Claims involved in the proof of Edmonds-Gallai Decomposition Theorem

Consider the maximum cardinality matching produced by Edmonds' blossom algorithm applied to an undirected graph $G = (V, E)$. Let X be the set of vertices which are unmathced in M. Define the following three subsets of V:

$$\text{Even} := \{v \in V\colon\ \exists\, x \in X \text{ and an } M\text{-alternating path of even length from } x \text{ to } v\}$$

$$\text{Odd} := \{v \in V\colon\ \exists\, x \in X \text{ and an } M\text{-alternating path from } x \text{ to } v\} \setminus \text{Even}$$

$$\text{Free} := \{v \in V\colon\ \forall x \in X \text{ there exists no } M\text{-alternating path from } x \text{ to } v\}$$

Let *shrunk graph* G_0 be the graph obtained in the final iteration of Edmonds' algorithm on G.

Claim 1 If $(u, v) \in E$ and $u \in$ Even, then there is an M-alternating walk of odd length from X to v and there is an M-alternating path from X to v.

Corollary There is no edge between Even and Free in G.

Claim 2 In G_0 there is no edge between two even vertices.

Claim 3 Even $= D(G) := \{v \in V\colon$ there exists a maximum cardinality matching in G which does not match $v\}$

Claim 4 Odd $= A(G) := \{v \in V\colon$ v is neighbor of some $u \in D(G)$ but $v \notin D(G)\}$

Claim 5 Free $= C(G) := V \setminus (D(G) \cup A(G)$

Claim 6 $|M \cap C(G)| := \frac{|C(G)|}{2}$

Claim 7 For every connected component H of $(G \setminus A(G) \cap D(G)$:

(a) either $|X \cap H| = 1$ and $M \cap \delta(H) = \emptyset$ or $X \cap H = \emptyset$ and $|M \cap \delta(H)| = 1$, where $\delta(H)$ is the set of edges with exactly one endvertex in H.

(b) H is factor-critical.

Claim 8 $|M| = \frac{1}{2}(|V| + |A(G)| - o(G \setminus A(G)))$