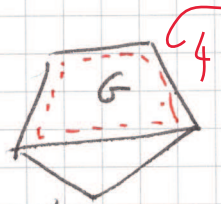
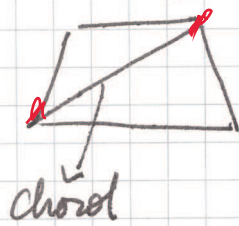


Chordal graphs

(1)

Def (Hajnal, Surányi 1958)

A graph G is called chordal if it contains no induced cycle C_k with $k \geq 4$ vertices. Equivalently, every cycle of length $k \geq 4$ in a chordal graph contains a chord, i.e. an edge connecting two nonconsecutive vertices of the cycle.

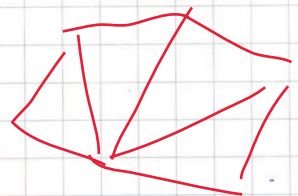


G is not chordal



G' is chordal

Further example: K_n is chordal, $\forall n \in \mathbb{N}$



What about bipartite graphs?

A bipartite graph G is chordal $\Leftrightarrow G$ is a forest

Chordal graphs are sometimes called triangulated graphs or triangle graphs. Be careful: "triangulated graphs" is different from "triangulation of the plane".

Next we will characterize chordal graphs:

K_n is not a triangulation of the plane.

Lemma (Dirac 1961)

A graph G is chordal iff every inclusion-minimal separator induces a clique in G .

Proof First of all observe that a graph is chordal if and only if all its connected components are chordal. So it is enough to prove the lemma for every connected component of a chordal graph.

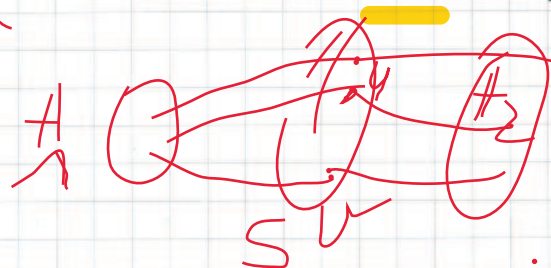
Assume w.l.o.g. that G is connected.

$\Rightarrow$) Let G be chordal (and connected)

(2)

Let S be a separator in G which is minimal with respect to set inclusion, i.e. $\forall s \in S, S \setminus \{s\}$ is not a separator in G .

Then $G \setminus S$ has at least 2 connected components H_1, H_2 , and each vertex in S has neighbors in both H_1 and H_2 (otherwise $S \setminus \{s\}$ would also be a separator).



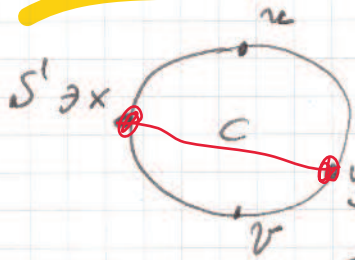
We can assume w.l.o.g. that $|S| > 1$, otherwise there would be nothing to show. Let $u, v \in S, u \neq v$.

Let P_i be a u - v -path of minimum length all of whose inner vertices belong to $H_i, i \in \{1, 2\}$. Then the cycle $C = P_1 \cup P_2$ has length ≥ 4 and it contains a chord (since G is chordal). As there is no edge between H_1 and H_2 the chord has to be $\{u, v\} \in E(G)$. Thus any two vertices in S are connected by an edge, thus S induces a clique in G .

$\Leftarrow$) Assume now that every separator S in G such that S is inclusion-minimal induces a clique in G . We show that G is chordal.

Let C be a cycle of length at least 4 in G . Let u, v be two vertices in $V(C)$ which are not consecutive in C , i.e. $\{u, v\} \notin E(C)$. If $\{u, v\} \in E(G)$, then C has a chord. Otherwise let S be an inclusion-minimal set separating u and v in G . S must contain

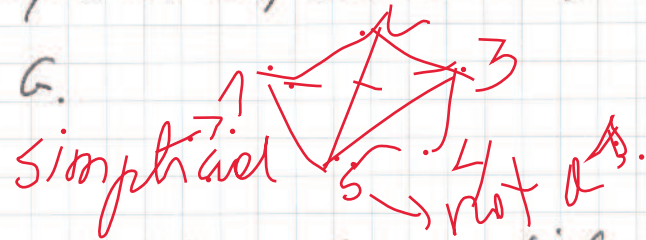
at least one vertex from each $u-v$ -path in C . (3)



Since S is a clique by assumption that then two vertices, say x and y are connected by an edge. Thus $\{x, y\}$ is a chord in C . $\square$

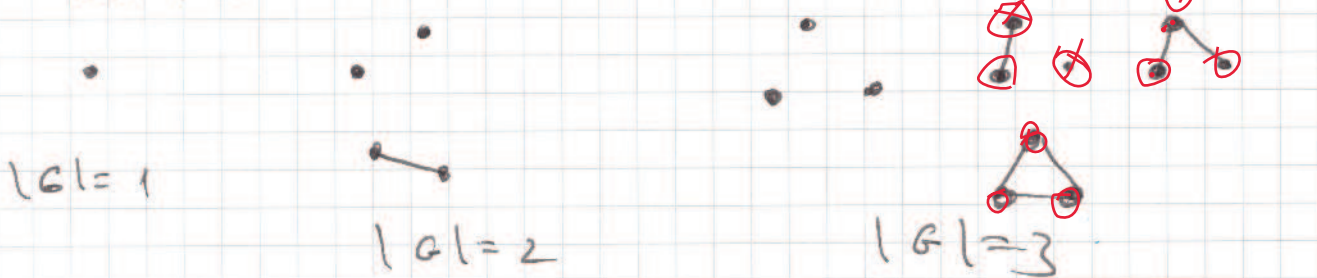
Def A vertex v in a graph G is called a simplicial vertex iff $N(v)$ induces a clique in G , where $N(v)$ is the set of neighbors of v in G .

Lemma (Dirac 1961)



Every chordal graph G contains a simplicial vertex and if G is not a complete graph then G contains 2 non-adjacent simplicial vertices.

Proof By induction on $|G|$. The statement is trivial for $|G| \in \{1, 2, 3\}$



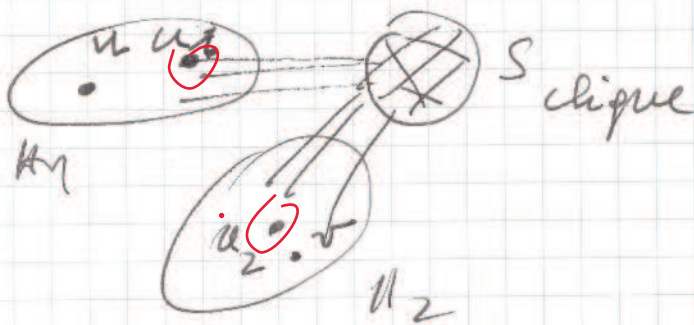
The statement is also trivial iff $G \cong K_n$ for some $n \in \mathbb{N}$.

Assume thus w.l.o.g. that $u, v \in V(G)$ and $\{u, v\} \notin E(G)$

Let S be an inclusion-minimal $u-v$ -separator in G , and let $G-S$ contain the connected components $H_i, i \in \mathbb{I}$, with $|\mathbb{I}| \geq 2$. Due to the previous lemma S induces a clique in G .

$\forall i \in \mathbb{I}, G \setminus H_i \cup S$ is chordal and contains a simplicial vertex by the induction's assumption. Indeed, H_i is in S . Why in H_i ? H_i is not in S .

if $G[H \cup S]$ is not a complete graph, then it contains ⁽⁴⁾ two non-adjacent simplicial vertices and since S induces a clique in G , at least one of those simpl. vertices is in H . If, otherwise, $G[H \cup S]$ is a complete graph, then every vertex is a simplicial vertex, so there is a simplicial vertex in H .



Since $\text{N}(H_i) \subseteq H \cup S$, the simplicial vertex u_i is also a simplicial vertex in G . Different u_i are clearly nonadjacent. □

Theorem (Berge 1961, Hajnal, Surányi 1958)

Chordal graphs and their complements are perfect.

Proof Observe that the property of being chordal is a hereditary property, i.e. every induced subgraph of a chordal graph is chordal. Thus for the perfectness of a chordal graph it would be enough to show that $\chi(G) = \omega(G)$.

We show this by induction on $|G|$.

Ind. basis: $|G| = 1$. $\chi(G) = \omega(G) = 1$ is trivial in this case.

Consider G a ~~chordal~~ chordal graph with $|G| = n$. If $G = K_n$ we are done. Otherwise $\exists u, v \in V(G)$ with $\{u, v\} \notin E(G)$. Then by the first lemma of Dirac the inclusion-minimal $u-v$ -separator induces