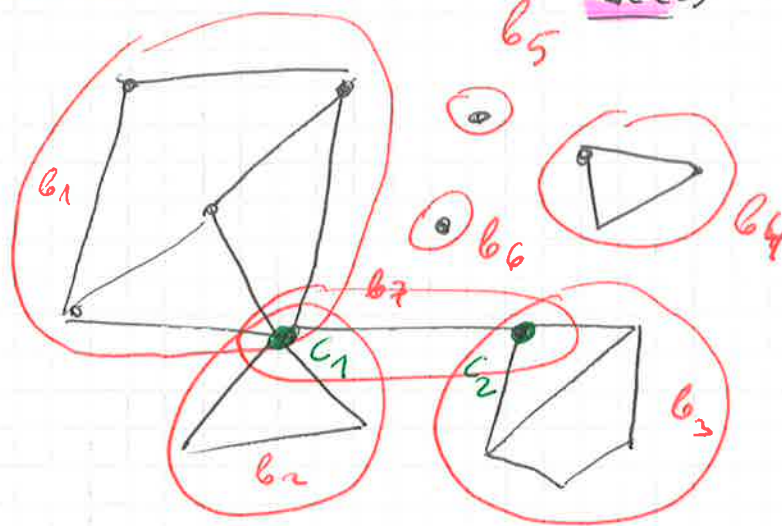


Example: The block-cut-vertex graph of a given graph G



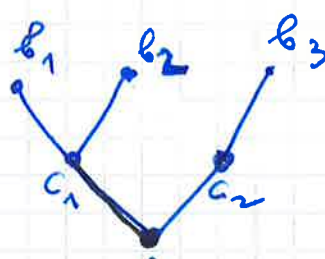
①

$$B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$$

$$C = \{c_1, c_2\}$$

$bc(G)$

$$V = \{b_1, \dots, b_7, c_1, c_2\}$$



$$E = \left\{ \{b_1, c_1\}, \{b_2, c_1\}, \{b_2, c_2\}, \{b_3, c_2\}, \{b_3, c_1\}, \{b_4, c_1\}, \{b_4, c_2\}, \{b_5, c_2\}, \{b_5, c_1\}, \{b_6, c_1\}, \{b_6, c_2\} \right\}$$

Use the example above to illustrate Observations 4 and 5

Observe: (A) Each block is either

- a max 2-connected subgraph (b_1, b_2, b_3, b_4)
- a bridge (b_7)
- an isolated vertex (b_5, b_6)

(B). If two blocks intersect, then they have a single vertex in common and this is a cut-vertex

$$b_1 \cap b_2 = \{c_1\} \quad b_2 \cap b_3 = \{c_2\} \quad b_7 \cap b_3 = \{c_2\}$$

- Every edges of G lies in a unique block
- G is the union of its block.

Proof of Theorem 6

(2)

Consider a connected graph G and show that $bc(G)$ is a tree, i.e. $bc(G)$ is connected and cycle-free.

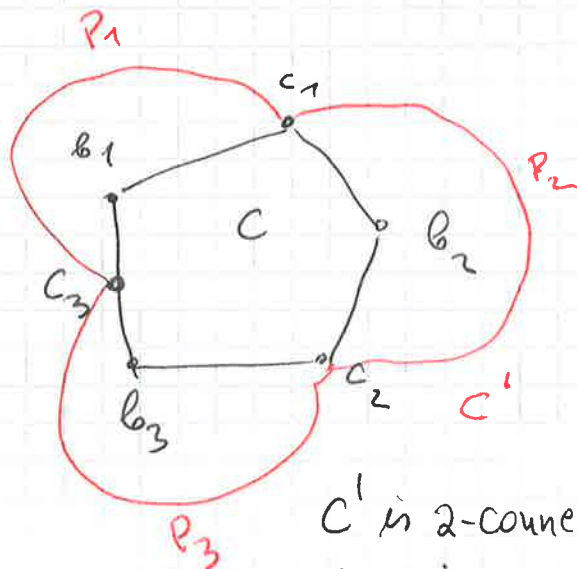
Since G is connected also $bc(G)$ is connected ($\forall s, t \in G, \exists s-t$ -path in G , so $\forall x, y \in B \cup E$ choose $s \in x$ and $t \in y$ and consider that every edge of G is contained in some ~~of the~~ block of G)

Now show that $bc(G)$ is cycle-free.

Assume we have a cycle in $bc(G)$.

From C we can construct a cycle C' in G by including

in C' a non-empty path P_b for each block b in C namely the path connecting the 2 cut-vertices which belong to b



Now C' is a 2-connected subgraph of

$$b_1 \cup b_2 \cup \dots \cup b_m$$

($m=3$ in the picture)

C' is 2-connected $\Rightarrow C'$ has no cut-vertex

$\Rightarrow b_1 \cup b_2 \cup \dots \cup b_m$ is 2-connected without a cut vertex. This contradicts the definition of blocks in terms of maximality.

Proof of Theorem 7 (Ear decomposition)

(3)

← Assume $G = C \cup P_1 \cup \dots \cup P_k$ for some $k \in \mathbb{N}$
show that G is 2-Connected.

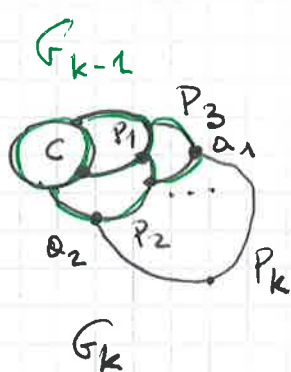
Induction over k

For $k=0$: $G = C$ and every cycle is 2-conn.

Let $k \geq 1$ and let $G_{k-1} := C \cup P_1 \cup \dots \cup P_{k-1}$

We show that $G_k = G_{k-1} \cup P_k$ is 2-Connected.

This means $\forall$ 2 vertices s, t in G_k there
exist 2 independent s - t -paths in G_k



Let a_1, a_2 be the end-vertices
of P_k in G_{k-1}

W.l.o.g. let $s \in G_{k-1}$ and
 $t \in P_k - a_1 - a_2$

From the fan theorem,

there exist 2 independent paths from
 s to a_1 and a_2 in G_{k-1} , respectively.

Combine those with the 2 paths from t to a_1
and from t to a_2 respectively

(4)

→ Assume G is 2-connected ~~and~~
 We show that G has an ear decomposition,
 i.e. a representation of the form

$$G = C \cup P_1 \cup \dots \cup P_k$$

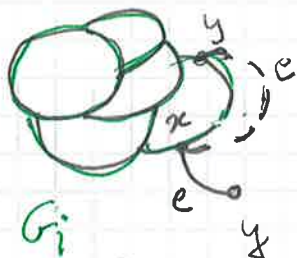
Since G is 2-connected it contains a cycle C
 Set $G_0 := C$ Assume we have constructed

$$G_i = C \cup P_1 \cup \dots \cup P_i \text{ for some } i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

If $G_i = G$, we are done.

Otherwise $G_i \neq G$. Then $\exists e \in E(G) \setminus E(G_i)$
 (because G is connected) s.t. e is incident
 to some node $x \in V(G_i)$. Let $e = \{x, y\}$

If $y \in V(G_i)$, let $P_i := \{x, y\}$



and set $G_{i+1} := G_i \cup P_{i+1}$

Otherwise consider $G - x$.

Since G is 2-connected, $G - x$ is connected

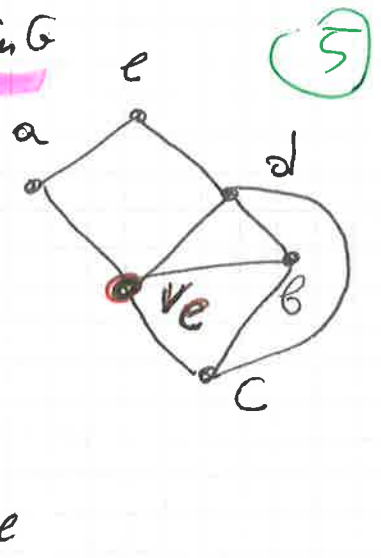
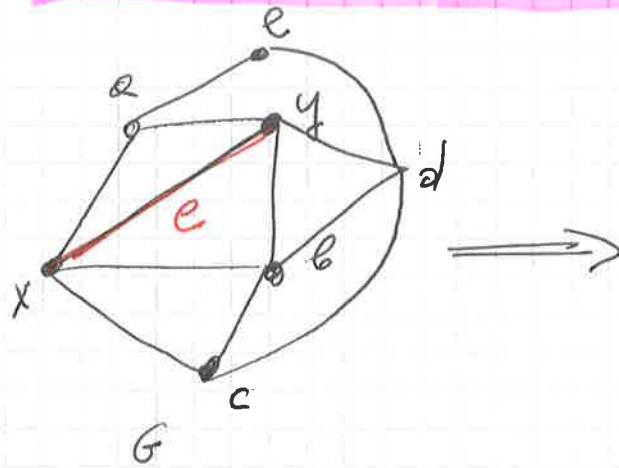
$\Rightarrow \exists$ an $y \in V(G_i) \setminus \{x\}$ - path P in $G - x$

Set $P_{i+1} = P \cup e$ and $G_{i+1} := G_i \cup P_{i+1}$

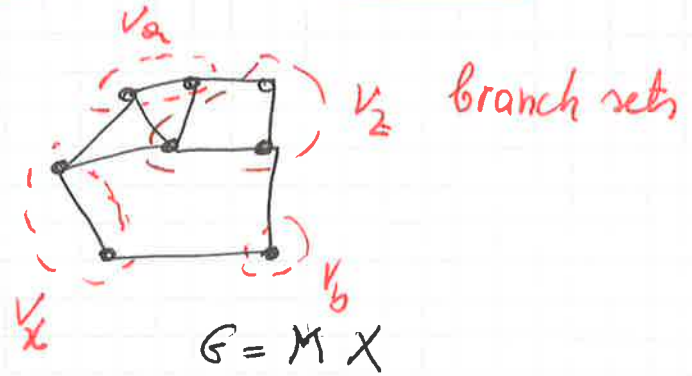
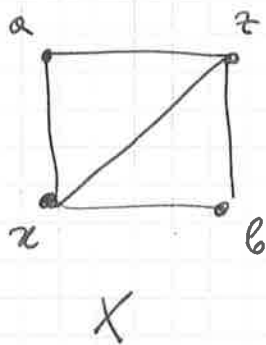
Repeat this procedure until $G = G_i$ for some
 $i \in \mathbb{N}$. This process will terminate because
 G has a finite number of edges



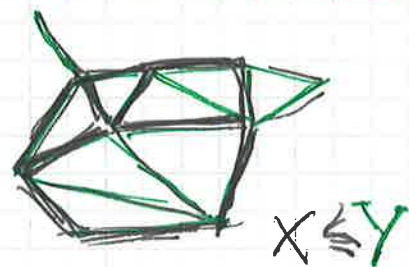
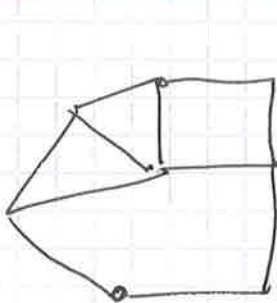
Contraction of an edge e in G



Example of $MX = G$ for a given X

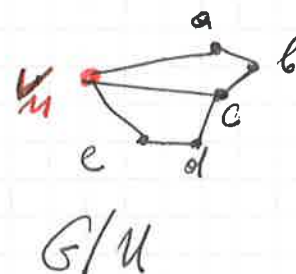
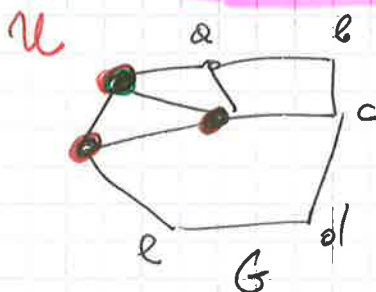


Example of a minor X of Y , $X \preceq Y$



$G = MX$ (where X as above)

Contraction of a subset of vertices $U \subset V(G)$



Proof of Lemma 1

6

Consider a 3-connected graph G with $|G| > 4$.
We want to show the existence of an edge
 $e \in E(G)$, s.t. that G/e is again 3-connected

Assume such an edge does not exist.

Thus $\forall \{x, y\} \in E(G)$, $G/\{x, y\}$ contain a separating
set S with $|S| \neq 2$.

Since G is 3-connected, S does not separate G

$$\Rightarrow \forall \{x, y\} \in E(G) \quad \text{and} \quad |S| = 2^{(*)}$$

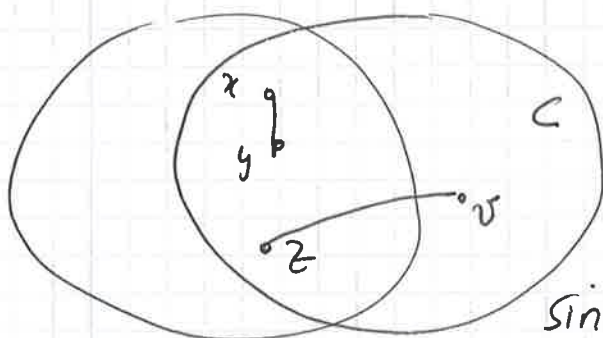
$\rightarrow$ the vertex into which e is contracted

(*) $|S| = 2$ holds because $|S| = 1 \Rightarrow S = \{v_{x, y}\}$

and then $S' = \{x, y\}$ would separate G if G is
 $\downarrow$ 3-conn.

Thus $\exists z \in V(G) \setminus \{x, y\}$ s.t. that $S = \{z, v_{x, y}\}$

separates $G/\{x, y\}$. let $T = \{x, y, z\}$.



Then any 2 vertices separated
by $\{v_{x, y}, z\}$ in $G/\{x, y\}$ are
separated by T in G

Since no proper subset of T
separates G , every vertex in T has a neighbor

in any component C of $G - T$.
Now choose edge $\{x, y\} \in E(G)$, the vertex z and the component
 C of $G - \{x, y\}$ such that C has smallest cardinality.
Let v be a neighbor of z in C , i.e.

$\{v, z\} \in E(G)$, and $v \in V(C)$.

(7)

$G/\{z, v\}$ is not 3-connected (by our assumption),
so $\exists W$ such that $\{z, W, v\}$ separates G and
every vertex in $\{z, v, W\}$ has a neighbor
in every component of $G - \{z, v, W\}$

As $\{x, y\} \in E(G)$, $G - \{z, v, W\}$ has a component
 D such that $D \cap \{x, y\} = \emptyset$ (because x and y
are both contained in the same component of $G - \{z, v, W\}$)

Then every neighbor of v in D lies in C (since
 $v \in C$). So $D \cap C \neq \emptyset$. Now this implies

$D \subseteq C$ because $\forall t \in D$ consider $t' \in N(v) \cap D$
and the path P connecting t and t' in D . Since
 P contains none of the vertices z, y, x ~~Path~~
and $v \in C$ then vP lies also in C , so $t \in C$.

On the other hand $v \in C$ but $v \notin D$.

So $D \subsetneq C$. But this contradicts the choice
of $\{x, y\}$, z and C such $|C|$ is as small
as possible $\downarrow$.

The contradiction shows that $\exists e \in E(G)$
such that G/e is 3-connected.



Proof of Theorem 8 (Tutte 1961)

8

$\Rightarrow$ Assume that G is 3-connected.

Apply lemma 1 iteratively as long as $|G| > 4$.
In every iteration i find an edge e_i such
that $G_{i-1} := G_i \setminus e_i$ is 3-connected.
The process ends when a 3-connected graph G_0

with 4 vertices is reached. But then $k(G_0) \geq 3$

$\Rightarrow \delta(G_0) \geq \lambda(G_0) \geq k(G_0) \geq 3$ (cf. Theorem 2)

So the min. degree is at most 3 and $|G_0| = 4$

$\Rightarrow G_0 = K_4$

$\Leftarrow$ Assume $G_0, G_1, \dots, G_n$ is a sequence of graphs
as stated in Theorem 8: $G_0 = K_4, \dots, G_n = G$,

and G_{i+1} has an edge $\{x, y\}$ with $d(x) \geq 3, d(y) \geq 3$,

2. that $G_i = G_{i+1} \setminus \{x, y\} \quad \forall 0 \leq i \leq n-1$

We claim that: If $G_i = G_{i+1} \setminus \{x, y\}$ is 3-connected, so is

G_{i+1} . This would complete the proof because K_4 3-con

$\Rightarrow G_1$ 3-con $\Rightarrow \dots \Rightarrow G_n = G$ 3-connected.

So let us prove the claim:

Assume the claim is not true, i.e. $\exists$ separator S
with $|S| \leq 2$ in G_{i+1} . let C_1, C_2 be 2 components
of $G_{i+1} - S$. Since $\{x, y\} \in E(G_{i+1})$

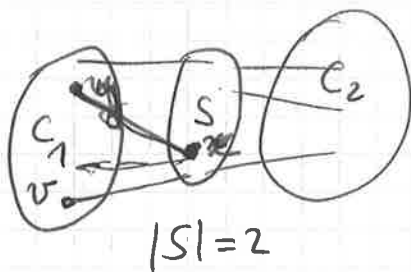
x and y cannot belong to different comp of
 $G_{i+1} - S$ because $\{x, y\} \in E(G_{i+1})$

But x, y cannot belong to the same component, either

because otherwise S would be a separator of $\textcircled{9}$
 cardinality 1 for G_i . ($\downarrow$)

On the other hand, $S = \{x, y\}$ would imply a
 separator S' of cardinality 1 in G_i (namely
 $S' = \{v_{xy}\}$). ($\downarrow$)

So the only possibility is $x \in S, y \notin S$
 (or $y \in S, x \notin S$). Say e.g. $y \in C_1$.



Assume $\exists v \in C_1 \setminus \{y\}$
 But then v is separated
 from C_2 by a set of
 cardinality 2 in G_i . ($\downarrow$)

So $C_1 = \{y\}$ and all its neighbors in
 G_{i+1} lie in S and $|S|=2$.

This contradicts $\deg_{G_{i+1}}(y) \geq 3$. ($\downarrow$)

and completes the proof of Theorem 8. $\square$