Combinatorial Optimization 2
Summer term 2019

1st work sheet

In this working sheet we use the following abbreviations and notations on a graph $G = (V, E)$ with edges weights $c\colon E \to \mathbb{R}$ and some mapping $x\colon E \to \mathbb{R}$.

- MWMP - the Maximum Weight Matching Problem,
- MWPMP - the Minimum Weight Perfect Matching Problem,
- $\delta(v)$ - the set of edges with an endvertex in v, for any vertex $v \in V$,
- $\delta(S)$ - the set of edges with only one endvertex in S, i.e. $\delta(S) := \{e \in E\colon |e \cap S| = 1\}$, for any set of vertices $S \subseteq V$,
- $x(F) := \sum_{e\in F} x_e$ - for any set of edges $F \subseteq E$,
- $\gamma(S)$ - the set of edges with both endvertices in S, i.e. $\gamma(S) := \{e \in E\colon |e \cap S| = 2\}$, for any set of vertices $S \subseteq V$,
- $\mathcal{S}$ - the set of subsets of V of odd cardinality,
- $\mathcal{S}^o$ - the set of subsets of V of odd cardinality larger than 1.

1. Find a max-size matching (i.e. a matching with maximum cardinality) and a minimizing set of vertices in the Tutte-Berge Formula for the graph in Figure 1.
2. Find the Edmonds-Gallai Decomposition for the graph in Figure 1.
3. *Slither* is a two-person game played on a graph $G = (V, E)$. The players, called First and Second, play alternatively with First playing first. At each step the player whose turn it is chooses a previously unchosen edge. The only rule is that at every step the set of chosen edges forms a simple path, i.e. un unclosed walk without any vertex repetitions. The loser is the first player unable to make a legal play at his or her turn.

   Let $(D(G), A(G), C(G))$ be the Edmonds-Gallai Decomposition of G (see the lecture to recall the meaning of these notations). Prove that, if $C(G) \neq \emptyset$, then First can force a win at slither.
4. Consider the MWMP on the weighted graph G given in Figure 2 (the numbers next to the edges are their weights). Construct a graph $\hat{G}$ with twice as many vertices as G and edge weights $\hat{c}$ in $\hat{G}$ such that the MWMP on (G, c) is equivalent to the MWPMP on $(\hat{G}, \hat{c})$ (see the lecture).
5. Consider the MWPMP on a graph G with edge weights $c\colon E \to \mathbb{R}$ and the following linear optimization problem

   $$\min\left\{\sum_{e\in E} c_e x_e \colon x(\delta(v)) \geq 1, \forall v \in V,\ x_e \geq 0, \forall e \in E\right\}.$$

   This linear optimization problem will be denoted by LP_FPMP. (The restrictions of the above linear optimization problem define the *fractional perfect matching polytope*, see the lecture.)

   (a) Show that in general the optimal value of LP_FPMP is a non-tight lower bound LB on the optimal value of the MWPMP on (G, c). This can be done by specifying an instance (G, c) of MWPMP with an optimal value which is strictly larger than the optimal value of the corresponding LP_FPMP.

(b) Consider the characteristic vector $x \in \{0,1\}^{|E|}$ of some perfect matching M in G given by $x_e = 1$ iff $e \in M$; it clearly fulfills the so-called *blossom inequalities* $x(\delta(S)) \geq 1$, for any $S \in \mathcal{S}$. Consider now the linear optimization problem $LP\_FPMP(\mathcal{S}_1)$ obtained by adding to LP_FPMP the blossom inequalities for all $S \in \mathcal{S}_1$, where $\mathcal{S}_1 \subseteq \mathcal{S}^o$. Observe that the optimal value $LB(\mathcal{S}_1)$ of this problem is also a lower bound on the optimal value of the MWPMP on (G,c) and that $LB(\mathcal{S}_1) \geq LB$. For the instance constructed as a solution of (a) find a set $\mathcal{S}_1$ such that $LB(\mathcal{S}_1) > LB$ holds.

6. Consider the MWMP on a graph $G = (V,E)$ with edge weights c_e, for $e \in E$. Show that the optimal value of the following linear optimization problem equals the optimal value of the MWMP on (G,c):

$$\max\left\{\sum_{e\in E} c_e x_e \colon x(\delta(v)) \leq 1\,, \forall v \in V, x(\gamma(S)) \leq (|S|-1)/2, \forall S \in \mathcal{S}^o\,, x_e \geq 0\,, \forall e \in E\right\}.$$

Hint: Recall from the lecture an analogous theorem discussing the relationship between MWPMP and an analogous linear programming problem. Apply that theorem to the equivalent formulation of MWMP as an MWPMP on some graph $(\hat{G},\hat{c})$.

7. Consider a graph $G = (V,E)$ with $c_e = 1$ for all $e \in E$ and let $(D(G), A(G), C(G))$ be the Edmonds-Gallai Decomposition of G. Define z_v to be 0 if $v \in D(G)$, 1 if $v \in A(G)$, and 1/2 if $v \in C(G)$. Define Z_S to be 1 if $S \in \mathcal{S}^o$ is a connected component of the subgraph induced in G by $D(G)$, and 0 otherwise. Show that z is an optimal solution of the dual of the linear programming problem given in Exercise 6. Derive properties of maximum matchings by applying the complementary slackness theorem of linear programming in this case.

8. Apply Edmonds' Blossom Algorithms for MWPMP on the instance in Figure 3 and find a minimum weight perfect matching as well an optimal solution of the involved dual linear optimization problem. The letters next to the vertices in Figure 3 are the labels of the verticea and the numbers next to the edges are their weights.

A more challenging exercise

Suppose that, in a game of slither the nodes of a path of played edges so far are all in $A(G)$, where $(D(G), A(G), C(G))$ is the Edmonds-Gallai Decomposition of G. Suppose moreover that the next player now chooses an edge having an endvertex not in $A(G)$. Prove that the other player can force a win.

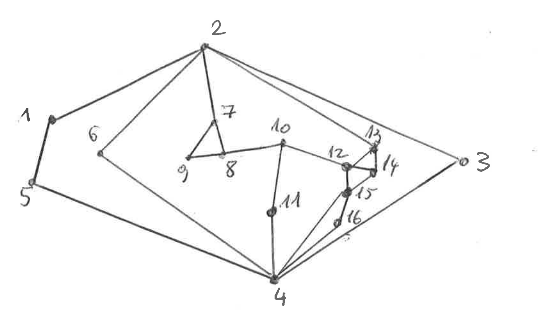

Figure 1: Inputgraph for Exercise 1 and Exercise 2

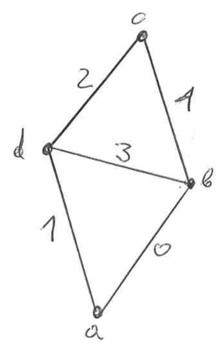

Figure 2: Inputgraph for Exercise 4

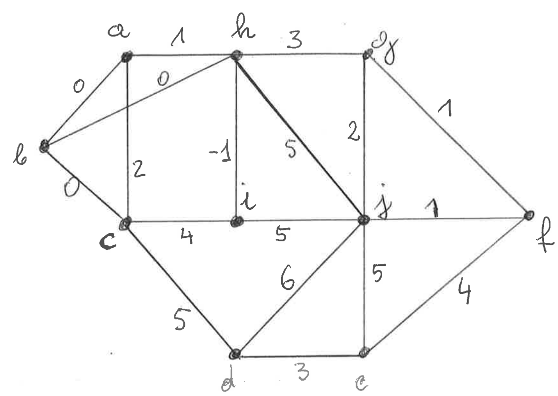

Figure 3: Inputgraph for Exercise 8