

Theorem of König

①

Every bipartite graph satisfies $\chi'(G) = \Delta(G)$.

A $|E(G)|$ -edge coloring
 $=: m$ can be constructed in
 $O(m \cdot n)$ time (where $n := |V(G)|$).

An algorithmic proof

Assume w.l.o.g. that $G = (V, V, E)$ is connected. (Otherwise color every connected component).

Consider a partial feasible edge-coloring, i.e. a coloring of some edges such that colored edges which share a vertex receive different colors.

Start with all edges being not-colored.

Let $e = \{x, y\} \in E(G)$ be a yet non-colored edge with $x \in U, y \in V$. The theorem would be proved if we can show that we can color every edge in $O(n)$ time while using not more than $\Delta(G)$ colors, namely $\{1, 2, \dots, \Delta(G)\}$.

Consider the colored neighboring edges at x , i.e.

the colored edges incident with x . Since $\deg(x) \leq \Delta(G)$ and $\{x, y\} \in E(G)$, these edges hold at most $\Delta(G) - 1$

colors. We say "there is at least one color missing in x ".

By analogy there is at least one color missing in y .

If there is a common color f_e (free at edge e) missing in both x and y then we can color e by f_e .

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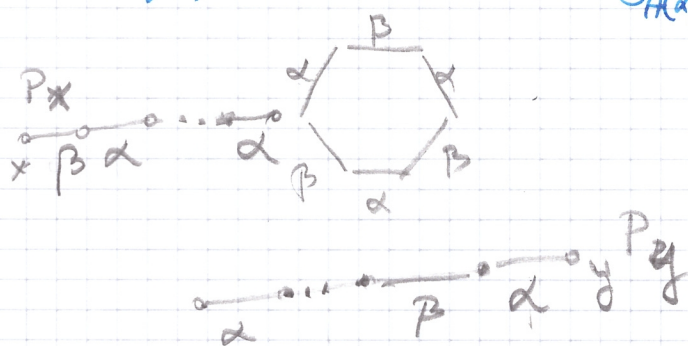
Otherwise there is no common color missing both in x and y .

Let α be a color missing in x and let β be a color missing in y . Then there exist an edge of color α incident with y , and there exist an edge of color β incident with x . These edges are called α -edge and β -edge, respect.

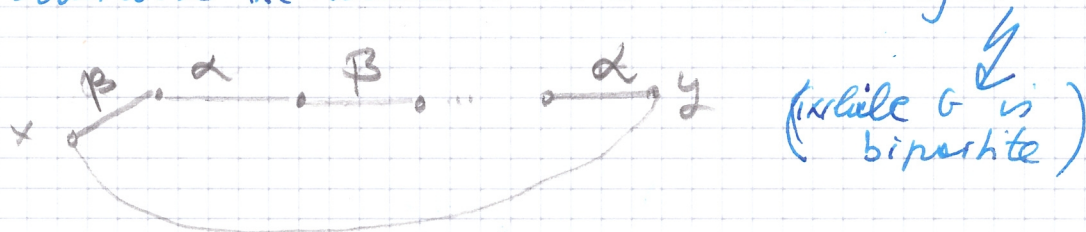
Let $H(\alpha, \beta)$ be the subgraph of G consisting of edges colored by α or β . As every vertex in $H(\alpha, \beta)$ has degree ≤ 2 , $H(\alpha, \beta)$ is a disjoint union of cycles and paths with edges colored alternately by α and β .

x and y can only be end-vertices of paths in this graph

Since $\deg_{H(\alpha, \beta)}(x) = \deg_{H(\alpha, \beta)}(y) = 1$, but not of the same path. Why not?

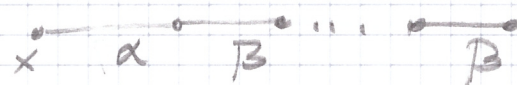


Because otherwise we would have an odd cycle in G



Now we swap the colors in one of the paths, e.g. P_x .

So P_x looks now as



We still have a feasible partial coloring after this swap and now β is also missing at x . So we can color $\{x, y\}$ with β .

We continue like this and repeat this procedure until all edges are colored by using a color from $\{1, \dots, \Delta(G)\}$. (3)

Finally we show that this process takes $O(n)$ elementary operations to color an edge.

The colors f_e, α, β can be determined in $O(\deg(x) + \deg(y))$

$= O(n)$ time. To compute path P_x in $H(\alpha, \beta)$ in

$O(n)$ time we use a data-structure $\text{neighbour}(v, f)$

defined by
$$\text{neighbour}(v, f) = \begin{cases} w & \text{if } \{v, w\} \in E(G) \text{ and} \\ & \text{color of } \{v, w\} \text{ is } f \\ 0 & \text{color } f \text{ is free in } v \end{cases}$$

$\forall v \in V(G), \forall f \in \{1, 2, \dots, \Delta(G)\}$.

We initialize $\text{neighbour}(v, f) = 0 \quad \forall v \in V(G), \forall f \in \overline{1, \Delta(G)}$

Once we have $\text{neighbour}(v, f)$ we can check in $O(1)$ time whether there is a common free color at x and y and determine α, β as well as P_x otherwise.

Furthermore after coloring an edge $e = \{x, y\}$ $\text{neighbour}(v, f)$ can be updated in linear time ($\overset{\text{this takes}}{\sqrt{|P_x|+1}}$ value assignments).



Theorem of Vizing (Vizing 1964)

(4)

Every graph satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

An algorithmic proof.

We show the upper bound by induction on nr. of edges m .

If G is the empty graph ($m=0$) we have $\chi'(G)=0 \leq 0+1$
 $\Delta(G)$.

Let $\Delta := \Delta(G)$ and $e = \{x, y\} \in E(G)$ be an arbitrary edge in G .

Assume $\exists$ feasible $(\Delta(G)+1)$ -edge-coloring of $G-e$, i.e. it exists

$$c: E(G) \setminus \{e\} \rightarrow \{1, 2, \dots, \Delta(G)+1\}.$$

We describe a procedure to color e by a color in $\{1, 2, \dots, \Delta(G)+1\}$.

In a vertex v there are $\Delta+1 - \deg(v) \geq 1$ colors missing at v .

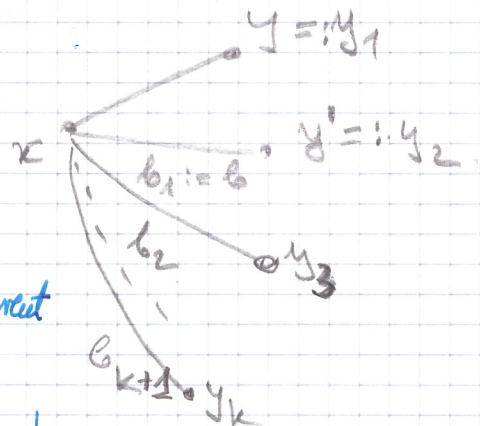
Denote by F_v the set of colors missing at v .

Let $b \in F_y$. If $b \in F_x$ holds, then we color e by b .

Otherwise $\exists$ an edge $\{x, y'\} \in E(G)$ with $c(\{x, y'\}) = b$ and the problem can be looked at vertex y' , i.e. we check whether $\{x, y'\}$ can be colored by a color $\neq b$ and if yes we color $\{x, y\}$ by b .

Construct a sequence $\{y_j \mid 1 \leq j \leq k\} \subseteq \Gamma(x)$ of neighbors of x such that

- $y_1 = y$, $b_1 = b$, $b_j \in F_{y_j}$
- $\forall 1 \leq j \leq k$ $c(\{x, y_{j+1}\}) = b_j$
- $y_1, \dots, y_k$ are pairwise different



We will see how to construct this sequence, but first let us consider when/how terminates this construction.

(5)
If in some step k we see that edge $\{x, y_k\}$ can be colored by color $f \in F_x \cap F_{y_k}$, $f \in \{b_1, \dots, b_{k-1}\}$ then we change the coloring by applying colorshift below:

colorshift (k, f)

begin

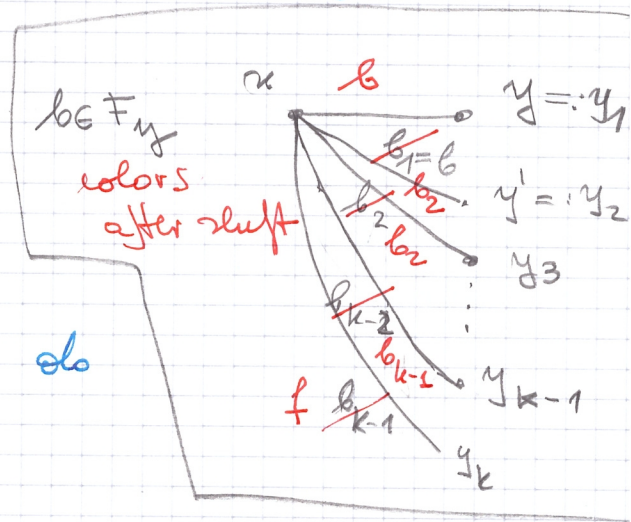
$c(\{x, y_k\}) = f$;

for $j = k-1$ down to 1 do

$c(\{x, y_j\}) = b_j$

end for

end



then after colorshift the edge $\{x, y\}$ gets a colored among $\{1, 2, \dots, \Delta(G)+1\}$ and the procedure terminates.

The following procedure constructs the sequences $(y_j)_{1 \leq j \leq k}$ and

$(b_j)_{1 \leq j \leq k}$:

$k=0$ $y_1=y$

repeat

$k:=k+1$

choose $b_k \in F_{y_k}$ $\# b_k \notin \{b_1, \dots, b_{k-1}\}$ holds $\#$

if $b_k \in F_x$ then colorshift (k, b_k); STOP;

else $\# b_k$ is not missing in x $\#$

let y_{k+1} be such that $y_{k+1} \in \Gamma(x)$ and $c(\{x, y_{k+1}\}) = b_k$

end if

until $y_{k+1} \in \{y_1, \dots, y_k\}$

Notice that the repeat loop can be repeated at most $\deg(x)$ times because $y_1, y_2, \dots, y_k$ are pairwise different

neighbours of x . If repeat is terminated by STOP we are done because we can apply colorshift (k, b_k) with eventually colors $c = \{x, y\}$ by b .

Assume now that repeat will not terminate with "Stop"

but because we reach $y_{k+1} \in \{y_1, \dots, y_{k-1}\}$.

let $y_{k+1} = y_i$ for some $i \in \{1, \dots, k-1\}$. Then $b_k = b_{i-1}$

$(\{y_{k+1}\})$ has color b_k and $\{x, y_i\} = \{x, y_{k+1}\}$ has color b_{i-1}

let $a \in F_x$ be a free color in x . Denote by $H(a, b_k)$ the subgraph of G containing ^{all vertices of G and} only edges colored by a or b_k .

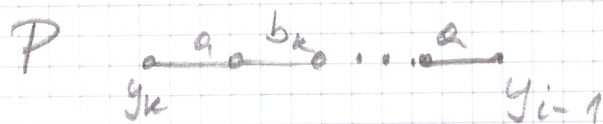
Every vertex in $H(a, b_k)$ has degree $\leq 2 \Rightarrow$

$H(a, b_k)$ is a disjoint union of color-alternating paths and cycles. Clearly switching the colors in one of the components of $H(a, b_k)$ yields a proper edge coloring.

Since color b_k is missing in y_k , $\deg_{H(a, b_k)}(y_k) \leq 1$ holds. So there is a (possibly degenerated) path P starting at y_k (a degenerated path would consist just of vertex y_k). Let z be the other endvertex of P .

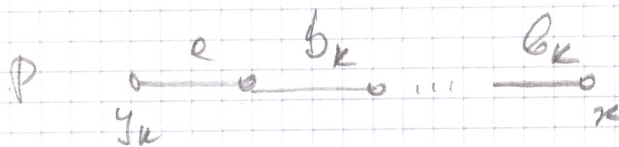
We distinguish a couple of cases:

1) $z = y_{i-1}$ (recall that $b_k = b_{i-1}$ is missing at y_{i-1})



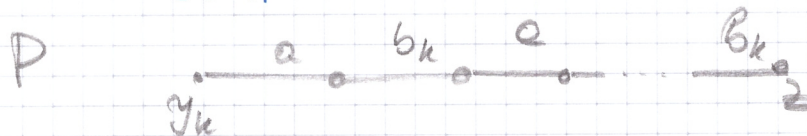
switch a and b_k along P to obtain an edge-coloring in which a is missing in y_k and y_{i-1} . Complete the coloring by colorshift $(i-1, a)$

2) $z = x$ (recall that a is missing at x , thus $\deg_{H(a, b_k)}(x) = 1$)



After switching the colors along P complete the coloring by colorshift $(i-1, b_{i-1})$

3) $z \notin \{y_{i-1}, x\}$



After switching the colors along P (as long as $E(P) \neq \emptyset$) a will be missing in y_k . Complete the coloring by $\text{colorshift}(k, a)$

4) If $E(P) = \emptyset$ we apply $\text{colorshift}(i-1, a)$

Corollary There exists an $O(m \cdot n)$ algorithm to color the edges of a graph G with at most $\Delta(G) + 1 \leq \chi'(G) + 1$ colors, where $m := |E(G)|$ and $n := |V(G)|$

Proof of the corollary:

It is enough to show that we can color every edge in $O(n)$ time.

This can be done by using the data structure $\text{neighbour}(v, f)$ we introduced in the proof of König's theorem.

$\forall i \in \overline{1, k}$ we need to (a) find a free color b_i at y_i and this can be done in $O(\deg(v_i))$, so in $O(n)$ in total,

and (b) check whether $b_k \in F_x$, which can be done in constant time $\checkmark$ so also in $O(n)$ time $\checkmark$ because repeat can be applied at most $\deg(x)$ times for the fixed edge $\{x, y\}$.

Then colorshift can also be applied in $O(\deg(x))$ time

because it changes the color of $O(\deg(x))$ edges at most and each change results in $O(1)$ update of the neighbor-array.

Recall that $\text{neighbour}(v, f)$ allows us to determine the path P

$O(n)$ time similarly as we argued in the proof

of König's theorem. Of course we need to save the whole

sequence $(y_1, y_2, \dots, y_k)$ for each edge order to back-propagate the color in colorshift . But also this can be done in $O(k)$