Game Theory, summer term 2018

Exercise Sheet 1

1. Consider a two-persons zero-sum game and check whether the following statements are true or false. Prove the true statements and a give counterexamples for the false statements.
   (a) All sadlle points (assumning that there is at least one) result in the same payoff to player I.
   (b) If the game has a saddle point in every 2×2 submatrix than it has a saddle point.

2. Find the value of the zero-sum game given by the following payoff matrix and determine some optimal strategies for each of the players:
$$A = \begin{pmatrix} 0 & 9 & 1 & 1 \\ 5 & 0 & 6 & 7 \\ 2 & 4 & 3 & 3 \end{pmatrix}.$$
(A solution by linear programming is not desirable.) Hint: Use domination techniques to reduce the size of the game.

3. Find the value of the zero-sum game given by the following payoff matrix and determine <u>all</u> optimal strategies for each of the players:
$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \\ 2 & 2 \end{pmatrix}.$$
(A solution by linear programming is not desirable.)

4. (a) Given an $m \times n$ matrix of a two-persons zero-sum game. How would you quickly determine by hand if it has a saddle point?
   (b) Define a two-persons zero-sum game in which one player's unique optimal strategy is pure and all of the other player's optimal strategies are mixed.

5. Two companies plan to open restaurants in three candidate locations with coordinates $(1, 0)$, $(2, 0)$ and $(3, 0)$ on a system of coordinates in the plane. The three locations are called Left, Central and Right, respectively. Company I opens one restaurant at one of these locations and company II open two restaurants (both restaurants can be at the same location). A customer is located at a uniformly random point $(x, 0)$ for $x \in [0, 4]$. He walks to the next closest location at which there is a restaurant and then into one of the restaurants there, chosen uniformly at random. The payoff of company I or company II is the probability that the customer visits a company I restaurant or a company II restaurant, respectively. Represent this situation a two-persons zero-sum game, determine the value of the game and find some optimal mixed strategies for the companies.

6. Find a two-persons zero-sum game which has at least one Nash equilibrium in pure strategies, but in which iterative elimination of dominated strategies yields a game with no Nash equilibria in pure strategies.

7. A Nash equilibrium (x^*, y^*) in a two-persons zero-sum game with an $m \times n$ payoff matrix A, $m, n \in \mathbb{N}$, is called strict if any deviation of player I (player II) from the strategy x^* (y^*) yields a smaller gain of player I (larger loss of player II), i.e. $x^t Ay^* < (x^*)^t Ay^*$, $\forall x \in \Delta_m \setminus \{x^*\}$ $((x^*)^t Ay > (x^*)^t Ay^*$, $\forall y \in \Delta_n \setminus \{y^*\}$. Further consider the following

definition of *strict dominated strateiges* [1]: a pure strategy i of player I is dominated by a strategy i' of player I iff for any strategy j of player II $a_{ij} < a_{i'j}$ holds, $i, i' \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$.

(a) Prove that if the process of iterative elimination of strictly dominated strategies results in a unique strategy vector (x^*, y^*), then the later is a strict Nash equilibrium and, it is the only Nash equilibrium of the game.

(b) Prove that if (x^*, y^*) is a pure strict Nash equilibrium, then none of the pure strategies x^*, y^* can be eliminated by iterative elimination of dominated strategies (under either strict or non-strict domination).

8. Two players each choose a positive integer. The player who chose the lower number pays one to the player who chose the larger number (with no payment in case of a tie). Show that this game has no Nash equilibrium. Show that the safety values for players I and II are -1 and 1, respectively.

9. Player I chooses a positive integer $x > 0$ and player II chooses a positive integer $y > 0$. The player with the lower number pays one unit to the player with the higher number unless the higher number is more than twice the lower number in which case the payments are reversed. Thus we have

$$A(x, y) = \begin{cases} 1 & \text{if } y < x \leq 2y \text{ or } x < y/2 \\ -1 & \text{if } x < y \leq 2x \text{ or } y < x/2 \\ 0 & \text{if } x = y \end{cases}$$

Find the unique optimal strategy in this game.

10. [2] Show that the optimal value of a two-persons zero-sum game with an $m \times n$ payoff matrix A can be determined by solving a linear program. Show moreover that the Minimax Theorem of von Neumann follows as a consequence of the strong duality theorem of the linear programming[3].

---

[1] Note that this definition of *strictly dominating strategies* is different from the definition given in the lecture. The definition given in the lecture coincides with the definition in the book `Game Theory, alive, by A.R. Karlin and Y. Peres, American Mathematical Society, 2017, Providence, Rhode Island, USA`, wheres the definition we are adopting in this exercise coincides with the definition in `Game Theory, by M. Maschler, E. Solan and S. Zamir, Cambridge University Press, 2013, Cambridge, UK`.

[2] Some background in linear programming is needed.

[3] The converse is also true, i.e. the strong duality theorem of linear programming can be derived as a corrollary of the Minimax Theorem of von Neumann, thus both theorems are equivalent.