

Theorem 14 (Galvin 1995, Proof according to Slivnik 1996) ①

Bipartite graphs are $\Delta(G)$ -edge-choosable.

Proof Slivnik proved a more general statement, which is better amenable to induction

Def Let $G=(L \cup R, E)$ be a bipartite graph and $f: E \rightarrow \mathbb{N}$. We say G is f -edge-choosable iff for any given lists $L(e) \neq \emptyset, e \in E(G)$, s.t. $\forall e \in E, |L(e)| \geq f(e)$ holds, there exist a list edge-coloring of G with lists $L(e), e \in E(G)$.

Lemma Let $G=(L, R, E)$ be a bipartite graph and let $C: E(G) \rightarrow \mathbb{N}$ be an edge coloring of G . Define a function

$$T_{G,C}(e) = |\{f \in E(G) : f_L = e_L \wedge C(f) > C(e)\}| + |\{f \in E(G) : f_R = e_R \wedge C(f) < C(e)\}|$$

where $f = \{f_L, f_R\}, e = \{e_L, e_R\}$ with $f_L, e_L \in L, f_R, e_R \in R$.

Then G is $(T_{G,C} + 1)$ -edge-choosable

If $f_L = e_L$ and $C(f) > C(e)$ for 2 edges $f, e \in E(G)$, we say that f dominates e from the left.

If $f_R = e_R$ and $C(f) < C(e)$ we say that f dominates e from the right.

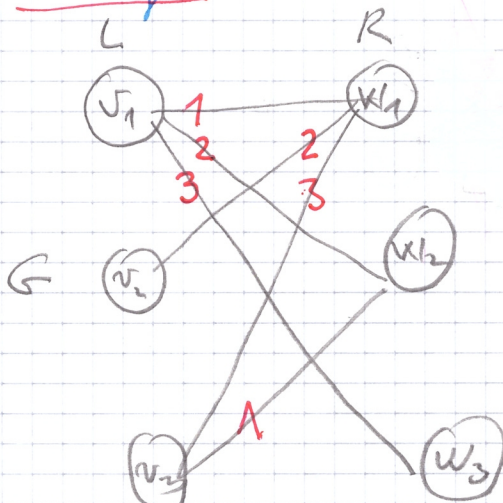
feasible edge coloring

the edge $\{v_1, w_2\}$ dominates $\{v_1, w_1\}$ from the left

Edge $\{v_1, w_1\}$ dominates $\{v_2, w_1\}$ dominates $\{v_3, w_1\}$ from the right

Question

$$\chi'(G) = ? \quad 3$$



Proof of the theorem assuming that the lemma holds: (2)

Let $c: E \rightarrow \{1, 2, \dots, \chi'(G) = \Delta(G)\}$ be an optimal edge coloring of the bipartite graph G .

Since

$$T_{G,c}(e) \leq \underbrace{(\chi'(G) - c(e))}_{\substack{\# \text{ edges down } e \text{ from the} \\ \text{left}}} + \underbrace{(c(e) - 1)}_{\substack{\# \text{ edges down } e \text{ from the} \\ \text{right}}} \leq \chi'(G) - 1$$

G is $(T_{G,c} + 1)$ -edge-choosable (due to the lemma)

Thus by def of f -edge-choosability: if we are given

lists $L(e), e \in E(G)$ s.t. $|L(e)| = \chi'(G), \forall e \in E(G)$,

we have $|L(e)| = \chi'(G) = (\chi'(G) - 1) + 1 \geq T_{G,c}(e) + 1, \forall e \in E(G)$

$\Rightarrow \exists$ a list-edge-coloring with lists $L(e), e \in E(G)$.

Then G is $\chi'(G)$ -edge-choosable $\Rightarrow \chi'_\ell(G) \leq \chi'(G)$.

Since $\chi'_\ell(G) \geq \chi'(G)$ holds always, the equality

$\chi'_\ell(G) = \chi(G)$ follows. 

Proof of the lemma by induction on $|E(G)|$.

(Recall that the edge coloring c in the lemma does not need to be optimal, but simply feasible).

For $E(G) = \emptyset$ there is nothing to be shown.

($T_{G,c}(e) = 0 \forall e \in E(G)$ and the graph has no edges, so nothing needs to be colored.)

Let $E(G) \neq \emptyset$ and let $L(e), e \in E(G)$ be list of colors with $|L(e)| \geq T_{G,c}(e) + 1, \forall e \in E(G)$.

We show, $\exists$ list edge coloring of G , $\ell: E \rightarrow \bigcup_{e \in E(G)} L(e)$ with

$\ell(e) \in L(e), \forall e \in E$, and $\ell(e) \neq \ell(f)$ if $e \cap f \neq \emptyset, \forall e, f \in E(G)$.

Choose a color $\alpha \in \bigcup_{e \in E(G)} L(e)$ and let $L_\alpha := \{e \in E: \alpha \in L(e)\}$

We will construct a matching M in (V, E_a) , such that 3
 $\forall e \in E_a \setminus M$ the following holds:

(•) either $e \in E_a \setminus M$ is dominated from the left by an edge $f \in M$, i.e. $f_L = e_L$ and $c(f) > c(e)$

(••) or $e \in E_a \setminus M$ is dominated from the right by an edge $f \in M$, i.e. $f_R = e_R$ and $c(f) < c(e)$.

Once we have this matching M the proof continues as follows:

color $e \in M$ with color α , remove edges of M from E , remove color α from all list $L(e)$ of the remaining edges and apply the induction hypothesis to color the remaining edges. Clearly, the resulting coloring is a list-edge coloring of G .

It remains to check that the conditions for the or induction hypothesis really fulfilled. More precisely, let $G' = (L, R, E \setminus M)$ and let $L'(e) = L(e) \setminus \{\alpha\}$, $\forall e \in E(G')$. Then we have

$$|L'(e)| = |L(e)| - 1 \geq T_{G, c}(e) - 1 + 1 \geq T_{G', c}(e) + 1, \quad \forall e \in E \setminus E_a$$

$$|L'(e)| = |L(e)| - 1 \geq T_{G, c}(e) - 1 + 1 \geq T_{G', c}(e) + 1, \quad \forall e \in E_a \setminus M$$

where the last inequality holds because for an edge $e \in E_a \setminus M$ the edge $f \in M$ which dominates e from the right or from the left does not count any more (see condition

(•), (••)) to be fulfilled by the matching M . $T_{G, c}(e) \geq T_{G', c}(e) + 1$

It remains now to construct the matching.

An edge $e \in E_a$ is called left-maximal with respect to $\leq$ iff $\nexists f \in E_a$ with $f_L = e_L$ and $c(f) > c(e)$, i.e. there is no edge in E_a which dominates e from the left.

Thus a left maximal edge $e \in E_L$ dominates from (4) the left all other edges from E_L which are incident to e_L .

maximal

The following procedure chooses successively left-~~max~~ edges from E_L and adds them to M by taking care that M remains a matching.

Initialize: $M = \emptyset$

while $\exists$ a left maximal $e \in E_L \setminus M$ do

begin

if $\exists f \in M$ such that $f_R = e_R \wedge c(f) < c(e)$ then
 $\quad$ /* $f \in M$ dominates e from the right */

$E_L := E_L - e$ /* removing edges from E_L */

else

/* there is no edge $f \in M$ dominating e from the right */

$M := M \cup \{e\}$ /* adding edges to M */

/* Is M still a matching */

if $\exists f \in M$ with $f_R = e_R \wedge c(f) > c(e)$

/* $e \in M$ dominates f from the right */

then $E_L := E_L \setminus f$; $M := M \setminus \{f\}$ /* removing edge from M */

end if

end if

end while

Observe that the relation " f dominates e from the right"

is transitive.

$$\left[\begin{array}{l} (f \text{ dom } e) \wedge (e \text{ dom } g) \Leftrightarrow (f_R = e_R, c(f) < c(e)) \wedge \\ (e_R = g_R, c(e) < c(g)) \Rightarrow (f_R = g_R, c(f) < c(g)) \end{array} \right]$$

This implies: if an edge e is dominated from the right by some edge $f \in M$, it will always remain dominated from the right by some other edge in M , because we remove an edge in M only if it is domin. from the right by some other edge in M .

Hence, at the termination of while all edges removed from E_0 will be dominated from the right by some edge in M . Moreover at the termination of while all left maximal edges ^{of E_0} which are not dominated from the right by some edge in M belong to M .

Therefore all remaining edges in $E_0 \setminus M$ are dominated from the left by some edge in M .

The while loop terminates because the cardinality of $|E_0 \setminus M|$ decreases by at least one in every iteration. At the beginning we had $|E_0 \setminus M| \leq |E| - 0 = |E|$.

At the end we have $|E_0 \setminus M| = |E_0 \setminus M| \geq 0$. Thus the while loop is repeated at most n times. 