \in \sigma_p(T_\alpha)$ we can assume that there are $\kappa' + 2$ eigenvalues in one of the open half planes. The corresponding eigenvectors $f_1, \ldots, f_{\kappa'+2}$ are mutually orthogonal and it follows as in [Proof of Proposition 1.1, Ćurgus and Langer (1989)] that there exist vectors $g_1, \ldots, g_{\kappa'+2}$ in $\operatorname{dom}(T_\alpha)$ such that $[T_\alpha f_i, g_j] = \delta_{ij}$, $i, j = 1, \ldots, \kappa' + 2$, holds. Since

$$\mathcal{L} := \Big(\operatorname{sp}\{f_1, \ldots, f_{\kappa'+2}, g_1, \ldots, g_{\kappa'+2}\}, \big[T_\alpha \cdot, \cdot\big]\Big)$$

is a Krein space with a $(\kappa'+2)$-dimensional neutral subspace, $\mathcal{L}$ contains also a $(\kappa'+2)$-dimensional negative subspace. But this is impossible since T has κ' negative squares and therefore T_α has at most $\kappa' + 1$ negative squares, thus $\rho(T_\alpha) \neq \emptyset$.

We denote the γ-field and Weyl function corresponding to the boundary value space $\{\mathbb{C}, \Gamma'_1 - \alpha\Gamma'_0, -\Gamma'_0\}$ for T^+ by γ' and σ, respectively. Clearly σ is holomorphic on $\rho(T_\alpha)$. From

$$\mathcal{H} = \operatorname{clsp}\left\{\mathcal{N}_{\lambda,T^+} \mid \lambda \in \rho(T_\alpha)\right\} = \operatorname{clsp}\left\{\gamma'(\lambda) \mid \lambda \in \rho(T_\alpha)\right\}$$

and $\sigma(\lambda) - \sigma(\overline{\mu}) = (\lambda - \overline{\mu})\gamma'(\mu)^+\gamma'(\lambda)$, $\lambda, \mu \in \rho(T_\alpha)$, we conclude that σ is not identically equal to a constant.

It is easy to see that $\{\mathbb{C}^2,\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$, where

$$\widetilde{\Gamma}_0\{\hat{f}_1,\hat{f}_2\}:=\begin{pmatrix}\Gamma_0\hat{f}_1\\ \Gamma_1'\hat{f}_2-\alpha\Gamma_0'\hat{f}_2\end{pmatrix}\quad\text{and}\quad\widetilde{\Gamma}_1\{\hat{f}_1,\hat{f}_2\}:=\begin{pmatrix}\Gamma_1\hat{f}_1\\ -\Gamma_0'\hat{f}_2\end{pmatrix},$$

is a boundary value space for $A^+\times T^+$ with corresponding γ-field

$$\lambda\mapsto\widetilde{\gamma}(\lambda)=\begin{pmatrix}\gamma(\lambda)&0\\0&\gamma'(\lambda)\end{pmatrix},\quad\lambda\in\rho(A_0)\cap\rho(T_\alpha),\tag{10}$$

and Weyl function

$$\lambda\mapsto\widetilde{M}(\lambda)=\begin{pmatrix}M(\lambda)&0\\0&\sigma(\lambda)\end{pmatrix},\quad\lambda\in\rho(A_0)\cap\rho(T_\alpha).\tag{11}$$

The selfadjoint extension of $A\times T$ corresponding to $\Theta=\left(\begin{smallmatrix}-\alpha&1\\1&0\end{smallmatrix}\right)\in\mathcal{L}(\mathbb{C}^2)$ via (2) and (3) is given by

$$\ker\bigl(\widetilde{\Gamma}_1-\Theta\widetilde{\Gamma}_0\bigr)=\Bigl\{\{\hat{f}_1,\hat{f}_2\}\in A^+\times T^+\,|\,\Gamma_0\hat{f}_1+\Gamma_0'\hat{f}_2=\Gamma_1\hat{f}_1-\Gamma_1'\hat{f}_2=0\Bigr\}$$

and coincides with $\widetilde{A}$ (cf. (8)). By (5) $(\Theta-\widetilde{M}(\lambda))$ is invertible for all points λ in $\rho(\widetilde{A})\cap\rho(A_0)\cap\rho(T_\alpha)$. Then we have

$$(\widetilde{A}-\lambda)^{-1}=\bigl((A_0\times T_\alpha)-\lambda\bigr)^{-1}+\widetilde{\gamma}(\lambda)\bigl(\Theta-\widetilde{M}(\lambda)\bigr)^{-1}\widetilde{\gamma}(\overline{\lambda})^+\tag{12}$$

(cf. (6)) and, as σ is not equal to a constant, we obtain

$$\bigl(\Theta-\widetilde{M}(\lambda)\bigr)^{-1}=\bigl(M(\lambda)-\sigma(\lambda)^{-1}+\alpha\bigr)^{-1}\begin{pmatrix}-1&-\sigma(\lambda)^{-1}\\-\sigma(\lambda)^{-1}&-\sigma(\lambda)^{-1}(\alpha-M(\lambda))\end{pmatrix}\tag{13}$$

for all $\lambda\in\rho(\widetilde{A})\cap\rho(A_0)\cap\rho(T_\alpha)$. Setting $\tau(\lambda):=-\sigma(\lambda)^{-1}+\alpha$ we conclude from (10), (12) and (13) that the formula

$$P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}|_{\mathcal{K}}=(A_0-\lambda)^{-1}-\gamma(\lambda)\bigl(M(\lambda)+\tau(\lambda)\bigr)^{-1}\gamma(\overline{\lambda})^+$$

holds. It is not hard to see that τ is the Weyl function corresponding to the boundary value space $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ for T^+. As $\ker\Gamma_0'$ is a selfadjoint extension of T it follows that $\ker\Gamma_0'$ has κ' or $\kappa'+1$ negative squares. Now [Lemma 3.7, Behrndt et al. (2005)] implies that τ belongs to some class $D_{\widehat{\kappa}}$, where

$$0\le\widehat{\kappa}\in\bigl\{\widetilde{\kappa}-\kappa-2,\dots,\widetilde{\kappa}-\kappa+1\bigr\}.$$

For a function τ in the class $D_{\widehat{\kappa}}$ it was shown in [§4, Behrndt et al. (2005)] that there exists a Krein space $\mathcal{H}$ and a minimal selfadjoint extension $\widetilde{A}\in\widetilde{\mathcal{C}}(\mathcal{K}\times\mathcal{H})$ such that the formula (7) holds and $\widetilde{A}$ has

$$0\le\widetilde{\kappa}\in\bigl\{\kappa+\widehat{\kappa}-1,\dots,\kappa+\kappa'+2\bigr\}$$

negative squares. □

References

Behrndt, J. & C. Trunk (2005). Sturm-Liouville operators with indefinite weight functions and eigenvalue depending boundary conditions, *submitted*.

Ćurgus, B. & H. Langer (1989). A Krein space approach to symmetric ordinary differential operators with an indefinite weight function. *J. Differential Equations* 79, 31-61.

Derkach, V. (1998) On Krein space symmetric linear relations with gaps, *Methods of Funct. Anal. Topology* 4, 16-40.

Derkach, V. (1999) On generalized resolvents of hermitian relations in Krein spaces. *J. Math Sciences* 97, 4420-4460.

Derkach, V., S. Hassi, M. Malamud & H. de Snoo (2000). Generalized resolvents of symmetric operators and admissibility. *Methods of Funct. Anal. Topology* 6, 24-53.

Derkach, V., S. Hassi, M. Malamud & H. de Snoo (2005). Boundary relations and their Weyl families. *to appear in Trans. Amer. Math. Soc.*.

Dijksma, A. & H. de Snoo (1987). Symmetric and selfadjoint relations in Krein spaces I. *Operator Theory: Advances and Applications*, 24, Birkhäuser Verlag Basel, 145-166.

Hassi, S., M. Kaltenbäck & H. de Snoo (1997). Selfadjoint extensions of the orthogonal sum of symmetric relations, I. *Operator theory, operator algebras and related topics (Timişoara, 1996)*, Theta Found. Bucharest, 163-178.

Hassi, S., M. Kaltenbäck & H. de Snoo (1998). Selfadjoint extensions of the orthogonal sum of symmetric relations, II. *Operator Theory: Advances and Applications*, 106, Birkhäuser Verlag Basel, 187-200.

Jonas, P. (2000). Operator representations of definitizable functions. *Ann. Acad. Sci. Fenn. Math.* 25, 41-72.

Krein, M. & H. Langer (1977). Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen. *Math. Nachr.* 77, 187-236.

Strauss, A. (1962). On selfadjoint operators in the orthogonal sum of Hilbert spaces. *Dokl. Akad. Nauk SSSR* 144, 512-515.