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Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$ be constants, and let $\psi\colon [0, \infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T \Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X - \mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu, \Sigma, \psi)$.

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Theorem:(Stochastic representation)
A d-dimensional random vector X is elliptically distributed, $X \sim E_d(\mu, \Sigma, \psi)$ with *rang*$(\Sigma) = k$, iff there exist a matrix $A \in \mathbb{R}^{d\times k}$, $A^T A = \Sigma$, a nonnegative r.v. R and a k-dimensional random vector U uniformly distributed on the unit ball $\mathcal{S}^{k-1} = \{z \in \mathbb{R}^k\colon z^T z = 1\}$, such that R and U are independent and $X \stackrel{d}{=} \mu + RAU$.

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Remark: An elliptically distributed random vector X ist *radial symmetric*, i.e. $X - \mu \stackrel{d}{=} \mu - X$.

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Definition: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with c.d.f. F and continuous marginal distributions $F_1, F_2, \dots, F_d$. The unique copula C of X (or F) with $C(u) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))$, is called an *elliptical copula*.

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Let C_R^{Ga} be the copula of a d-dimensional normal distribution with correlation matrix R. Then $C_R^{Ga}(u) = \phi_R^d(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d))$ holds, where ϕ_R^d is the c.d.f. of a d-dimensional normal distribution with expected vector 0 and correlation matrix R, and ϕ^{-1} is the inverse of the standard normal distribution function.

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In the bivariate case we have:

$$C_R^{Ga}(u_1, u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ \frac{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)} \right\} dx_1 dx_2,$$

where $\rho \in (-1, 1)$.

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Definition: Let $X \stackrel{d}{=} \mu + \frac{\sqrt{\alpha}}{\sqrt{S}} AZ \sim t_d(\alpha, \mu, \Sigma)$, where $\mu \in \mathbb{R}^d$, $\alpha \in \mathbb{N}$, $\alpha > 1$, $S \sim \chi^2_\alpha$, $A \in \mathbb{R}^{d \times k}$ with $AA^t = \Sigma$, $Z \sim N_k(0, I_k)$, and S and Z independent. We say that X has a d-dimensional t-distribution with expectation μ (for $\alpha > 1$) and covariance matrix $Cov(X) = \frac{\alpha}{\alpha - 2}\Sigma$. ($\alpha > 2$ should hold, $Cov(X)$ does not exist for $\alpha \leq 2$.)

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Definition: The (unique) copula $C^t_{\alpha,R}$ of X is called t-copula:

$$C^t_{\alpha,R}(u) = t^d_{\alpha,R}(t^{-1}_\alpha(u_1), \ldots, t^{-1}_\alpha(u_d)).$$

$R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$, $i, j = 1, 2 \ldots, d$, is the correlation matrix of AZ.

$t^d_{\alpha,R}$ is the cdf of $\frac{\sqrt{\alpha}}{\sqrt{S}} Y$, where $S \sim \chi^2_\alpha$, $Z \sim N_k(0, R)$, and S, Y are independent. t_α are the marginal distributions of $t^d_{\alpha,R}$.

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In the bivariate case ($d = 2$):

$$C^t_{\alpha,R}(u_1, u_2) = \int_{-\infty}^{t^{-1}_\alpha(u_1)} \int_{-\infty}^{t^{-1}_\alpha(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\alpha(1-\rho^2)}\right\}^{-\frac{\alpha+2}{2}} dx_1 dx_2$$

for $\rho \in (-1, 1)$. R_{12} is the linear correlation coefficient of the corresponding bivariate t_α-distribution for $\alpha > 2$.

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Definition: (Radial symmetry)
A d-dimensional random vector X (or a d-variate distribution function) is called *radial symmetric around a*, for some $a \in \mathbb{R}^d$, iff $X - a \stackrel{d}{=} a - X$.

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A copula C is called radial symmetric iff

$$(U_1 - 0.5, \ldots, U_d - 0.5) \stackrel{d}{=} (0.5 - U_1, \ldots, 0.5 - U_d) \Longleftrightarrow U \stackrel{d}{=} 1 - U,$$

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The Gumbel and Clayton Copulas are not radial symmetric. Why?

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Let C be the copula of a distribution F with differentiable marginal distributions $F_1, \ldots, F_d$. By differentiating

$$C(u_1, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$$

we obtain the density c of C:

$$c(u_1, \ldots, u_d) = \frac{f(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \ldots f_d(F_d^{-1}(u_d))}$$

where f is the density function of F, f_i are the marginal density functions, and F_i^{-1} are the inverse functions of F_i, for $1 \leq i \leq d$,

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For such a copula $C(u_1, u_2, \ldots, u_d) = C(u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(d)})$ holds for any permutation $(\pi(1), \pi(2), \ldots, \pi(d))$ of $(1, 2, \ldots, d)$.

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Examples of exchangeable copulas:

Gumbel, Clayton, and also the Gaussian copula C_P^{Ga} and the t-Copula $C_{\nu,P}^t$, if P is an *equicorrelation matrix*, i.e. $R = \rho J_d + (1-\rho) I_d$.

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For bivariate exchangeable copulas we have:

$$P(U_2 \leq u_2 | U_1 = u_1) = P(U_1 \leq u_2 | U_2 = u_1).$$

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Theorem: Let $(X_1, X_2)^T$ be a normally distributed random vector. Then $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

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Theorem: Let $(X_1, X_2)^T \sim t_2(0, \nu, R)$ be a random vector with a t-distribution and ν degrees of freedom, expectation 0 and linear correlation matrix R. For $R_{12} > -1$ we have

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$$

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The proof is similar to the proof of the analogous theorem about the Gaussian copulas.
Hint:

$$X_2 | X_1 = x \sim \left(\frac{\nu+1}{\nu+x^2}\right)^{1/2} \frac{X_2 - \rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}$$

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