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This fact (all paths constructed in the proof of the above corollary have length 2) is due to a fundamental phenomenon for constant edge probabilities (p independent on n).

From some results in logic, it follows that any statement about graphs expressed by quantifying over vertices (a fixed number of vertices) only (and not over sets or sequences of vertices of varying length) is either almost surely true or almost surely false.

All such statements (or their negations) are in fact consequences of an assertion that the graph has the property P_{ij} for some suitably chosen i or j .

Next we show that almost every graph has a surprisingly high chromatic number.

Proposition 7.13 $\forall p \in (0,1), \forall \varepsilon > 0$ almost every graph $G \in \mathcal{G}(n, p)$ fulfills

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$$X(G) > \frac{\log(1/p)}{2+\varepsilon} \frac{n}{\log n}$$

$X(G) < \frac{\log(1/p)}{2-\varepsilon} \frac{n}{\log n}$ holds a.s.

Proof: $\forall n \geq k \geq 2$ fixed we apply Lemma 7.2 and get

$$\begin{aligned} P(\alpha(G) \geq k) &\leq \binom{n}{k} q^{\binom{k}{2}} \text{ and then} \\ &\leq n^k q^{\binom{k}{2}} = (q^{\log_q n})^k q^{\binom{k}{2}} = \left(q^{\frac{\log n}{\log q}}\right)^k q^{\binom{k}{2}} \\ &= q^{k \frac{\log n}{\log q} + \frac{k(k-1)}{2}} = q^{k \left(\frac{\log n}{\log q} + \frac{k-1}{2}\right)} \\ &= q^{\frac{k}{2} \left(-\frac{2 \log n}{\log(1/q)} + k-1\right)} \quad (*) \end{aligned}$$

Now let $k := (2+\varepsilon) \frac{\log n}{\log(1/q)}$ and observe

that the exponent of the last expression in (*) tends to ∞ hence the expression itself tends to 0.

$$\begin{aligned} \text{Indeed } \frac{k}{2} \left(-\frac{2 \log n}{\log(1/q)} + k-1\right) &= \\ &= (2+\varepsilon) \frac{\log n}{\log(1/q)} \left[-\frac{2 \log n}{\log(1/q)} + (2+\varepsilon) \frac{\log n}{\log(1/q)} - 1 \right] \\ &= (2+\varepsilon) \frac{\log n}{\log(1/q)} \left[\frac{\varepsilon \log n}{\log(1/q)} - 1 \right] \xrightarrow{n \rightarrow \infty} \infty \end{aligned}$$

$\Rightarrow P(\alpha(G) \geq k) \leq q^{\infty} \xrightarrow{n \rightarrow \infty} 0$
for $k \geq m(n)$

Thus almost every graph $G \in \mathcal{G}(n, p)$ is such that in any vertex coloring of G no k vertices ^{can} have the

some color, no every coloring uses more than

$$\frac{n}{k} = \frac{n}{(2+\varepsilon) \frac{\log n}{\log(1/q)}} = \frac{\log(1/q)}{(2+\varepsilon)} \frac{n}{\log n} \text{ colors.}$$

Remarks: It has been shown by Bollobas in 1988 that if we replace ε by $-\varepsilon$ in the statement of proposition 7.13 then the lower bound turns into an upper bound!

Threshold functions and second moments

Observation In ^{all} the results we saw in the previous section the concrete value of p played no role as long as it was constant, i.e. independent on n . For example if almost every graph $G \in \mathcal{G}(n, p)$ had a certain property for $p=0.99$, this is also true for $p=0.01$. How can this happen? This is due to the fact that all this type of results (of the previous section) basically follow from this property $P_{i,j}$.

The situation is different however for many other properties which exist or not for sure in $G \in \mathcal{G}(n, p)$ depending on a p which is not constant; the threshold itself depends on n . E.g. in the proof of Erdős' theorem the threshold was $\frac{1}{n}$.

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Def A function $t: \mathbb{N} \rightarrow \mathbb{R}_{0,+}$ is called a threshold function
 $n \mapsto t(n)$

for a graph property $\mathcal{P}$ if the following

holds $\forall p = p(n)$ and $G \in \mathcal{G}(n, p)$: $p(n) = o(t(n))$

$$\lim_{n \rightarrow \infty} \mathbb{P}(G \in \mathcal{P}) = \begin{cases} 0 & \text{if } \frac{p}{t} \rightarrow 0 \text{ as } n \rightarrow \infty \\ 1 & \text{if } \frac{p}{t} \rightarrow \infty \text{ as } n \rightarrow \infty \end{cases}$$
 $p(n) = \Omega(t(n))$

If $\mathcal{P}$ has a threshold function t , then clearly
any ct for some $c > 0$, c fixed, is also a threshold
function for $\mathcal{P}$. So the threshold as defined above can
only be unique up to a multiplicative ^{positive} constant.

It has been shown that all increasing graph properties
have threshold functions (Bollobás and Thomason 1987).

Increasing properties are properties that are closed
under the addition of edges, eg. $\mathcal{P} = \{G: G \supseteq H\}$
for some fixed graph H . We will see some examples
of thresholds for this kind of properties:

We can consider the graph property as

$\mathcal{P} = \{G: X(G) = 1\}$ where X is

a suitably defined nonnegative random variable in $\mathcal{G}(n, p)$.

X could be e.g. the indicator random variable of $\mathcal{P}$ in $\mathcal{G}(n, p)$.

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Increasing property

But also other choices of X are allowed, e.g.

If $\mathcal{P}$ is connectedness we might take $X(G)$ to be the number of spanning trees in G .

Assume we want to prove that some given function t is a threshold function of $\mathcal{P}$. We will have to prove 2 parts

(1) almost no $G \in \mathcal{G}(n, p)$ has property $\mathcal{P}$ when p is small when compared to t $p(n) = o(t/n)$

(2) almost every $G \in \mathcal{G}(n, p)$ has property $\mathcal{P}$ when p is large when compared to t . $\lim_{n \rightarrow \infty} \frac{p(n)}{t(n)} = \infty$

(1) Since $X \geq 0$ we may use Markov's inequality for (1):

find an upper bound on $E(X)$, if this is small enough

We get $P(X \geq 1) \leq E(X)$ and if $E(X) \rightarrow 0$ as $n \rightarrow \infty$ we are done. $\neg X(G) \geq 1 \Leftrightarrow G \notin \mathcal{P}$

Computing expectations is much easier than computing probabilities, in general. This is due to the linearity of expectation.

(2) This part is more complicated.

in order to show $P[X \geq 1]$ is large it is not enough to bound $E[X]$ from below, if X is not bounded from above. X might be very large for very few graphs $G \in \mathcal{G}(n, p)$ and still be 0 for most of $G \in \mathcal{G}(n, p)$, so in order to prove $P(X \geq 1) \rightarrow 1$ as $n \rightarrow \infty$

we must show that X cannot deviate from the mean a lot too often. To this end we use Chebyshev's inequality.

Notations: let $\mu := E(X)$ and $\sigma^2 := E((X-\mu)^2)$
 variance (second moment) of X

We have $\sigma^2 = E(X^2) - \mu^2$.

Recall: μ and σ^2 refer to a random variable on some fixed probability space $\mathcal{G}(n, p)$. When n varies μ and σ^2 are function of n . $\mu(n)$

Lemma 7.13 (Chebyshev's inequality) $\sigma^2(n)$

$\forall \lambda > 0 \quad P[|X - \mu| \geq \lambda] \leq \frac{\sigma^2}{\lambda^2}$ given that the variance σ^2 exists

Proof: $P[|X - \mu| \geq \lambda] = P[(X - \mu)^2 \geq \lambda^2] \leq \frac{\sigma^2}{\lambda^2}$ for λ
 $=: Y$ Markov's meq.

Lemma 7.14 If $\mu > 0$ for all large n and $\frac{\sigma^2}{\mu^2} \rightarrow 0$ as $n \rightarrow \infty$, then $X(G) > 0$ for almost all $G \in \mathcal{G}(n, p)$

Proof: Any $G \in \mathcal{G}(n, p)$ with $X(G) = 0$ satisfies
 $|X(G) - \mu| = \mu \Rightarrow P(X(G) = 0) \leq P[|X - \mu| \geq \mu]$
 $\leq \frac{\sigma^2}{\mu^2} \rightarrow 0$ since $X \geq 0$ this means that $X > 0$ almost surely $\Rightarrow X(G) > 0$ for almost all $G \in \mathcal{G}(n, p)$
 Chebyshev's meq

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The main result of this section: we prove a Threshold for all graph properties $\mathcal{P}_H$ which is the property of containing a fixed graph H (a copy of it) as a subgraph.

Let H be given, and let $k := |V(H)|$ and $l := |E(H)|$.

Assume $l \geq 1$. Let $X(G)$ be the number of subgraphs of G that are isomorphic to H . $G \in \mathcal{P}_H \Leftrightarrow H \subseteq G \Leftrightarrow X(G) \geq 1$

For a given $n \in \mathbb{N}$, let $\mathcal{H}$ denote the set of all graphs isomorphic to H on subsets of $\{0, 1, \dots, n-1\}$.

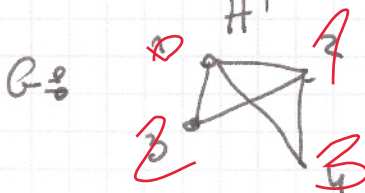
$$\mathcal{H} := \{H' : H' \cong H, V(H') \subseteq \{0, 1, \dots, n-1\}\}.$$

Given $H' \in \mathcal{H}$ and $G \in \mathcal{G}(n, p)$ we write $H' \subseteq G$

to express that H' itself (not just an isomorphic copy of H') is a subgraph of G .

Example $H =$  $n=4$

$$\mathcal{H} = \left\{ \begin{array}{c} \text{triangle on } \{0,1,2\} \\ \text{triangle on } \{0,1,3\} \\ \text{triangle on } \{0,2,3\} \\ \text{triangle on } \{1,2,3\} \end{array} \right\}$$



$H' \subseteq G$ but $H'' \not\subseteq G$

Double counting:
$$E(X) = \sum_{H' \in \mathcal{H}} X_{H'} = \sum_{H' \in \mathcal{H}} P(H' \subseteq G) \quad (*)$$

and $\forall H' \in \mathcal{H} \quad P(H' \subseteq G) = p^{|E(H')|} = p^l$

let h be the nr. of isomorphic copies of H on a fixed k -set; we have $h \leq k!$

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As there are $\binom{n}{k}$ vertex sets possible for the graphs in $\mathcal{H}$ we have $|\mathcal{H}| = \binom{n}{k} \cdot k \leq \binom{n}{k} k! \leq n^k$

let $p := p(n)$ be an edge probability and let $t = t(n)$ be a candidate for a threshold function; we write $\gamma := \frac{p}{t}$. The first lemma considers the case $\gamma \rightarrow 0$.

$$E(\mathcal{H}) = \frac{l}{k}$$

lemma 7.15 If $t = n^{-\frac{1}{E(\mathcal{H})}}$ and p is such that $\gamma \xrightarrow{n \rightarrow \infty} 0$, then almost no $G \in \mathcal{G}(n, p)$ lies in $\mathcal{P}_{\mathcal{H}}$.

Proof: Goal: find an upper bound for $E(X)$ and show that this tends to 0 as $n \rightarrow \infty$.

We have $p = \gamma t = \gamma n^{-\frac{1}{E(\mathcal{H})}} = \gamma n^{-\frac{k}{l}}$ ($E = \frac{l}{k}$)

Thus $E(X) = |\mathcal{H}| \cdot p^l \leq n^k (\gamma n^{-\frac{k}{l}})^l = \gamma^l$

Then $P(G \in \mathcal{P}_{\mathcal{H}}) = P[X \geq 1] \leq E(X) \leq \gamma^l \xrightarrow{n \rightarrow \infty} 0$
 Markov's inequality (because $\gamma \xrightarrow{n \rightarrow \infty} 0$ and $l \geq 1$)