Hamiltonicity: sufficient conditions (contd.)

Theorem 4.
(Chvátal, Erdös 1972)
Consider a graph G with $n := |G| \geq 3$. The inequality $\kappa(G) \geq \alpha(G)$ implies the hamiltonicity of G.
($\alpha(G)$ is the stability number of G, i.e.
$\alpha(G) := \max\{|S| : S \subseteq V(G)$ and $G[S]$ has no edges$\}$.)

The result of Theorem 4 is best possible:

(i) There exists a graph G with $\kappa(G) = \alpha(G) - 1$ which is not Hamiltonian (e.g. the Petersen graph or $K_{r,r+1}$, as discussed in the practicaL)

(ii) The statement also does not hold if $\kappa(G)$ is replaced by $\lambda(G)$.

Corollary 5.
There exists an $O(mn)$ algorithm which either constructs a Hamiltonian cycle in G (with $n := |V(G)$, $m := |E(G)|$) or finds a stable set S and a separating set T in G such that $|T| < |S|$ holds.

Hamiltonicity: further sufficient conditions

Theorem 6.
(Bondy, 1978)
Let G be a graph with $|G| \geq 3$ and $deg(u) + deg(v) \geq |G|$ for all $u, v \in V(G)$ with $\{u, v\} \notin E(G)$. Then $\kappa(G) \geq \alpha(G)$ holds.

Finding long cycles in graphs

The problem of determining the length of a longest cycle in a graph G is NP-hard.

We assume w.l.o.g. that G is 2-connected.

Theorem 7.
(Ore 1960, Bermond 1976, Linial 1976)
Let $d \in \mathbb{N}$ and let G be a 2-connected graph such that $\forall u, v \in V(G)$ with $\{u, v\} \notin E(G)$, $deg(u) + deg(v) \geq d$ holds. Then there exists a cycle of length at least $\min\{n, d\}$ in G, where $n := |G|$.

Theorem 8.
(Bauer, Broersma, Veldman and Rao 1989)
If G is a 2-connected graph with n vertices and connectivity $\kappa(G)$ such that $deg(x) + deg(y) + deg(z) \geq n + \kappa(G)$ for any triple of independent vertices $x, y, z \in V(G)$, then G is Hamiltonian.

Proof in *Journal of Combinatorial Theory, Series B* **47(2)**, 1989, 237–243.

Hamiltonian cycles and degree sequences

Definition 1.
If G is a graph with $n := |G|$ and vertex degrees $d_1 \leq d_2 \leq \ldots \leq d_n$, then the n-tuple $(d_1, d_2, \ldots, d_n)$ is called **the degree sequence** *of G. An arbitrary sequence of natural numbers $(a_1, a_2, \ldots, a_n)$ is called a* **Hamiltonian sequence** *if every graph with n vertices and degree sequence $(d_1, d_2, \ldots, d_n)$ which is pointwise not smaller than $(a_1, a_2, \ldots, a_n)$ (i.e. $d_i \geq a_i$, $\forall i \in 1, n$), is Hamiltonian.*

The degree sequence of a graph is unique while there may be different enumerations of the vertices all of them leading to the same degree sequence.

Theorem 9.
(Chvátal 1972)
A sequence $(a_1, a_2, \ldots, a_n)$ of natural numbers with $0 < a_1 \leq a_2 \leq \ldots \leq a_n < n$ and $n \geq 3$ is Hamiltonian iff $a_i \leq i$ implies $a_{n-i} \geq n - i$, for every $i < \frac{n}{2}$.