The maximum flow problem (MFP) with a given source-sink pair

Definition 1 (s-t-flow and its value)

Consider a capacitated network (G, u, s, t)*, with a directed graph* G*, capacities* $u\colon E(G) \to \mathbb{R}_+$ *and two specific vertices* $s, t \in V(G)$*, where* s *is the source and* t *is the sink. An* s*-*t*-flow in* (G, u, s, t) *is a mapping* $f\colon A(G) \to \mathbb{R}_+$ *such that (i)* $f(e) \leq u(e)$, $\forall e \in A(G)$*, and (ii)* $ex_f(i) := \sum_{j \in V(G)} f(j, i) - \sum_{j \in V(G)} f(i, j) = 0$*, for all* $i \in V(G) \setminus \{s, t\}$*.*
The value $val(f)$ *of* f *is defined as* $val(f) := -ex_f(s)$*.*

The maximum flow problem (MFP)
Input: a network (G, u, s, t), with a directed graph G, capacities $u\colon E(G) \to \mathbb{R}_+$, a source $s \in V(G)$ and a sink $t \in V(G)$.
Output: an s-t-flow of maximum value.

Theorem 1 (Dantzig and Fulkerson 1956)

MFP always has an optimal solution. If the capacities are integer, there exists an optimal solution f^* *with* $f^*(e) \in \mathbb{Z}_+$ *for all* $e \in A(G)$*.*

The minimum s-t cut problem (MCP)

Definition 2 (s-t-cut and it capacity)

Consider a capacitated network (G, u, s, t), with a directed graph G, capacities $u\colon E(G) \to \mathbb{R}_+$ and two specific vertices $s, t \in V(G)$, where s is the source and t is the sink. Any subset $X \subset V(G)$ with $s \in X$ and $t \notin X$ defines an s-t-cut $\delta(X) := \{(x, y) \in A(G)\colon x \in X, y \notin X\}$. The capacity of the cut $\delta(X)$ is given as $u(\delta(X)) := \sum_{e \in \delta(X)} u(e)$.

The minimum cut problem (MCP)
Input: a network (G, u, s, t), with a directed graph G, capacities $u\colon E(G) \to \mathbb{R}_+$, a source $s \in V(G)$, and a sink $t \in V(G)$.
Output: an s-t-cut cut of minimum capacity.

Theorem 2 (max-flow min-cut theorem, Ford and Fulkerson 1956)

In a network (G, u, s, t), with a directed graph G, capacities $u\colon E(G) \to \mathbb{R}_+$, a source $s \in V(G)$ and a sink $t \in V(G)$, the maximum value of an s-t-flow equals the minimum capacity of an s-t-cut.

Decomposition of flows

Theorem 3 (Gallai 1958, Ford and Fulkerson 1962)

Consider a capacitated network (G, u, s, t), with a directed graph G, capacities $u\colon E(G) \to \mathbb{R}_+$ and two specific vertices $s, t \in V(G)$, where s is the source and t is the sink. Every s-t-flow can be decomposed into flows along directed s-t-paths from a familiy of paths $\mathcal{P}$ and flows along directed cycles from a family of cycles $\mathcal{C}$ in G such that the following conditions hold:

(i) *there exists a weighting function $w\colon \mathcal{C} \cup \mathcal{P} \to \mathbb{R}_+$ with*

$$f(e) = \sum_{P \in \mathcal{P}\colon\, e \in A(P)} w(P) + \sum_{C \in \mathcal{C}\colon\, e \in A(C)} w(C), \text{ for all } e \in A(G),$$

(ii) $val(f) = \sum_{P \in \mathcal{P}} w(P)$,

(iii) $|\mathcal{P}| + |\mathcal{C}| \leq |A(G)|$.

Moreover, if f is integral, w can be chosen to be integral.

Algorithmics for MFP and MCP

Efficient algorithms for MFP:
Augmenting shortest paths $O(|V(G)||A(G)|^2)$ (Edmonds and Karp 1972),
Push-Relabel $O(|V(G)|^2\sqrt{|A(G)|})$ (Goldberg and Tarjan 1988),
...

Efficient algorithm for the flow decomposition:
constructive algorithm $O(|V(G)||A(G)|)$ (Gallai 1958, Ford and Fulkerson 1962)

Efficient algorithm for MCP: Constructive algorithm based on depth first search starting at s on the residual graph of an optimal flow, same time complexity as the corresponding MFP algorithm (Ford and Fulkerson 1956)