Advanced and Algorithmic Graph Theory
TU Graz, Summer term 2024

1 General terminology and notations

Definition 1 **An undirected graph** *(often also simply called* **graph***) G is an ordered pair (V, E), where V is the* **set of vertices** *and $E \subseteq \{\{u, v\}: u, v \in V, u \neq v\} =: \mathcal{P}_2(V)$ is the* **set of edges***. Thus the edges of a graph are two-element subsets of its set of vertices.*

Notation: Given a set V, we denote by $\mathcal{P}_2(V)$ as above the set of two-element subsets of V.

A directed graph*, briefly* **digraph***, G is an ordered pair (V, E), where V is the* **set of vertices** *and $E \subseteq V \times V$ is the* **set of arcs***. For a given graph (digraph) G we denote by $V(G)$ and $E(G)$ its set of vertices and its set of edges (arcs), respectively.*

The **order** *of a graph (digraph) G is the cardinality $|V(G)|$ of its vertex set.*

An edge e of a graph is denoted by $e := \{u, v\}$ and an arc of a digraph is denoted by (u, v); u and v are called **end-vertices** *of e. We say that vertex u (v) is* **incident** *with edge (arc) e and that the vertices u and v are* **adjacent vertices** *or* **neighbors***. Moreover, we say that e is an edge at u (v) or that arc e starts at u and ends at v respectively. Two edges e, f are called* **adjacent edges** *if $e \neq f$ and $e \cap f \neq \emptyset$. The set of* **neighbors of a vertex** *v in G is denoted by $N_G(v)$, or briefly by $N(v)$, if there is no ambiguity about the underlying graph G.*

Definition 2 *The* **degree** *$deg_G(v)$* **(or** *$deg(v)$***) of a vertex** *v* **in** *G is defined as $deg(v) := |N(v)|$. A vertex with degree equal to 0 is called an* **isolated vertex***.*

The **minimum degree** *$\delta(G)$* **of a graph** *G is defined as $\delta(G) := \min_{v \in V(G)} deg(v)$.*

Analogoulsy, the **maximum degree** *$\Delta(G)$* **of a graph** *G is defined as $\Delta(G) := \max_{v \in V(G)} deg(v)$.*

The **average degree** *$d(G)$* **of a graph** *G is defined as $d(G) := \frac{1}{|V(G)|} \sum_{v \in V(G)} deg(v)$.*

The **density** *$\varepsilon(G)$* **of a graph** *G is defined as $\varepsilon(G) := \frac{|E(G)|}{|V(G)|}$.*

Exercise 1 *Recall the handshake lemma and derive a simple relationship between $d(G)$ and $\varepsilon(G)$.*

Definition 3 *For some $k \in \mathbb{Z}_+$, a* **k-regular graph** *G is a graph with $deg(v) = k$, for all $v \in V(G)$. A 3-regular graph is also called a* **cubic graph***.*

A graph G is called **a complete graph** *if all vertices in $V(G)$ are pairwise adjacent. For some $n \in \mathbb{N}$, the complete graph of order n is denoted by K_n.*

Definition 4 *A set $A \subseteq V(G)$ is called an* **independent set** *in G if no two vertices of A are adjacent. The* **independence number of a graph** *G is denoted by $\alpha(G)$ and is defined as the largest $k \in \mathbb{N}$ such that there exists an independent set of cardinality k in G, i.e.*

$$\alpha(G) := \max\{|A|: A \subseteq V(G), A \text{ is an independent set in } G\}.$$

A set $A \subseteq V(G)$ is called an **clique** *in G if any two vertices of A are adjacent. The* **clique number of a graph** *G is denoted by $\omega(G)$ and is defined as the largest $k \in \mathbb{N}$ such that there exists a clique of cardinality k in G, i.e.*

$$\omega(G) := \max\{|A|: A \subseteq V(G), A \text{ is a clique in } G\}.$$

Definition 5 *The graphs G and G' are called* **isomorphic** *iff there exists a bijection $\phi\colon V(G) \to V(G')$ such that $\{u,v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(G')$. In this case ϕ is called an* **isomorphism** *of G and G'. If G and G' are isomporphic we denote $G \simeq G'$. Frequently. we make no difference between two isomorphic graphs and write $G = G'$ instead of $G \simeq G'$. If $G = G'$, then an isomorphism ϕ of G and G' is called an* **automorphism**.

A mapping taking graphs as arguments is called **a graph invariant** *iff it assigns equal images (values) to isomorphic graphs. For example the number of vertices and the number of edges in a graph are graph invariantes.*

Exercise 2 *Specify some other graph invariants.*

Definition 6 *Given two graphs $G = (V, E)$ and $G' = (V', E')$, we define the* **union of** G **and** G' *as $G \cup G' := (V \cup V', E \cup E')$ and* **the intersection of** G **and** G' *as $G \cap G' := (V \cap V', E \cap E')$.*
The graphs G and G' are called **disjoint graphs** *if $V(G) \cap V(G') = \emptyset$.*
The **product** $G * G'$ **of two disjoint graphs** G *and G' is obtained from $G \cup G'$ by adding to it the edges of the form $\{v, v'\}$ for all $v \in V(G)$, $v' \in V(G')$.*

Exercise 3 *Draw the graph $K_2 * K_3$? Is this product a complete graph?*

Definition 7 *The* **complement** G^C **of a graph** G *is defined as $G^C := \big(V(G), \mathcal{P}_2(V(G)) \setminus E(G)\big)$.*
The **line graph** $L(G)$ **of a graph** G *is defined as*

$$L(G) := \big(E(G), \big\{\{e, f\}\colon e, f \in E(G), e \cap f \neq \emptyset\big\}\big) \ .$$

Definition 8 *A graph G is a* **subgraph** *of a graph G' if $V \subseteq V'$ and $E \subseteq E'$. We denote $G \subseteq G'$ and say that G' is a* **supergraph** *of G. If $G \subseteq G'$ and $G \neq G'$ we say that G is a fproper subgraph of G' and denote $G \subsetneq G'$.*
If $G \subseteq G'$ and $E(G)$ contains all edges $\{u, v\}$ of $E(G')$ with $u \in V(G)$ and $v \in V(G)$ we call G an **induced subgraph** *of G'. In this case we say that V* **induces** G **in** G' *and denote $G = G'[V]$. A subgraph G is called* **a spanning subgraph** *of G' iff $G \subseteq G'$ and $V(G) = V(G')$.*

Question 1 *Given that G is an induced subgraph of G'. Does the following equality hold*

$$E(G) = \big\{\{u, v\}\colon u \in V, v \in V\big\} \cap E(G')\,?$$

Definition 9 *Given a graph G and a set $U \subseteq V(G)$ we denote $G - U := G[V \setminus U]$. If U is a* **singleton**, *i.e. $U = \{v\}$ for some $v \in V(G)$ we write $G - v$ instead of $G - \{v\}$. We also write $G - G'$ instead of $G - V(G')$. For some $F \subseteq \mathcal{P}_2(V(G))$ we write $G - F := (V(G), E(G) \setminus F)$ and $G + F := (V(G), E(G) \cup F)$. If F is a singleton, i.e. $F = \{e\}$ for some $e \in \mathcal{P}_2(V(G))$, we write $G + e$ and $G - e$ instead of $G + \{e\}$ and $G - \{e\}$, respectively.*

Definition 10 *A graph G is called* **edge-maximal** *with respect to some graph property P iff G itself has the property P, but the graph $G + \{u, v\}$ does not have property P, for all $u, v \in V(G)$ such that $\{u, v\} \notin E(G)$. We say that a* **graph G is maximal (minimal) with respect to some property** *P if G has the property P but no supergraph (subgraph) of G has it. Analogosly, if we speak of a* **maximal or minimal set of vertices of edges** *with a certain propertxy, the considered order relation is simply the set-inclusion.*

Question 2 *Let P be the property "contains a triangle" and let P' the property "does not contain a triangle" or equivalently "is triangle-free". Consider the set $\mathcal{G}_6$ of all graphs of order 6. Give a graph which is minimal with respect to P and a graph which maximal with respect to P' in $\mathcal{G}_6$.*

Definition 11 *A* **path** *P is a graph $P = (V, E)$ with $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \big\{\{x_{i-1}, x_i\} \colon i \in \{1, 2, \ldots, k\}\big\}$, where $k \in \mathbb{Z}_+$ and all x_i, $i \in \{0, 1, \ldots, k\}$, are pairwise disjoint, except for may be x_0 and x_k. Analogoulsy, a* **directed path (dipath)** *P is defined as a digraph $P = (V, E)$ with $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \big\{(x_{i-1}, x_i) \colon i \in \{1, 2, \ldots, k\}\big\}$, where all x_i, $i \in \{0, 1, \ldots, k\}$, are pairwise disjoint, except for may be x_0 and x_k. In the latter case the path (dipath) is called* **closed**.
The vertices $x_1, \ldots, x_{k-1}$ are called **internal vertices** *of the path (dipath) P. x_0 and x_k are called* **start-vertex of** *P and* **end-vertex of** *P, respectively, if P is a dipath, and* **end-vertices of** *P, otherwise. We say that the end-vertices of path P are joined (connected) by P and P joins (connects) its end-vertices.*
The **length of the path (dipath)** *is the number k of its edges (arcs). Thus a path (dipath) of length 0 is just a single vertex (or the complete graph K_1; such a path (dipath) is called a* **trivial path** *(***trivial dipath***). Often we refer to the path (dipath) P as above by the natural sequence of its vertices and denote $P = x_0, x_1, \ldots, x_k$. Let $P = x_0, x_1, \ldots, x_k$.*

Definition 12 *We distinguish the following* **subpaths** *of P:*
$Px_i := x_0, x_1, \ldots, x_i$, $x_iP = x_i, x_{i+1}, \ldots, x_k$, $x_iPx_j := x_i, x_{i+1}, \ldots, x_{j-1}, x_j$, for $i, j \in \{0, 1, \ldots, k\}$, and $\mathring{P} = x_1, \ldots, x_{k-1}$.
Given a graph G and a path P we say that P is a path in G iff $P \subseteq G$, i.e. if the path P is a subgraph of G.

Definition 13 *A* **walk** *W* **of length** *k,* *$k \in \mathbb{N}$,* **in a graph** *G is a non-empty alternating sequence $v_0.e_0, v_1, e_1, \ldots, e_{k-1}, v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\} \in E(G)$, for all $i \in \{0, 1, \ldots, k-1\}$. If $v_0 = v_k$ the walk is called a* **closed walk**. *If the vertices of the walk W are pairwise distinct, then W defines a path in G.*
Consider a graph G and two subsets $A, B \subseteq V(G)$. A path $x_0, x_1, \ldots, x_k$ in G is called **an A-B-path** *if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$[1]. If $A = \{a\}$ and/or $B = \{b\}$ we write a-B-path, A-b-path or a-b-path, respectively. Two or more paths are* **independent paths** *iff none of them contains an inner vertex of another.*
Given a graph H we call a path $P = x_0, \ldots, x_k$ an **H-path** *if P is non-trivial (i.e. $k \geq 1$) and $V(P) \cap V(H) = \{x_0, x_k\}$.*

[1]This concept and the further concepts within this definition can be analogously defined for digraphs and dipaths.

Definition 14 *If $P = x_0, \ldots, x_{k-1}$ is a path and $k \geq 3$, then $C := P + \{x_{k-1}, x_0\}$ is called a* **cycle***. Sometimes we denote $C = x_0, x_1, \ldots, x_{k-1}, x_0$. The* **length of a cycle** *is the number of its edges. A* **cycle in a graph** *G is a subgraph of G which is a cycle.*
The minimum length of a cycle in a graph G is called the **girth of** *G and is denoted by $g(G)$. The maximum length of a cycle in a graph G is called the* **circumference of** *G and is denoted by $circ(G)$. If G contains no cycle we set $g(G) = \infty$ and $circ(G) = 0$. An edge joining two vertices of a cycle C which is not an edge of C is called a* **chord of the cycle** *C. Thus, an induced cycle in G (i.e. a cycle which is an induced subgraph of G) is a cycle without chords.*

Proposition 1 *Every graph G contains a path of length $\delta(G)$ and a cycle of length $\delta(G)+1$, provided that $\delta(G) \geq 2$.*[2]

Definition 15 *The* **distance** $d_G(u, v)$ **(or** $d(u, v)$**) of two vertices** *u and v in a graph G is the length of a shortest u-v-path in G; if there is no such a path we set $d_G(u, v) = \infty$.*
The **diameter of** *G is denoted by $diam(G)$ and is defined as $diam(G) := \max_{u,v \in V(G)} d_G(u, v)$.*

Given a graph G, a vertex $v \in v(G)$ is called a **central vertex** *if its greatest distance form any other vertex is as small as possible. This distance is called the* **radius of graph** *G and is denoted by $rad(G)$, i.e. $rad(G) := \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$.*

Exercise 4 *For $k \in \mathbb{Z}$, $k \geq 3$, consider a cycle C^k with k vertices. Determine $diam(C^k)$ and $rad(C^k)$.*

Exercise 5 *Can a graph have more than one central vertex?*

Exercise 6 *Do the inequalities $rad(G) \leq diam(G) \leq 2rad(G)$ hold? Are these inequalitioes best possible. i.e. are there particular graphs for which these inequalities are fulfilled with equality?*

Definition 16 *A graph G with at least one vertex is called* **connected** *iff for any two vertices $u, v \in V(G)$ there esists a u-v-path in G. Otherwise G is called* **disconnected***. A set $U \subseteq V(G)$ is called a* **connected set of vertices in** *G iff $G[U]$ is connected.*
G is called a **k-connected graph** *iff $|G| > k$ and $G - X$ is connected, for all $X \subset V$ with $|X| < k$. The largest nonnegative integer k such that G is k-connected is called* **the connectivity of graph** *G and is denoted by $\kappa(G)$.*
If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E(G)$ with $|F| < \ell$, then G is called an **ℓ-edge connected graph***. The largest nonnegative integer ℓ such that G is ℓ-edge connected is called the* **edge connectivity of graph** *G and is denoted by $\lambda(G)$.*

Question 3 *Which graphs are 0-connected? Does 1-connected mean connected? For which graph does $\kappa(G) = 0$ hold?*

Exercise 7 *Specify a graph G with $|G| = n$ and $\kappa(G) = n - 1$, for $n \in \mathbb{N}$.*

[2] In contrast to this statement, the minimum degree and the girth are not related to each other.

Exercise 8 *Specify $\lambda(G)$ for a disconnected graph G. Specify a 2-edge connected graph G with $|G| = n$, for $n \in \mathbb{N}$.*

Definition 17 *A maximal connected subgraph of a graph G is a* **component of graph** *G. By definition a graph without vertices has no components.*

Question 4 *Are the components of G induced subgraphs of G? Do their vertex sets partition $V(G)$?*

If $A, B \subseteq V(G)$ and $X \subseteq V \cup E$ are such that every A-B-path in G contains a vertex or an edge from X, we say that X **separates the sets** *A* **and** *B* **in** *G. Notice that this implies $A \cap B \subseteq X$.*
We say that $X \subset V(G) \cup E(G)$ **separates** *G if $G - X$ is disconnected, i.e. if X separates in G some two vertices that are not in X. A separating set of vertices is called a* **separator**. *Separator sets of edges have no generic name, but some of them do (e.g. cuts and bonds, to be defined later).*
A vertex which separates two other vertics from the same component is called a **cut-vertex**. *An edge which separates its own end-vertices is called a* **bridge**.

Exercise 9 *Draw some examples of cut-vertices and bridges. Can you characterize bridges by means of cycles?*

Definition 18 *A graph G is called an* **acyclic graph** *or a* **forest** *iff it contains no cycles. A connected forest is called a* **tree**. *A vertex of degree one in a tree is called a* **leaf of the tree**. *A* **rooted tree** *is a tree T with a special vertex r in it, $r \in V(T)$, called* **root of the tree**.
In a rooted tree T with root r we define a partial order $\leq$ on $V(T)$, called **the tree-order**, *such that $x \leq y$ iff x lies on the unique r-y-path in T, for all $x, y \in V(T)$.*

We think of the tree-order as expressing **height**, *i.e. if $x < y$ (which means that $x \leq y$ and $x \neq y$ hold), we say that x* **lies below** *y* **in** *T. We denote by $\lceil y \rceil := \{x \in V(T) \colon x \leq y\}$ the* **down-closure of** *y; analogously, we denote by $\lfloor x \rfloor := \{y \in V(T) \colon y \geq x\}$ the* **up-closure of** *x. A set $X \subseteq V(T)$ that equals its up-closure, i.e. which satisfies $X = \lfloor X \rfloor := \cup_{x \in X} \lfloor x \rfloor$, is said to be* **closed-upwards** *or an* **up-set**. *A set which is* **closed-downwards** *or a* **down-set** *is defined analogously. The vetices at distance k from the root are said to* **have height equal to** *k and form the* **k-th level of** *T.*

Question 5 *Given a tree T with root $r \in V(T)$. Are the end-vertices of any edge $\{x.y\}$ of T* **comparable** *in terms of the tree-order, i.e. does $x \leq y$ or $y \leq x$ hold, for all $e = \{x, y\} \in E(T)$? Is the down-closure of every vertex a* **chain**, *i.e. a set of pairwise comparable elements?*

Definition 19 *A rooted tree T with root r contained in a graph G is called a* **normal tree in** *G iff for every T-path in G its end-vertices are comparable in the tree-order of T.* **Normal spanning trees of a graph** *G are the so called* **depth first search trees of** *G (because of the way they arise in computer searches on graphs).*

Exercise 10 *Give the example of a graph G with $|G| = 15$ and a normal spanning tree in it. Give the example of another spanning tree of G which is not normal.*

Observe that the following statement holds:

Proposition 2 *Every connected graph contains a normal spanning tree, with any specified vertex as its root.*