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Then M is the copula of $(X, T(X))^T$ and W is the copula of $(X, S(X))^T$.

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Theorem: Assume that W or M is a copula of $(X_1, X_2)^T$. Then there exist two monotone functions $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}$ and a r.v. Z, such that

$$(X_1, X_2) \stackrel{d}{=} (\alpha(Z), \beta(Z)) \,.$$

If M is the copula of $(X_1, X_2)^T$, then both α and β are monotone increasing, if W is the copula of $(X_1, X_2)^T$, then one of the functions α, β is monotone increasing and the other one is monotone decreasing.

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If C is the copula of (X_1, X_2) and the marginal d.f. F_1 and F_2 of (X_1, X_2) are continuous, then the following hold:

$C = W$ iff $X_2 \stackrel{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ (1 - F_1)$ monotone decreasing,
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Proof: In McNeil et al., 2005.

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Theorem: Let $(X_1, X_2)^T$ be a random vector with marginal d.f. F_1, F_2 and some unknown copula. Let $var(X_1), var(X_2) \in (0, \infty)$ hold. Then the following statements hold:

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The proof uses the equality of Höffding:
Lemma: (The Höffding equality)
Let $(X_1, X_2)^T$ be a random vector with c.d.f. F and marginal d.f. F_1, F_2.
If $cov(X_1, X_2) < \infty$ then the following equality holds:

$$cov(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1)F_2(x_2))dx_1dx_2 \,.$$

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Example: Let X_1, X_2 be two random variables with $X_1 \sim \mathit{Lognormal}(0,1)$, $X_2 \sim \mathit{Lognormal}(0,\sigma^2)$, $\sigma > 0$. Determine Sie $\rho_{L,min}(X_1,X_2)$ und $\rho_{L,max}(X_1,X_2)$.

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Hint: Observe that $X_1 \stackrel{d}{=} \exp(Z)$ and $X_2 \stackrel{d}{=} \exp(\sigma Z) \stackrel{d}{=} \exp(-\sigma Z)$. Moreover e^Z, $e^{\sigma Z}$ are co-monotone and e^Z,$e^{-\sigma Z}$ are anti-monotone.

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Example: Determine two random vectors $(X_1,X_2)^T$ and $(Y_1,Y_2)^T$ with different c.d.f.s such that $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds while $X_1,X_2,Y_1,Y_2 \sim N(0,1)$ and $\rho_L(X_1,X_2)=0$, $\rho_L(Y_1,Y_2)=0$ also hold.

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If $(X_1,X_2)^T$, $(Y_1,Y_2)^T$ represent the asset returns of two different portfolios consisting of two assets each, then we have two portfolios with the same marginal distributions of their assets and the same linear correlation coefficient, respectively, but having different value at risk.

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Conclusion: The marginal distributions of the assets in a portfolio and the linear correlation between the assets do not determine the loss distribution, in particular, they do not determine the risk measure of the portfolio.

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Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two samples of a random vector $(X, Y)^T$. $(x, y)^T$ und $(\tilde{x}, \tilde{y})^T$ are called *concordant* if $(x - \tilde{x})(y - \tilde{y}) > 0$ and *discordant* if $(x - \tilde{x})(y - \tilde{y}) < 0$.

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The sample Kendall's Tau:

Let $\{(x_1, y_1)^T, (x_2, y_2)^T, \ldots, (x_n, y_n)^T\}$ be a sample of size n of the random vector $(X, Y)^T$ with continuous marginal distributions. Let c be the number concordant pairs in the sample and let d be the number of discordant pairs. Then the sample Kendall's Tau is given as

$$\tilde{\rho}_\tau(X, Y) = \frac{c - d}{c + d} \overset{a.s.}{=} \frac{c - d}{n(n-1)/2}$$

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Equivalent definition (without a proof):
Let F_1 und F_2 be the continuous marginal distributions of $(X_1, X_2)^T$. Then $\rho_S(X_1, X_2) = \rho_L(F_1(X_1), F_2(X_2))$ holds, i.e. the Spearman's Rho is the linear correlation of the unique copula of $(X_1, X_2)^T$.

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In the d-dimensional case $X \in \mathbb{R}^d$:
$\rho_S(X) = \rho(F_1(X_1), F_2(X_2), \ldots, F_d(X_d))$ is the correlation matrix of the unique copula of X, where F_1,F_2,...,F_d are the continuous marginal distributions of X.

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Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

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- X_1, X_2 are co-monotone iff $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = 1$. X_1, X_2. X_1, X_2 are anti-monotone iff $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = -1$.

Properties of ρ_τ and ρ_S.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

$$\rho_\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1$$

$$\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2 =$$
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- Let F_1, F_2 be the continuous marginal distributions of $(X_1, X_2)^T$ and let T_1, T_2 be strictly monotone functions on $[-\infty, \infty]$. Then the following equalities hold $\rho_\tau(X_1, X_2) = \rho_\tau(T_1(X_1), T_2(X_2))$ and $\rho_S(X_1, X_2) = \rho_S(T_1(X_1), T_2(X_2))$.

(See Embrechts et al., 2002).

Tail dependence coefficients

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Definition: Let $(X_1, X_2)^T$ be a random vector with marginal distributions F_1 und F_2.

The coefficent $\lambda_U(X_1, X_2)$ *of the upper tail dependency* of $(X_1, X_2)^T$ is defined as $\lambda_U(X_1, X_2) = \lim_{u \to 1^-} P(X_2 > F_2^{\leftarrow}(u) | X_1 > F_1^{\leftarrow}(u))$, provided that the limit exists.

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If the limit exists and $\lambda_U > 0$ ($\lambda_L > 0$) we say that $(X_1, X_2)^T$ have an upper (a lower) tail dependence.

Tail dependency and survival copulas

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Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \ldots, U_d)$ with $U_i \sim U[0, 1]$, $i = 1, 2, \ldots, d$. The c.d.f. of $(1 - U_1, 1 - U_2, \ldots, 1 - U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

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Lemma: Let X be a random vector with multivariate tail distribution function $\bar{F}$ ($\bar{F}(x_1, x_2, \ldots, x_d) := Prob(X_1 > x_1, X_2 > x_2, \ldots, X_d > x_d)$) and marginal distributions F_i, $i = 1, 2, \ldots, d$. Let $\bar{F}_i := 1 - F_i$, $i = 1, 2, \ldots, d$. Then the following holds

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Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a unique copula C. The following equalities hold $\lambda_U(X_1, X_2) = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u}$ and $\lambda_L(X_1, X_2) = \lim_{u \to 0^+} \frac{C(u,u)}{u}$, provided that the limits exist.

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The Gumbel family of copulas:

$$C_\theta^{\mathrm{Gu}}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right),\ \theta \geq 1$$

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The Clayton family of copulas:

$$C_\theta^{\mathrm{Cl}}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{1/\theta},\ \theta > 0$$

We have $\lambda_U = 0$, $\lambda_L = 2^{-1/\theta}$.