

(Chapter 5) Edge coloring

Observation: Trivial/simple lower bounds for $\chi'(G)$

$$\chi'(G) \geq \Delta(G) \qquad \chi'(G) \geq \left\lceil \frac{|E(G)|}{\nu(G)} \right\rceil,$$

where $\nu(G)$ is the matching number of G .

Proposition 8.

(König 1916)

Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$. A $\Delta(G)$ -edge coloring of G can be constructed in $O(|E(G)||V(G)|)$ time.

Theorem 9.

(Vizing 1964)

Every graph satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

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Corollary 10.

There exists an $(O(|E(G)||V(G)|))$ algorithm to color the edges of an input graph G with at most $\Delta(G) + 1$ colors.

Class I graphs: $\chi'(G) = \Delta(G)$; Class II graphs: $\chi'(G) = \Delta(G) + 1$

Examples: bipartite graphs, even cycles, K_{2l} for $l \in \mathbb{N}$, are Class I graphs; odd cycles, K_{2l+1} for $l \in \mathbb{N}$, are Class II graphs.

The **Chromatic Index problem** (ChIP)

Input: a graph $G = (V, E)$

Question: Is G a Class I graph or a Class II graph?

Theorem 11.

(Holyer 1981)

The Chromatic Index Problem is NP-complete even if the input graph is a cubic bridge-free graph.

I. Holyer, The NP-completeness of edge coloring, *SIAM J. on Computing* **10**, 1981, 718–720.