

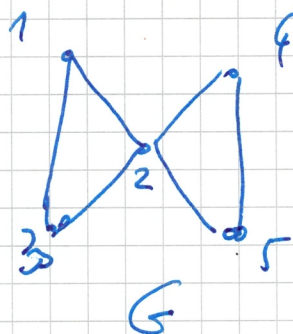
Das Theorem 4 (Chvátal, Erdős 1972) is best possible:

1

In particular $\exists$ graph G with $\lambda(G) \geq \alpha(G)$ and G not Hamiltonian.

Thus in the Theorem of Chvátal and Erdős $K(G)$ cannot be substituted by $\lambda(G)$.

Indeed



(•) $\lambda(G) = 2$ (because $G - e$ is connected, $\forall e \in E(G)$ and $G - \{1, 2\}, \{2, 3\}$ is disconnected).

(••) $\alpha(G) = 2$ because $S = \{4, 3\}$ is a stable set and any subset of 3 vertices contains an edge.

However G is not Hamiltonian.

It contains a cut-vertex, namely vertex 2.

So if $A = \{2\}$ we have that

$G - A$ has more than $|A| = 1$ Conn. Comp.

so a necessary cond. for hamiltonicity is not fulfilled.

Corollary 5

(2)

Assume w.l.o.g. that G is 2-connected.

[If G is not 2 connected we can find an edge pair of vertices u and v s.t. $\{u, v\} \notin E(G)$.

Then $S = \{u, v\}$ is a stable set.

Since G is not 2 connected it has a cut-vertex v_0 (which can be found by DFS); set $T = \{v_0\}$.

Then we have $|T| < |S|$.]

Start with an arbitrary cycle C in G .

If $|V(C)| = n$ we are done.

Otherwise let S be a conn. comp. of $G - V(C)$.

Let T be the neighbors of S in $C = \{v_1 \dots v_k\}$

If $\exists j \in \overline{1, k}$ with $\{v_j, v_j^*\} \subseteq T$, then Extension 1,

If $\exists i, j \in \overline{1, k}$ with $\{v_i^*, v_j^*\} \in E$ then Extension 2,

otherwise set $S_i = T^* \cup \{v_i\}$ for any $v_i \in S^*$.

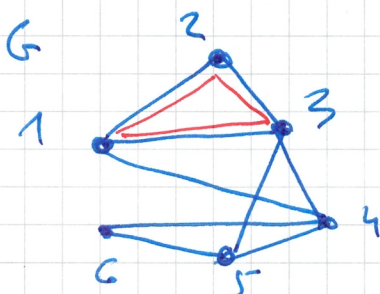
Then T is a separator and S is a stable set with $|T| < |S|$.

Algorithm to find a cycle C with enlargement:

Time complexity: C will be enlarged at most n (actually $n-3$) times.

Every check for a possible enlargement and the enlargement itself can be done in $O(m)$ time.

Example



$$C = (1, 2, 3)$$

$$V(G - C) = \{4, 5, 6\}$$

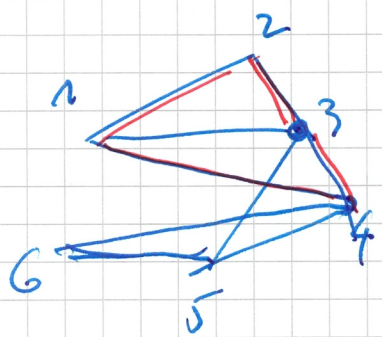
$$T = \{1, 3\}$$

$$T^* = \{2, 1\} \Rightarrow \text{Extension 1}$$

$$C' = (1, 2, 3, 4)$$

reverse C in
clockwise
order

$$(T \cap T^* \neq \emptyset)$$

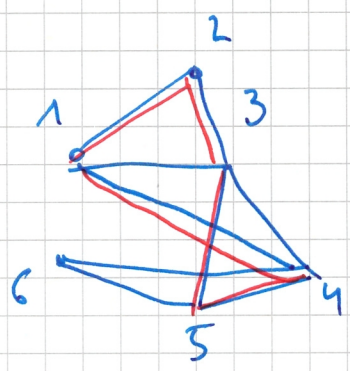


$$V(G - C') = \{6, 5\}$$

$$T = \{3, 4\} \quad T^* = \{4, 1\}$$

$$T \cap T^* \neq \emptyset \quad \text{Extension 1}$$

$$C'' = (1, 2, 3, 5, 4)$$



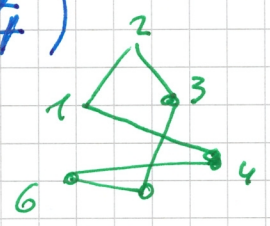
$$V(G - C'') = \{6\}$$

$$T = \{4, 5\} \quad T^* = \{5, 3\}$$

$$T \cap T^* \neq \emptyset \Rightarrow \text{Extension 1}$$

$$C''' = (1, 2, 3, 5, 6, 4)$$

Hamiltonian cycle



Homework: Construct yourself an example where the procedure would produce a separator T and a stable S with $|T| < |S|$.

Proof of Theorem 6

(4)

Let G be a graph with $|G| \geq 3$.

Let $\deg(u) + \deg(v) \geq |G|$, $\forall u, v \in V(G)$, $\{u, v\} \notin E(G)$

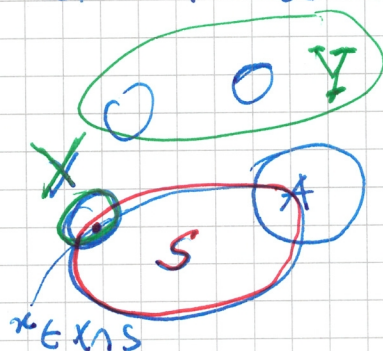
We show: $|S| \leq |A|$ $\forall$ stable set S and

$\forall$ separator A in G .

If $S \subseteq A$ there is nothing to show.

Assume $S \not\subseteq A$.

Let X be a conn. comp. of $G-A$ with $S \cap X \neq \emptyset$



Let $x \in X \cap S$.

Let $Y := V(G) \setminus (A \cup X) \neq \emptyset$

$\downarrow$
because A is a separator

Distinguish 2 cases

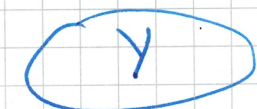
Case I: $S \cap Y = \emptyset$

Let $y \in Y$. Then x and y belong to 2 different comp. of $G-A$

$$\Rightarrow \{x, y\} \notin E \Rightarrow |G| \leq \deg(x) + \deg(y) \\ = |N_G(x)| + |N_G(y)| \leq \underbrace{|(X \cup A) \setminus S|}_{\deg(x)} + \underbrace{|Y| + |A| - 1}_{\deg(y)}$$

$$\stackrel{(*)}{=} |G| - |S| + |A| - 1 \Rightarrow |S| \leq |A| - 1 \quad \checkmark$$

(*) holds because



$\equiv S$

$(X \cup A) \setminus S$ and $Y = V(G) \setminus (A \cup X)$ are disjoint

$$\text{Thus } |(X \cup A) \setminus S| + |Y| = |(X \cup A) \setminus S \cup Y| \\ = |V(G) \setminus S| = |G| - |S|$$

Case II $S \cap Y \neq \emptyset$

(5)

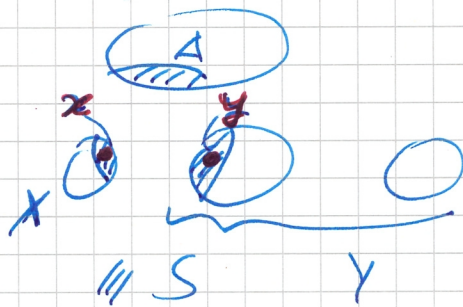
Let $y \in S \cap Y$. Since $\{x, y\} \notin E(G)$ we in Case I

we obtain $|G| \leq \deg(x) + \deg(y) = |N_G(x)| + |N_G(y)|$

$$\leq \underbrace{|X \cap S| + |A \cap S|}_{\substack{= \\ N_G(x)}} + \underbrace{|Y \cap S| + |A \cap S|}_{\substack{\leq N_G(y)}} \quad (\text{adding the first 3 summands})$$

$$\leq |V \cap S| + |A| = |G| - |S| + A$$

$$\Rightarrow |S| \leq |A| \quad \checkmark$$

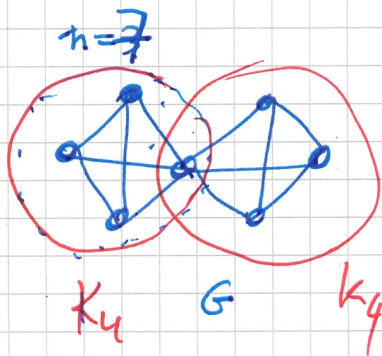


Proof of Theorem 7

G is 2-connected, $n := |G|$, $\forall u, v \in V(G)$ with $\{u, v\} \notin E(G)$ we have $\deg(u) + \deg(v) \geq d$ for some $d \in \mathbb{N}$.

We show that $\exists$ a cycle of length $\min\{4, d\}$

Remark: The assumption of 2-connectivity is crucial



2 copies of $K_{\frac{n+1}{2}}$ with a common vertex

$$\deg(u) + \deg(v) \geq 6 \quad \forall u, v \in V(G)$$

but there is no cycle of length 6 in this graph.

the largest cycle has length 4

Proof according to Euler (1978)

(6)

Step 1: Construction of a long path in G

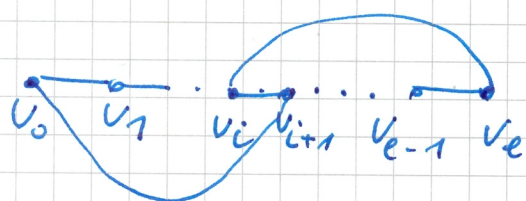
Start with an arbitrary path $P = (v_0, \dots, v_{\ell-1}, v_{\ell})$ $\ell+1 \neq n$
 P can be extended in 2 manners

(1) P can be extended on both ends until
 $N(v_0) \subseteq V(P)$, $N(v_{\ell}) \subseteq V(P)$ hold.

(2) If $\exists i \in \{1, \dots, \ell-2\}$ s.t. that

$$\{v_0, v_{i+1}\} \in E(G)$$

$$\text{and } \{v_i, v_{\ell}\} \in E(G), \text{ then}$$

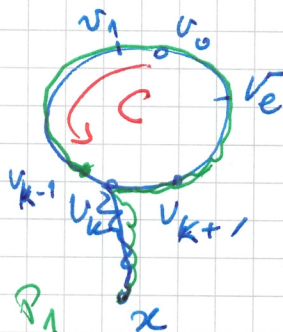


the cycle

$$C = (v_0, v_1, \dots, v_i, v_{\ell}, v_{\ell-1}, \dots, v_{i+1}, v_0)$$

contains all vertices of P (and no other vertices).

Since $V(P) \neq V(G)$ ($|N(P)| \neq n$), $\exists x \in V \setminus V(P)$
which is connected to C via an x - C path
(since G is connected). Let $v_k, 0 \leq k \leq \ell$, be the
first vertex of this path
belonging to C .



Then $P' = x, v_k, v_{k+1}, \dots, v_{\ell}, v_0, v_1, \dots, v_{k-1}$
is a path longer than P

Rename P' to P and repeat this
extending operations until $V(P) = V(G)$
or none of the extensions can be performed.

Step 2 Construction of a long cycle by using

the path P constructed in Step 1

If $P = (v_0, \dots, v_{\ell})$ and $\{v_0, v_{\ell}\} \in E(G)$ then $C = P \cup \{v_0, v_{\ell}\}$ would
be a Hamiltonian cycle, because otherwise we could
open extension of Step 1

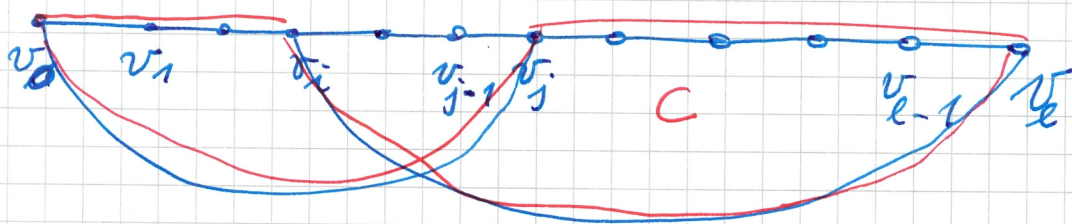
So we assume w.l.o.g. $\{v_0, v_\ell\} \notin E(G)$.

(7)

Since also operation (extension) 1 of Step 1 cannot be performed, the inclusions $N(v_0) \subseteq P$, $N(v_\ell) \subseteq P$ should hold.

Distinguish 2 cases:

Case ① $\exists i, j$ with $1 \leq i < j \leq \ell-1$ and $\{v_0, v_i\} \in E(G)$, $\{v_\ell, v_j\} \in E(G)$



Choose such a pair (v_i, v_j) of vertices with minimum $(j-i)$. Denote $\text{dist} = j-i$.

Since extension (2) of Case 1 cannot be performed $\text{dist} = j-i \geq 2$ must hold.

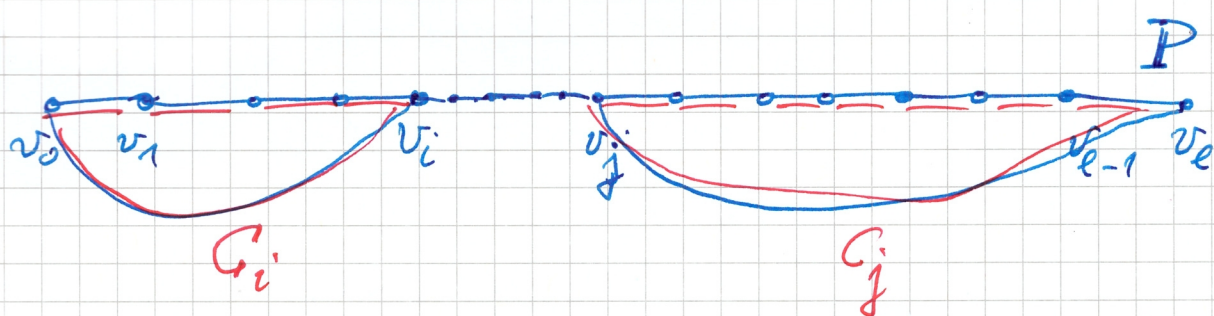
Because of the selection of (v_i, v_j) with the ^{smallest} ~~max~~ possible $j-i$ none of the vertices between v_i and v_j is adjacent to v_0 or v_ℓ .

Consider the cycle $C = (v_0, v_1, \dots, v_i, v_\ell, \dots, v_j, v_0)$
 C contains the vertices v_ℓ , $N(v_\ell)$ and all vertices of P which are predecessors of neighbors of v_0 except for $v_{j-1} \neq v_i$.

There are $1 + \deg(v_\ell) + (\deg(v_0) - 1) \geq d$ vertices and those are pairwise different because otherwise we could perform extension 2 of Step 1. Thus $|V(C)| \geq d$.

Case 2

$$i := \max \{k : v_k \in N(v_0)\} \leq \min \{k : v_k \in N(v_\ell)\} = j$$



Consider the cycles C_i and C_j :
 $C_i := (v_0, v_1, \dots, v_i, v_0)$
 $C_j := (v_j, v_{j+1}, \dots, v_{\ell-1}, v_j)$

Due to 2-connectedness, $\exists$ 2 vertex-disjoint C_i - C_j -paths P_1, P_2 which have only their end-vertices in common with C_i, C_j , respectively.

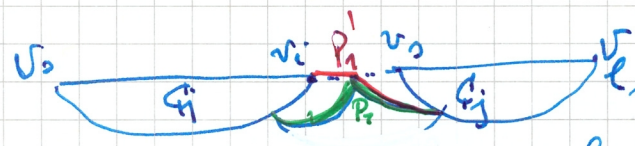
ray P_1 ,

We can assume w.l.o.g. that one of P_1, P_2 starts at v_i .

(If this not the case start at v_i and move toward v_j along P until either v_j or one of the paths P_1, P_2 is reached.

~~If this~~ In the first case replace one of P_1, P_2 by $P[v_i, v_j]$, the later case, if P_1 is met before reaching v_j , then

replace P_1 by P_1' , i.e.

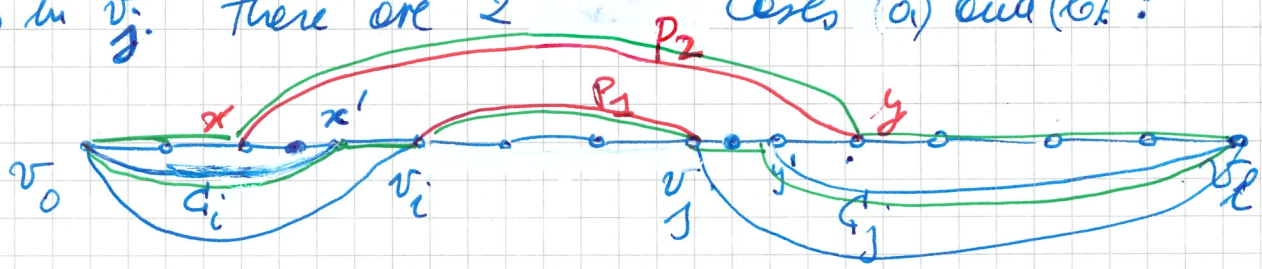


replace the "first part of P_1 " by the subpath of P starting at v_i

and ending at the first common vertex of P_1 and $P[v_i, v_j]$

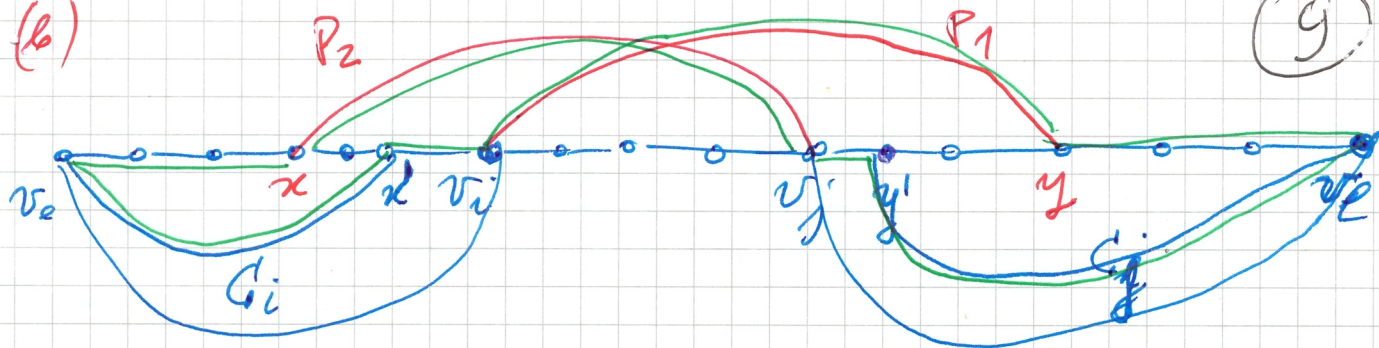
Analogously we can assume that one of the paths, ends in v_j . There are 2 cases (a) and (b):

(a)



(b)

(9)



In (a) let x, y be the endvertices of P_2 which is not v_j .
 In (b) let $x \neq v_i$ be the endvertex of P_2 and $y \neq v_j$ be the endvertex of P_1 which is not v_i .

Let x' be the smallest neighbour of v_0 larger than x and let y' be the largest neighbour of v_k smaller than y .

We construct a cycle ^{green} containing $v_0, N(v_0), v_k, N(v_k)$ and the 2 paths

Since v_0 and v_k have at most one neighbor in common (if $i=j$) we have

$$|C| \geq 1 + \deg(v_0) + 1 + \deg(v_k) - 1 \geq d+1$$

and this completes the proof. □