Theorem 1 ((Menger 1927))

(a) *(**The edge version**) Let s and t be two arbitrary vertices of a graph G. The maximum number of s-t-paths which do not share an edge equals the minimum cardinality of a set of edges which separates s and t in G*

(b) *(**The vertex version**) Let s and t be arbitrary vertices of a graph G such that $\{s, t\} \notin E(G)$. The maximum number of independent s-t-paths equals the minimum cardinality of an s-t-separator G.*

Corollary 2

Menger's Theorem implies an equivalent definition of connectivity:

(a) *(**Edge connectivity**) A graph G with $|G| > 1$ is ℓ-edge connected, $\ell \in \mathbb{N}$, iff (i) every edge set which separates G has at least ℓ elements, or equivalently (ii) $\forall s, t \in V(G)$ there exists ℓ edge-disjoint s-t-paths.*

(b) *(**Vertex connectivity**) A graph G with $|G| > k$, $k \in \mathbb{N}$, is k-edge connected iff (i) every separator of G has at least k elements, or equivalently (ii) $\forall s, t \in V(G)$ there exists k independent s-t-paths.*

Theorem 3

If the graph G is non-trivial, i.e. $|G| > 1$, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$ holds.

Thus high connectivity implies high minimum degree.

The converse is not true: there are graphs with large $\delta(G)$ which have a small $\lambda(G)$. However a high $\delta(G)$ implies the existence of a highly connected subgraph.

Proposition 4 ((Mader 1972), no proof)

Let $k \in \mathbb{N}$, $k \neq 0$. Every graph G with $d(G) \geq 4k$ has a (k+1) connected subgraph H such that $\epsilon(H) = \frac{|E(H)|}{|V(H)|} > \epsilon(G) - k$ holds.

Observation 5

The vertices of a connected graph G can always be enumerated as $v_1, v_2, \ldots, v_{|G|}$ such that the induced subgraph $G_i := G[\{v_1, v_2, \ldots, v_i\}]$ is connected, for all $i \in \overline{1, |G|}$.

(The proof is a homework.)

The block-cut-vertex graph

Definition 1

A maximal connected subgraph without a cut vertex is called a **block**.

Observation 6

Every block is either (i) a maximal 2-connected subgraph, or (ii) a bridge, or (iii) an isolated vertex. Conversely, every such graph is a block.

Observation 7

(a) Different blocks of a graph G overlap in at most one vertex which is then a cut-vertex, (b) every edge of G lies in a unique block, and (c) G is the union of its blocks.

Definition 2

The **block-cut-vertex graph** *$bc(G)$ of a given graph G is a graph with vertex set $\mathcal{B} \cup \mathcal{C}$, where $\mathcal{B}$ is the set of blocks of G and $\mathcal{C}$ is the set of cut-vertices of G. The edge set of $bc(G)$ consists of edges $\{b, c\}$ with $b \in \mathcal{B}$, $c \in \mathcal{C}$, and $c \in b$.*

Theorem 8 ((Gallai 1964, Harary and Prins 1966))

The block-cut-vertex graph $bc(G)$ of a connected graph C is a tree.

2-connectivity and 3-connectivity

Theorem 9 ((Ear decomposition, Whitney 1932, Halin & Jung 1963))

A graph G is 2-**connected** *iff it can be represented as $G = C \cup P_1 \cup P_2 \cup \ldots \cup P_k$, for some $k \in \mathbb{N}$, where C is a cycle and P_i $1 \leq i \leq k$, are paths which have only their end-vertices in common with $C \cup P_1 \cup \ldots \cup P_{i-1}$.*

Definition 3 (Contraction of an edge)

Let $e = \{x, y\}$ be an edge of a graph $G = (V, E)$. We denote by G/e the graph obtained from G by contracting the edge e into a new vertex v_e which becomes adjacent to all the former neighbors of x and of y. Formally $G/e := (V', E')$, where

$$V' := V \setminus \{x, y\} \cup \{v_e\} \text{ and}$$

$$E' := \big\{\{v, w\} \in E : \{v, w\} \cap \{x, y\} = \emptyset\big\} \cup \big\{\{v_e, u\} : \{x, u\} \in E \text{ or } (y, u\} \in E\big\}.$$

Definition 4 (Minor)

Let X, G be graphs and let $\{V_x : x \in V(X)\}$ be a partition of $V(G)$ such that (i) $G[V_x]$ is connected for all $x \in X$ and (ii) for any two vertices $x, y \in X$ there is a V_x-V_y-edge in G iff $\{x, y\} \in E(X)$. Then we denote $G = MX$. The sets V_x are called the branch sets of $G = MX$.

Thus, X is obtained from G by contracting every branch set to a single vertex and deleting any parallel edges or loops that may arise.

If $MX = G$ is a subgraph of Y, i.e. $G \subseteq Y$, we call X a minor of Y and denote $X \preceq Y$.

Definition 5 (Contraction of a set of vertices)

If $V_x = U \subset V(G)$ is one of the branch sets and every other branch set consists just of a single vertex, we also write G/U for the graph X and V_U for the vertex $x \in V(X)$ to which U contracts and think of the rest of X as an induced subgraph of G. The contraction of edge $\{u, u'\}$ corresponds to the simple case $U = \{u, u'\}$.

Proposition 10

G is an MX iff X can be obtained from G by a series of edge contractions, i.e. iff there exist an $n \in \mathbb{N}$, the graphs G_0, G_1, ..., G_n and the edges $e_i \in E(G_i)$ such that $G_0 = G$, $G_n \simeq X$ and $G_{i+1} = G_i/e_i$ for $0 \leq i \leq n-1$.

Proof: Induction on $|G| - |X|$, homework.

Lemma 11

If G is 3-connected and $|G| > 4$, then G has an edge e such that G/e is again 3-connected.

Theorem 12 ((Tutte 1961))

A graph G is 3-connected iff there exits an $n \in \mathbb{N}$ and sequence of graphs G_0, G_1, ..., G_n with the following properties:

(i) *$G_0 = K_4$ and $G_n = G$*

(ii) *G_{i+1} has an edge $\{x, y\}$ with $deg_{G_{i+1}}(x) \geq 3$, $deg_{G_{i+1}}(y) \geq 3$, and $G_i = G_{i+1}/\{x, y\}$, for all $0 \leq i \leq n-1$.*