Combinatorial Optimization 2
Summer Term 2019

fifth work sheet

28. Consider the following local search algorithm for the Maximum Cut Problem (i.e. the problem of finding a cut with maximum cardinality in a given undirected graph). Start with an arbitrary nonempty proper subset $S \subset V(G)$ of the input graph G. Check iteratively if some vertex can be added to S or deleted from S such that $|\delta(S)|$ increases. Stop if no such improvement is possible.

    (a) Show that any undirected graph has a cut containing at least half of the edges.

    (b) Prove that the local search algorithm described above is a 2-approximation algorithm.

    (c) Can the algorithm be extended to the Maximum Weight Cut Problem with nonnegative weights on the edges?

    (d) Does the above local search algorithm always find an optimal solution for bipartite graphs?

29. Show that the following graphs are perfect:

    (a) bipartite graphs

    (b) interval graphs defined as graphs $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and

    $$E = \Big\{\{i, j\}: i, j \in \{1, 2, \dots, n\}, [a_i, b_i] \cap [a_j, b_j] \neq \emptyset\Big\},$$

    for some $n \in \mathbb{N}$ and a set of intervals $[a_i, b_i]$ on the real line with $a_i < b_i$, for $i \in \{1, 2, \dots, n\}$.

    (c) chordal graphs defined as graphs having no cycles of length at least 4 as induced subgraphs (an equivalent characterization of chordal graphs is given at the end of this work sheet).

30. Consider the linear time algorithm for the Weighted Median Problem discussed in the lecture.

    (a) Prove the correctness of the algorithm by showing that it outputs the correct weighted median in all cases of the "if" query.

    (b) Apply the algorithm to solve an instance of the Weighted Median Problem with input $n = 11$, $z = (3, 5, 2, 4, 10, 7, 9, 8, 11, 10, 6)$, $w = (1, 3, 2, 5, 2, 1, 4, 3, 2, 1, 3)$, $W = 16$.

31. Consider an instance of the Knapsack Problem with $n = 6$ objects, profit vector $c = (3, 2, 1, 2, 3, 1)$, weight vector $w = (3, 4, 2, 5, 4, 3)$ and weight limit $W = 14$.

    (a) Apply the greedy algorithm to solve the corresponding instance of the Fractional Knapsack Problem.

    (b) Give an instance of the Weighted Median Problem equivalent to the above instance of the Fractional Knapsack Problem.

    (c) Use the solution of the above instance of the Fractional Knapsack Problem to construct a 2-approximation of the optimal solution of the original Knapsack Problem.

    (d) Apply the exact algorithm discussed in the lecture to solve the above instance of the Knapsack Problem.

32. The k-Center Problem is defined as follows. Given an undirected graph G with edge lengths $c: E(G) \to \mathbb{R}_+$ and a number $k \in \mathbb{N}$ with $k \leq |V(G)|$, find a set $X \subseteq V(G)$ of cardinality k which minimizes

    $$\max_{v \in v(G)} \min_{x \in X} dist(v, x),$$

    where $dist(v, x)$ denotes the length of a shortest path connecting v and x in G. Let $OPT(G, c, k)$ denote the optimal value of this problem.

(a) Let S be a maximal stable set in the graph $\Big(V(G), \Big\{\{v,w\}: dist(v,w) \leq 2R\Big\}\Big)$ for some $R \in \mathbb{R}_+$. Show that then $OPT(G, c, |S|-1) > R$.

(b) Use (a) to describe a 2-factor approximation algorithm for the k-Center Problem [2].

33. Consider the m-Dimensional Knapsack Problem defined as follows. The instance consists of a natural number n and nonnegative integers c_i, w_{ij}, W_j, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The task is to find a subset $S \subseteq \{1, 2 \ldots, n\}$ such that $\sum_{i=1}^{n} w_{ij} \leq W_j$ holds, for all $j \in \{1, 2, \ldots, m\}$ and $\sum_{i=1}^{n} c_i$ is maximized. Give a pseudopolynomial time exact algorithm for the m-Dimensional Knapsack Problem.

    Hint: Construct first a pseudopolynomial time exact algorithm with time complexity $O(nW)$ for the knapsack problem. Then generalize it to a pseudopolynomial time exact algorithm for the m-Dimensional Knapsack Problem. Notice that for any $m \geq 2$ there exists a polynomial time approximation scheme but no fully polynomial time approximation scheme for the m-Dimensional Knapsack Problem as shown in [1] and [3], respectively.

Some properties and definitions related to chordal graphs.
An order $\{v_1, v_2, \ldots, v_n\}$ of the n vertices of an undirected graph G is called a *simplicial order* if $(v_i v_j) \in E(G)$ and $(v_i, v_k) \in E(G)$ imply $(v_j, v_k) \in E(G)$, for all $i < j < k$, $i, j, k \in \{1, 2, \ldots, n\}$.
An order $\{v_1, v_2, \ldots, v_n\}$ of the n vertices of an undirected graph G is called a *maximum adjacency order (MA order)* if for all $i \in \{2, \ldots, n\}$ the following holds:

$$\left|E\Big(\{v_1, v_2 \ldots, v_{i-1}\}, \{v_i\}\Big)\right| = \max_{j \in \{i, \ldots, n\}} \left|E\Big(\{v_1, v_2 \ldots, v_{i-1}\}, \{v_j\}\Big)\right|.$$

It can be shown that if $\{v_1, v_2, \ldots, v_n\}$ is an MA order of a chordal graph G than $v_n, v_{n-1}, \ldots, v_1$ is a simplicial order in G.
Further, a graph is chordal if and only if it has *a simplicial order* [4].

References

[1] A.M. Frieze and M.R.B. Clarke, Approximation algorithms for the m-dimensional 0-1 knapsack problem: worst case and probabilistic anylysis, *European Journal of Operations Research* **15**, 1984, 100–109.

[2] D. Hochbaum and D.B. Schmoys, A best possible heuristic for the k-center problem, *Mathematics of Operations Research* **10**, 1985, 180–184.

[3] B. Korte and R. Schrader, On the existence of fast approximation schemes, in *Nonlinear Programming, Vol. 4* (O. Mangasarian, R.R. Mayer, S.M. Robinson, eds.), Academic Press, New York, 1981, pp. 415–437.

[4] D.J. Rose, Triangulated graphs and the elimination process, *Journal of Mathematical Analysis and Applications* **32**, 1970, 597–609.