Combinatorial Optimization 2
Summer term 2019

Second work sheet

9. Consider a (finite) graph $G = (V, E)$ and the following definitions. A *stable set* in G is a subset $S \subseteq V$ of vertices such that there is no edge in E connecting two vertices in S. The *stability number* $\alpha(G)$ of G is the cardinality of the largest stable set in G. An *edge cover* $F \subseteq E$ in G is a set of edges such that for every vertex $v \in V$ there exists an edge $e \in E$ which is incident to v. The *edge cover number* $\xi(G)$ of G is the cardinality of an edge cover with minimum cardinality. A *vertex cover* in $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that every edge $e \in E$ is incident to at least one vertex in C. The *vertex cover number* $\tau(G)$ of G is the cardinality of a vertex cover with minimum cardinality in G. Recall finally that the *matching number* $\nu(G)$ of G is the cardinality of a matching with maximum cardinality in G.

   Consider the convex hull of the characteristic vectors of all

   (a) vertex covers
   (b) stable sets
   (c) edges covers

   in a bipartite graph. Exploit the total unimodularity of the vertex-edge incidence matrix of a bipartite graph to give a representation of the convex hulls mentioned above as pohyhedra defined by the corresponding linear inequalities. Conclude that in a bipartite graph without isolated vertices $\nu(G) = \tau(G)$ and $\xi(G) = \alpha(G)$ hold.

10. Formulate and prove a weighted version of the following Theorem of König “In a bipartite graph the maximum cardinality of a matching equals the minimum cardinality of a vertex cover” (see Exercise 9 for the difinition of a vertex cover).

    Hint: Use the fact that the fractional matching polytope of a bipartite graph is integral. Why ist that the case? Refresh your memory about the relationship between integral polytopes and total unimodular matrices.

11. The half-integrality of the fractional perfect matching polytope.

    Let $G = (V, E)$ be a graph and P the fractional perfect matching polytope of G. Prove that the vertices of P are excatly the vectors x with

$$x_e = \begin{cases} \frac{1}{2} & \text{für } e \in E(C_1) \cup E(C_2) \cup \ldots \cup E(C_k) \\ 1 & \text{für } e \in M \\ 0 & \text{sonst} \end{cases},$$

    where $C_1, \ldots, C_k$ are vertex-disjoint odd cycles in G and M is a perfect matching in $G \setminus (V(C_1) \cup \ldots \cup V(C_k))$. Recall that $x \in \mathbb{R}^m$ is a vertex of a polytope $P \subset \mathbb{R}^m$ iff x is an extremal point of P, i.e. it is not possible to represent x as a convex combination of points from $P \setminus \{x\}$.

12. Given the undirected graph $G = (V, E)$ with edge weights $c \colon E(G) \to \mathbb{R}_+$ and two vertices $s, t \in v(G)$, $s \neq t$, we look for a shortest s-t-path with an even (or with an odd) number of edges. Give an equivalent formulation of this problem as a Minimum Weight Perfect Matching Problem (MWPMP) in some auxiliary graph.

    Hint : Takes two copies of G, connect each vertex with its copy by an edge of zero length and delete s and t (or s and the copy of t).

13. In the bottleneck perfect matching problem we are given an undirected graph $G = (V, E)$ with edge weights $c: E \to \mathbb{R}$ and search for a perfect matching M^* in which the maximum weight over all matching edges is as small as possible, i.e. $b(M^*) = \min\{b(M): M \text{ is a perfect matching in } G\}$, where $b(M) := \max\{c(e): e \in M\}$ for any perfect matching M in G. Derive an efficient algorithm to solve this problem.

14. In the minimum weight edge cover problem we are given an undirected graph $G = (V, E)$ with edge weights $c: E \to \mathbb{R}$. The task is to find an edge cover F with minimum weight $c(F) := \sum_{e \in F} c(e)$ (see Exercise 9 for the definition of an edge cover). Derive an efficient algorithm to solve this problem.

15. Give a graph $G = (V, E)$ which does not contain a simple perfect 2-matching and give a concrete pair of sets $X, Y \subseteq V$ of vertices in G which violate the Tutte condition for the existence of a simple perfect 2-matching in G (cf. lecture for the corresponding definitions and the existence criterion for perfect b-matchings)

16. Let $G = (V, E)$ be an indirected graph with egde weights $c: E \to \mathbb{R}$ and $n := |V|$. Show that the problem of finding a simple perfect 2-matching of minimum weight in G can be equivalently formulated as an instance of the minimum weight perfect matching problem on an auxiliary graph with $O(n^2)$ vertices. Deduce then that the minimum weight simple 2-matching problem in G can be solved in $O(n^6)$ time.