

Advanced and algorithmic graph theory
Summer term 2020
Fifth worksheet

41. A graph is called *comparability graph* iff there exists a partial ordering of its vertex set such that two different vertices are adjacent if and only if they are comparable. Give an example of a comparability graph that is neither complete nor bipartite. Give an example of a comparability graph that is neither complete nor bipartite. Show that every comparability graph is perfect.
42. A graph G is called an *interval graph* iff there exists a set $\{I_v : v \in V(G)\}$ of intervals such that $I_v \cap I_u \neq \emptyset$ if and only if $\{u, v\} \in E(G)$.
- (a) Show that every interval graph is chordal.
 - (b) Show that the complement of every interval graph is a comparability graph.¹
 - (c) Give an example of a chordal graph that is not an interval graph.
43. The Berge mystery story.
- Six professors had been to the library on the day that the precious tractate was stolen. Each had entered once, stayed for some time and then left. If two professors were in the library at the same time, then at least one of them saw the other. Detectives questioned the professors and gathered the following testimony: Abe said that he saw Burt and Eddie in the library; Burt said that he saw Abe and Ida; Charlotte claimed to have seen Desmond and Ida; Desmond said that he saw Abe and Ida; Eddie testified to seeing Burt and Charlotte; Ida said that she saw Charlotte and Eddie. One of the professors lied!! Who was it?
44. (a) We know that a graph H obtained from a perfect graph G by expanding a vertex is perfect (cf. lecture). Is the converse “If the graph H obtained from a graph G by expanding a vertex is perfect, then G is also perfect” true?
- (b) The union of two graphs G_1 and G_2 is a graph denoted by $G_1 + G_2$ with vertex set $V(G_1 + G_2) := V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) := E(G_1) + E(G_2)$. Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be the set of all connected graphs of order 25 (for some $k \in \mathbb{N}$) and let $G = G_1 + G_2 + \dots + G_k$ be the union of these graphs. Prove or disprove the following statements: (i) $\chi(G) = \omega(G)$, (ii) the graph G is perfect.
45. (a) Show that if G is perfect, then $|V(H)| \leq \alpha(H)\omega(H)$ holds for every induced subgraph H of G .
- The converse is also true: a graph G is perfect if and only iff $|V(H)| \leq \alpha(H)\omega(H)$ holds for every induced subgraph H of G . This characterisation was proved by Lovász [2].
- (b) Applying the above characterisation of Lovász to prove the perfect graph theorem: a graph G is perfect iff $\bar{G}$ is perfect.
46. Let σ be a perfect elimination scheme for a chordal graph G . Show that the greedy algorithm yields a $\chi(G)$ -coloring of G if applied to the vertices ordered as $\sigma^{-1}(n), \sigma^{-1}(n-2), \dots, \sigma^{-1}(1)$.

¹Conversely, a chordal graph is an interval graph if its complement is a comparability graph; this is a theorem of Gilmore and Hoffman (1964).

The fact that interval graphs have a richer structure than general chordal graphs or comparability graphs has consequences for the theoretical complexity of certain prominent graph theoretical problems, e.g. the *Hamiltonian Cycle Problem*, i.e. the decision problem whether a graph has a Hamiltonian cycle or not, is NP-complete in general chordal graphs [1] and also in bipartite graphs [4], but is however polynomially solvable in interval graphs [3].

47. (a) Show that not every perfect graph is chordal by specifying a concrete graph which is perfect but not chordal.
- (b) Consider the graph G in Figure 1. Apply an efficient algorithm to check whether this graph is chordal (the straightforward application of the definition of chordal graphs is not considered to be efficient). Apply efficient algorithms to determine a maximum stable set, a maximum clique, an optimal coloring and an optimal clique covering of G as well as the invariants $\alpha(G)$, $\omega(G)$, $\chi(G)$ and $\theta(G)$, respectively.
48. Consider a tree T and a collection of subtrees of T . The so-called *subtree graph* related to this collection of subtrees is a (so-called *intersection*) *graph* that has one vertex per subtree and an edge connecting any two subtrees that overlap in one or more nodes of the tree. Show that such a subtree graph is chordal. Is the converse also correct, i.e. given an arbitrary chordal graph G , is there a tree T and a collection of subtrees of T such that the corresponding subtree graph is isomorphic to G ?
49. A graph $G = (V, E)$ is called a *split graph* iff there is a partition $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ such that the induced subgraphs $G[V_1]$ and $G[V_2]$ are a clique and a stable (i.e. a graph with no edges), respectively.
- (a) Show that the class of split graph is closed under complementation, i.e. the complement of a split graph is a split graph.
- (b) Show that G is a split graph iff G and its complement $\bar{G}$ are both chordal.
50. Determine whether $\bar{C}_8$, i.e. the complement graph of a cycle on 8 vertices, is perfect.

References

- [1] C.J. Colbourn and L.K. Stewart, Dominating cycles in series-parallel graphs, *Ars Combinatoria* **19A**, 185, 107–112.
- [2] L. Lovász, On chromatic number of finite set systems, *Acta. Math. Acad. Sci Hungar.* **79**, 1967, 59–67.
- [3] J.M. Keil, Finding Hamiltonian circuits in interval graphs, *Information Processing Letters* **20**, 1985, 201–206.
- [4] M.S. Krishnamoorthy, An NP-hard problem in bipartite graphs, *SIGACT News* **7**, 1975, pp. 26.

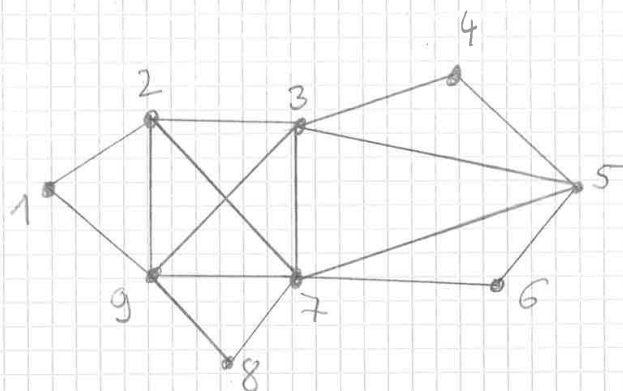


Figure 1: Graph for Exercise 47(b)